Introduction to Soft-Collinear Effective Theory (SCET)

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- Method of expanding Feynman integrals by momentum regions
- Soft-collinear effective theory
- Example of factorization : Sudakov form factor

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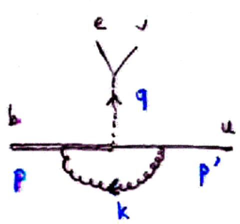
Expansion by regions [MB, Smirnov (1997)]



$p^2 = m^2$ $\quad p = m(1,\vec{0})$

$p'^2 = 0$ $\quad p' = E n_-$

$\quad = E(1,0,0,-1)$

$p \cdot p' = mE$

gluon mass λ

small scale

scalar integral for simplification

$$I = m^2 \frac{(4\pi)^2}{i} \int [dk] \frac{1}{[k^2-\lambda^2][k^2+2p\cdot k][k^2+2p'\cdot k]}$$

$$\equiv I\left(\frac{\lambda}{m}, \frac{E}{m}\right)$$

Aim :

Expansion in $\frac{\lambda}{m}$ for small energy $(E=0)$

and large energy $(E = \frac{m}{2})$

$\simeq$ factorization of the integral

$E = 0$

$$I(\tfrac{\lambda}{m}, 0) = \text{logs of square roots} = -\frac{\pi}{\lambda} + \left(-\frac{1}{2}\ln\lambda^2 + 1\right) + O(\lambda) \qquad [\text{set } m = 1]$$

Hard region $k \sim m$

$$I_h = m^2 \frac{(4\pi)^2}{i} \int [dk] \, \frac{1}{k^2 (k^2 + 2p\cdot k)\, k^2}$$

$+$ higher order in λ

$$= -\frac{1}{2\epsilon} - \frac{1}{2}\ln\mu^2 + 1 + O(\lambda^2)$$

↑ IR div

Taylor expansion of the integrand in λ^2

Soft region $k \sim \lambda$

$$I_s = m^2 \frac{(4\pi)^2}{i} \underbrace{\int [dk]}_{\lambda^4} \, \underbrace{\frac{1}{[k^2 - \lambda^2]\, k^2\, [2p\cdot k]}}_{1/\lambda^5} \left(\underbrace{1}_{1} - \underbrace{\frac{k^2}{2p\cdot k}}_{\lambda} + \ldots\right)$$

$$= -\frac{\pi}{\lambda} + \left[\frac{1}{2\epsilon} - \frac{1}{2}\ln\frac{\lambda^2}{\mu^2}\right] + O(\lambda)$$

↑ UV div

Taylor expansion of the integrand for soft k

→ $I = I_h + I_s$

→ single scale integrals

→ pre-determined λ-scaling

The rules

- Identify momentum regions: hard + regions where propagators become singular
- Expand the integrand in all quantities that are small in a given region
- Use dim or analytic reg, integrate over all d^dk in every region, drop scaleless integrals
- Add up everything

$E = \frac{m}{2}$

energetic, on-shell light

$$I\left(\frac{\lambda}{m}, \frac{1}{2}\right) = -\frac{1}{4}\left(\ln^2\lambda^2 + \pi^2\right) + \ldots\ldots$$

Hard

$$I_h = m^2 \frac{(4\pi)^2}{i} \int [dk] \frac{1}{k^2(k^2+2p\cdot k)(k^2+2p'\cdot k)} = -\frac{1}{2\epsilon^2} - \frac{\ln\mu^2}{2\epsilon} - \frac{1}{4}\ln^2\mu^2 - \frac{\pi^2}{24}$$

Soft $k \sim \lambda$

$$I_s = m^2 \frac{(4\pi)^2}{i} \int [dk] \frac{1}{[k^2-\lambda^2][2p\cdot k][2p'\cdot k]}$$

$$\propto \frac{1}{\epsilon}\mu^{2\epsilon} \int_0^\infty \frac{dx}{x}\,(x^2+\lambda^2)^{-\epsilon}$$

undefined in dim. reg.

→ analytic reg.

→ momentum region missing

Collinear $n_+ \cdot k \sim m$, $k^2 \sim \lambda^2$

$$I_c = m^2 \frac{(4\pi)^2}{i} \int [dk] \frac{1}{[k^2-\lambda^2][m n_+ k][k^2 + n_+ p' n_- k]}$$

→ also ill-def. in dim. reg.
but $I_s + I_c$ is def.

$$I = I_h + I_c + I_s$$

* works with more loops (presumably)

* Expansion of propagators + vertices in every region

↓

Introduce fields for every regions + kinetic terms and vertices that reproduce the expansion rules ⇒ effective field theory

* threshold expansion → NRQCD
hard / soft → OPE, HQET
hard / collinear / soft → SCET

Bauer, Pirjol, Stewart + Fleming, Manohar, Wise .... (2000-)
MB, Chapovsky, Diehl, Feldmann
Chay, Kim
Hill, Neubert + Becher, Lange, ...

Terminology

Short-distance scale
$Q \in \{Q, M, E\}$
$\gg \Lambda$

Power-counting parameter
$\lambda \equiv \sqrt{\Lambda/Q}$

Often set $Q \equiv 1$ for power counting

$p^\mu = n_+ p \, \frac{n_-^\mu}{2} + p_\perp^\mu + n_- p \, \frac{n_+^\mu}{2}$

$n_\pm^2 = 0 \qquad n_+ n_- = 2$
$(1,0,0,\pm 1)$

HARD $\quad p \sim Q\,(\overset{n_+p}{1}, \overset{p_\perp}{1}, \overset{n_-p}{1})$ — short-distance scale $p^2 \sim Q^2$

HARD-COLLINEAR $\quad p \sim Q\,(1, \lambda, \lambda^2)$ — intermediate scale $p^2 \sim Q\Lambda$ — still perturbative

[SEMI-HARD] $\quad p \sim Q\,(\lambda, \lambda, \lambda)$

COLLINEAR $\quad p \sim Q\,(1, \lambda^2, \lambda^4)$ — long-distance scale $p^2 \sim \Lambda^2$ — usually non-perturbative

SOFT $\quad p \sim Q\,(\lambda^2, \lambda^2, \lambda^2)$

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SCET Lagrangian

(hard-) collinear (light) quark $\qquad \xi \sim \lambda \qquad \not{n}_- \xi = 0 \qquad \psi = \underbrace{\frac{\not{n}_- \not{n}_+}{4}\psi}_{\xi} + \underbrace{\frac{\not{n}_+ \not{n}_-}{4}\psi}_{\eta \text{ small}}$

$$\xi(x)\bar{\xi}(y) \sim \underbrace{\int d^4p}_{\lambda^4} \, e^{-ip(x-y)} \underbrace{\frac{i n_+ p}{p^2}}_{1/\lambda^2} \sim \lambda^2$$

soft quark $\qquad q \sim \lambda^3$
$h_v \sim \lambda^3 \qquad \not{v} h_v = h_v \qquad$ (heavy quark near mass shell)

(hard-) collinear gluon $\qquad n_+ A_c \sim 1 \qquad A_{c\perp} \sim \lambda \qquad n_- A_c \sim \lambda^2$

soft gluon $\qquad A_s \sim \lambda^2$

[Gluon fields scale like the corresponding momenta.]

$\Longrightarrow$ Power counting of operators

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$$\mathcal{L}_{\text{light quark}} = \bar\Psi i\not{D}\Psi = \bar\xi \frac{\not{n}_+}{2} in_-D\,\xi + \bar\eta \frac{\not{n}_-}{2} in_+D_c\,\eta + \bar\xi\, i\not{D}_{\perp c}\,\eta + \bar\eta\, i\not{D}_{\perp c}\,\xi + \bar q\, i\not{D}_s\, q + \mathcal{O}(\lambda)$$

integrate out η in the path-integral $\longrightarrow$

$$\bar\xi\left(in_-D + i\not{D}_{\perp c}\frac{1}{in_+D_c}\, i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\xi + \bar q\, i\not{D}_s\, q$$

$$\text{QCD} \xrightarrow{\text{hard}} \text{SCET}(hc, c, s)$$

$$\eta = -\frac{\not{n}_+}{2}\frac{1}{in_+D_c + i\epsilon}\, i\not{D}_{\perp c}\,\xi + \ldots$$

$$= \bar\xi(x)\, in_-D \frac{\not{n}_+}{2}\xi(x) + i\int_{-\infty}^{0} ds\, [\bar\xi\, i\overleftarrow{\not{D}}_{\perp c} W_c](x)\, [W_c^\dagger i\not{D}_{\perp c}\frac{\not{n}_+}{2}\xi](x+sn_+) + \bar q\, i\not{D}_s\, q$$

$\uparrow$ soft gluon coupling to ξ only here

$$W_c \equiv P\exp\left(ig_s\int_{-\infty}^{0} ds\, n_+A_c(x+sn_+)\right) \quad \text{“Wilson line”}$$

$$\frac{1}{in_+D_c} = W_c \frac{1}{in_+\partial} W_c^\dagger, \quad W_c W_c^\dagger = 1, \quad [in_+D_c W_c] = 0$$

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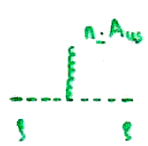
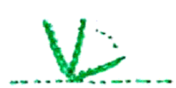
* SCET is a non-local EFT, because $n_+P_{(h)c} \sim 1$
local in $\perp$ and $n_-\partial$

* 



up to $2\,A_{\perp c}$
any n_+A_c

because $n_+A_c \sim 1$

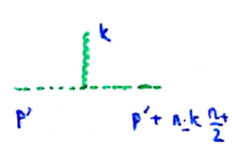
much simpler in light-cone gauge $n_+A_c = 0$ $(\rightarrow W_c = 1)$

* Multipole expansion
of soft field in products with hc fields $\int d^4x\, \bar\xi(x)\, n_-A_s(x)\, \xi(x)$

$\hookrightarrow$ $n_-x \sim 1$, $x_\perp \sim 1/\lambda$, $n_+x \sim 1/\lambda^2$

$$\phi_s(x) = \phi_s(x_-) + \overset{\lambda}{[x_\perp\partial\phi_s](x_-)} + \overset{\lambda^2}{\frac{n_-x}{2}[n_+\partial\phi_s](x_-)} + \overset{\lambda^2}{\frac{1}{2}[x_\perp^\mu x_\perp^\nu \partial_\mu\partial_\nu\phi_s](x_-)} + \ldots$$

$$x_-^\mu \equiv n_+x\,\frac{n_-^\mu}{2}$$



momentum non-conservation at vertices

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* Soft - collinear decoupling [Bauer et al.]

At leading power (hard-) collinear fields couple only to $n_- A_{us}(x_-)$.

Field redefinition:

$$\xi(x) = Y(x)\,\xi^{(0)}(x) \qquad\qquad Y \equiv P\exp\left(ig_s\int_{-\infty}^{0} ds\; n_- A_{us}(x_- + s n_-)\right)$$
$$A_c(x) = Y(x)\,A_c^{(0)}(x)\,Y^\dagger(x)$$

$$\bar\xi(x)\, in_- D\, \xi(x) = \bar\xi^{(0)}\, in_- D_c^{(0)}\, \xi^{(0)}$$

$$\Rightarrow \qquad \mathcal{L}^{(0)} = \underbrace{\mathcal{L}_\xi + \mathcal{L}_{YM}^{(c)}}_{\text{only collinear}} + \underbrace{\bar q\, i\not{D}_s q + \bar h_v\, iv\cdot D_s h_v}_{\text{only soft}}$$

This corresponds to the decoupling of soft gluons from jets in the "old" diagrammatic factorization proofs – the soft gluons are moved to the sources, see below.

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Power - suppressed interactions

- systematic derivation
- e.o.m, field redefinitions, multipole - expansion, ... $\longrightarrow$

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \;\ldots$$

$$\mathcal{L}^{(0)} = \bar\xi\left(in_- D + i\not{D}_{\perp c}\frac{1}{in_+ D_c} i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\xi + \bar q\, i\not{D}_s q \qquad (+\text{ YM})$$

$$\mathcal{L}^{(1)} = \bar\xi\left(x_\perp^\mu n_-^\nu\, W_c\, g_s F^s_{\mu\nu} W_c^\dagger\right)\frac{\not{n}_+}{2}\xi + \bar q\, W_c^\dagger\, i\not{D}_{\perp c}\,\xi - \bar\xi\, i\overleftarrow{\not{D}}_{\perp c} W_c\, q \qquad \text{[BCDF]}$$

$$\vdots$$

No soft-collinear decoupling at order λ after field redefinition

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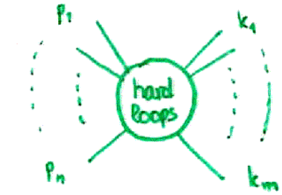
(No) Renormalization of the SCET Lagrangian

$\mathcal{L} = \sum_i C_i O_i$ $\qquad$ C_i short-distance coefficient : tree + hard loops



$$= \int \prod_{i=1}^{N} d^d l_i \prod_k \frac{1}{(L_k + P_k + K_k)^2} \times \text{polynomial}$$

$$= \int \prod_{i=1}^{N} d^d l_i \prod_k \frac{1}{(L_k^2 + n_+ P_k\, n_- L_k)^{a_k}} \times \text{polynomial} = 0$$

expand integrand for hard L_k

scaleless, only $(n_+ P_k)\frac{n_-^\mu}{2}$ and $n_-^2 = 0$

$\Rightarrow \quad C_i = C_i^{\text{tree}}$

Reason : the notion "collinear" (large Energy) acquire a Lorentz-invariant meaning only in the presence of external sources ; nothing integrated out up to now.

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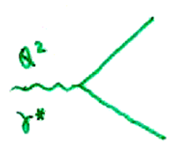
Sources - provide external Lorentz invariants

Simplest cases

* Electromagnetic current, large Q^2 DIS, e^+e^-, ...



$\longrightarrow$ two copies of (hard-) collinear fields

* Weak currents for heavy quark decay

... 4-Quark-Operators ....

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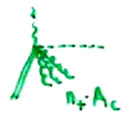
Heavy quark current in SCET ($J = \bar{\psi} Q$, scalar current)

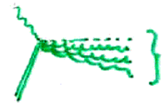
$$\bar{\psi} Q \longrightarrow \bar{\xi} W_c h_v + O(\lambda) \longrightarrow \int d\hat{s}\, \tilde{C}(\hat{s})\, (\bar{\xi} W_c)(s n_+)\, h_v(0) + O(\lambda)$$

$(m_b v + p)^2 \approx O(m_b)^2$
off-shell



effective vertex

integrate out hard loops



non-local $\rightarrow$ momentum space coefficient depends on $n_+ p'/m_b$

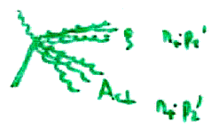
$C(n_+ p'/m_b, m_b/\mu)$

Power suppressed currents

$O(\lambda)$ $\quad (\bar{\xi}\, i\overleftarrow{D}_{\perp c} \frac{1}{n_- v\, i n_+ D_c} W_c)(s n_+)\, h_v(0)$

$\quad \frac{1}{m_b} (\bar{\xi} W_c)(s_1 n_+) (W_c^\dagger i \not{D}_{\perp c} W_c)(s_2 n_+)\, h_v(0)$

- known to 1-loop [MB, Kiyo, Yang]
- 3-body operator



$\vdots \quad O(\lambda^2)$

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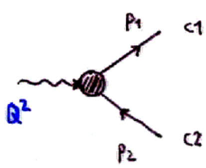
Sudakov form factor

[Collins, Mueller ~ 1980, Sen; Korchemsky, Radyushkin;]



$$F(Q^2) = \langle q(p_1) \bar{q}(p_2) \,|\, \bar{\psi} \gamma^\mu \psi(0) \,|\, 0 \rangle$$

$Q^2 \to \infty$ (e.g. at fixed small off-shellness $p_i^2 \sim \lambda^2 Q^2$ or internal masses $m^2 \sim \lambda^2 Q^2$)

Aim: sum large $\ln Q^2$

Step 1 $\quad \bar{\psi} \gamma^\mu \psi(0) = \int d\hat{s}\, d\hat{t}\; \tilde{C}(\hat{s}\hat{t})\, [\bar{\xi}_{c1} W_{c1}](s n_{1+})\, \gamma^\mu\, [W_{c2}^\dagger \xi_{c2}](t n_{2+}) + O(1/Q)$

$$\Rightarrow \quad F(Q^2) = C(Q^2/\mu^2)\; \langle q(p_1) \bar{q}(p_2) \,|\, \bar{\xi}_{c1} W_{c1} \gamma^\mu W_{c2}^\dagger \xi_{c2}(0) \,|\, 0 \rangle_{(\mu)}$$

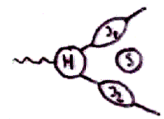
Step 2 $\quad$ Hard-collinear soft decoupling $\quad \xi_{c_i} \to Y_i \xi_{c_i}^{(0)}$, $\quad A_{c_i} \to Y_i A_{c_i}^{(0)} Y_i^\dagger$

$$F(Q^2) = C(Q^2/\mu^2) \; \langle q(p_1)\bar{q}(p_2) | \bar{\xi}^{(0)}_{c1} W^{(0)}_{c1} Y_1^{\dagger} \gamma^{\mu} Y_2 W^{(0)\dagger}_{c2} \xi^{(0)}_{c2} | 0 \rangle_{(0)}$$

$$= \underbrace{C(Q^2/\mu^2)}_{\substack{\text{hard} \\ H}} \cdot \underbrace{\langle q(p_1) | \bar{\xi}^{(0)}_{c1} W^{(0)}_{c1\,\alpha} | 0 \rangle}_{J_1} \underbrace{\langle \bar{q}(p_2) | W^{(0)\dagger}_{c2} \xi^{(0)}_{c2\beta} | 0 \rangle}_{J_2} \underbrace{\langle 0 | Y_1^{\dagger} Y_2 | 0 \rangle}_{S} \; \gamma^{\mu}_{\alpha\beta}$$

virtualities $\lambda^2 Q^2$

virtualities $\lambda^4 Q^2$



Step 3 Log summation

Use anomalous dimension of $\bar{\xi}_{c1} W_{c1} \gamma^{\mu} W^{\dagger}_{c2} \xi_{c2}$ to evolve C to $\mu \sim \lambda Q$

Use renormalization of Wilson lines to run S from $\lambda^2 Q$ to λQ

Warning: above is only a sketch - important details depend on the IR regularization (off-shell, masses, ...)