Supergravity from QCD Amplitudes

Z. Bern, L. Dixon, D. Dunbar, M. Perelstein and J. Rozowsky

KITP, January, 2004

Abstract

QCD and gravity are intimately linked. A good path for addressing the fundamental question of UV divergences in gravity is by starting with the gluonic QCD tree amplitudes described in Witten's talk.

Kawai, Lewellen and Tye, Nucl. Phys. B269:1,1986
F. Berends, W.T. Giele, and H. Kuijf, Phys. Lett. B211:91 (1988)
Z.B., L. Dixon, D. Dunbar, M. Perelstein and J. Rozowsky, hep-th/9802162
Z.B., L. Dixon, D. Dunbar, M. Perelstein and J. Rozowsky, hep-th/9809160, hep-th/9811140
Z.B. and A.K. Grant, hep-th/9904026
Z.B., A. De Freitas, and H.L. Wong, hep-th/9912033
Z.B., gr-qc/0206071
Z.B., L.Dixon, and D.A. Kosower, hep-ph/9602280
Z.B., L. Dixon, D.C. Dunbar, D.A. Kosower, hep-ph/9403226
P.S. Howe and K.S. Stelle, hep-th/0211279
N.E.J. Bjerrum-Bohr, hep-th/0302131, hep-th/0305062

Non-renormalizability of Quantum Gravity

Power counting strongly suggests that field theories of gravity are not renormalizable and therefore not fundamental quantum theories.

Non-renormalizability of pure Einstein gravity confirmed by explicit two-loop calculations. — Goroff and Sagnotti (1986); van de Ven (1992)

Supergravity better but still appears to be non-renormalizable.

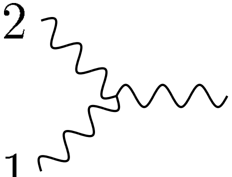
Various authors concluded that supergravity would diverge at three loops. Howe and Stelle; Green, Schwarz, Brink; etc.

Although very believable, arguments are based on power counting and not on direct calculations.

Possible loophole: Coefficient of divergences might vanish due to a hidden symmetry or structure.

Difficulty with perturbative gravity

It is not so easy to use standard Feynman rules. Why?

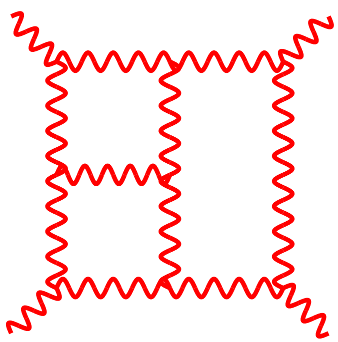


The three vertex, for example, is a mess:

$$G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) =$$

$$\mathrm{sym}[-\frac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \frac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \frac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma})$$
$$+ P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma})$$
$$+ P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma})$$
$$+ 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})]$$

Background field gauges or superspace do not help enough.



$\sim 10^{21}$ terms in diagram

At 10^9 terms/sec

$\sim 20,000$ years to complete 1 diagram

But we are really interested in 5 loops $\sim 10^{30}$ terms.

String Theory Intuition

Basic string theory fact:

$$\text{closed string} \sim (\text{left-mover open string}) \times (\text{right-mover open string})$$

In the field theory or infinite string tension limit this should imply

$$\text{gravity} \sim (\text{gauge theory}) \times (\text{gauge theory})$$

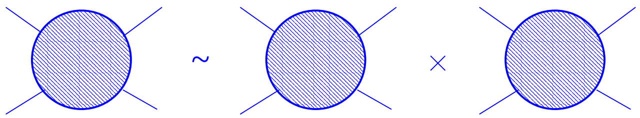
1) How do we make this precise?

2) How can we exploit this?

3) How can this be understood from the Einstein-Hilbert Lagrangian?

Kawai-Lewellen-Tye Tree-Level Relations

At tree-level, KLT (1985) presented some remarkable relations between closed and open string amplitudes.



In the field theory limit ($\alpha' \to 0$)

$$M_4^{\rm tree}(1,2,3,4) = s_{12} A_4^{\rm tree}(1,2,3,4)\, A_4^{\rm tree}(1,2,4,3)\,,$$

$$M_5^{\rm tree}(1,2,3,4,5) = s_{12}s_{34} A_5^{\rm tree}(1,2,3,4,5) A_5^{\rm tree}(2,1,4,3,5) + s_{13}s_{24} A_5^{\rm tree}(1,3,2,4,5)\, A_5^{\rm tree}(3,1,4,2,5)$$

$$s_{ij} = (k_i + k_j)^2$$

where we have stripped all coupling constants. M_n is gravity amplitude and A_n is color stripped gauge theory amplitude.

$$\mathcal{A}_4^{\rm tree} = g^2 \sum_{\rm non-cyclic} {\rm Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) A_4^{\rm tree}(1,2,3,4)$$

These relations hold for any external string states.

Explicit all n formula: hep-th/9811140 Appendix A

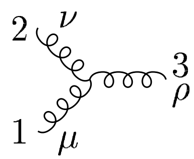
Also holds for classes of higher dimension operators.

Niels Emil Bjerrum-Bohr
hep-th/0302131, hep-th/0305062

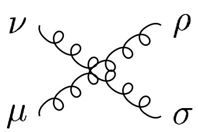
Gravity in Terms of Gauge Theory

We can describe gravity with color stripped gauge theory Feynman rules:

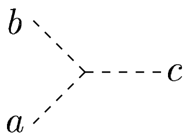
Z.B. and Abilio De Freitas and Henry Wong



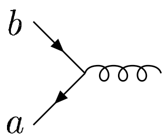
$$= \frac{i}{\sqrt{2}}(k_1 - k_2)^\rho \eta^{\mu\nu} + \text{cyclic}$$



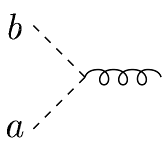
$$= i\eta^{\mu\rho}\eta^{\nu\sigma} - \frac{i}{2}\left(\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right)$$



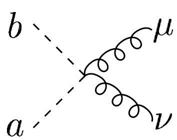
$$= f^{abc}$$



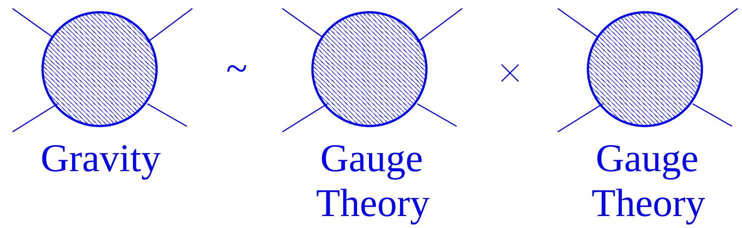
$$= i\delta^{ab}\gamma^\mu$$



$$= i\delta^{ab}(k_1 - k_2)^\mu$$



$$= \frac{i}{2}\delta^{ab}\eta^{\mu\nu}$$



Construction motivated by structure of heterotic string.

Construction checked with up to one fermion pair.

Lagrangians

For scattering amplitudes:

$$\text{gravity} \sim \sum(\text{gauge theory}) \times (\text{gauge theory}).$$

Consider the Lagrangians:

$$\mathcal{L}_{\rm gravity} = \sqrt{g}\, R\,, \qquad \mathcal{L}_{\rm YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu}$$

Not obvious from Lagrangians why gravity scattering amplitudes are a product of gauge theory ones. Z. Bern and A.K. Grant, hep-th/9904026

Even less obvious when you add matter or higher dimension operators.

Z. Bern, A. De Freitas, and H.L. Wong, hep-th/9912033
N.E.J. Bjerrum-Bohr,hep-th/0302131, hep-th/0302107

But it is true!

Spinor Helicity

Xu, Zhang and Chang
F.A.Berends, R.Kleiss, P.De Causmaecker,
R. Gastmans and T. T. Wu
J.F. Gunion and Z. Kunszt
& many others

Vector polarizations

$$\varepsilon_\mu^+(k;q) = \frac{\langle q^- | \gamma_\mu | k^- \rangle}{\sqrt{2} \langle q\, k \rangle}, \qquad \varepsilon_\mu^-(k,q) = \frac{\langle q^+ | \gamma_\mu | k^+ \rangle}{\sqrt{2} [k\, q]}$$

All required properties of polarization vectors satisfied:

$$\varepsilon_i^2 = 0\,, \qquad k \cdot \varepsilon(k,q) = 0\,, \qquad \varepsilon^+ \cdot \varepsilon^- = -1$$

Notation

$$\epsilon^{ab} \lambda_{ja} \lambda_{lb} \longleftrightarrow \langle j\, l \rangle = \langle k_{j\,-} | k_{l\,+} \rangle = \sqrt{2 k_j \cdot k_l}\, e^{i\phi}$$

$$\epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_j^{\dot{a}} \tilde{\lambda}_l^{\dot{b}} \longleftrightarrow [j\, l] = \langle k_{j\,+} | k_{l\,-} \rangle = -\sqrt{2 k_j \cdot k_l}\, e^{-i\phi}$$

Changes in reference momentum q are equivalent to gauge transformations.

Graviton polarization vectors are the squares of these!

$$\varepsilon_{\mu\nu}^{++} = \varepsilon_\mu^+ \varepsilon_\nu^+\,, \qquad 2 = 1 + 1$$

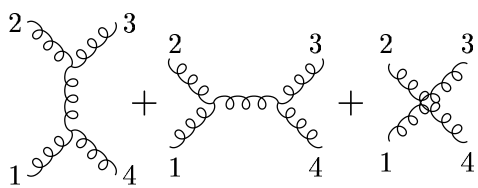
Simple Examples

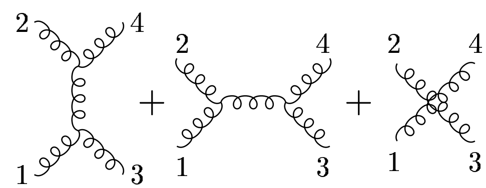
$$iM_4^{\rm tree}(1_h^-,2_h^-,3_h^+,4_h^+) = \left(\frac{\kappa}{2}\right)^2 s_{12} A_4^{\rm tree}(1_g^-,2_g^-,3_g^+,4_g^+) \times A_4^{\rm tree}(1_g^-,2_g^-,4_g^+,3_g^+)$$

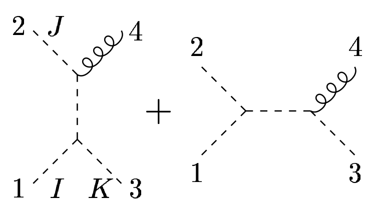
$$= \left(\frac{\kappa}{2}\right)^2 s_{12} \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle} \times \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,4\rangle\langle 4\,3\rangle\langle 3\,1\rangle}$$

$$M_4^{\rm tree}(1_g^-,2_g^-,3_g^+,4_h^+) = g\frac{\kappa}{2} s_{12} A_4^{\rm tree}(1_g^-,2_g^-,3_g^+,4_g^+) \times A_4^{\rm tree}(1_s^I,2_s^J,4_g^+,3_s^K)$$

$$= g\frac{\kappa}{2} s_{12} \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle} \times f^{IJK} \frac{[4\,3]\,\langle 3\,2\rangle}{\langle 2\,4\rangle}$$







All n generalizations

In QCD, the maximal helicity violation (MHV) pure gluon tree Parke-Taylor amplitudes are amazingly simple:

$$A_n(1^-, 2^-, 3^+, \ldots, n^+) = i \frac{\langle 1\, 2 \rangle^4}{\langle 1\, 2 \rangle \langle 2\, 3 \rangle \cdots \langle n\, 1 \rangle}$$

Gravity obtained by pushing above through KLT formulae.
After cleaning up:

Berends, Giele and Kuijf

$$M_n^{\rm tree}(1^-, 2^-, 3^+, \ldots, n^+) = -i \langle 1\, 2 \rangle^8$$
$$\times \left[\frac{[1\, 2]\,[n-2\;\; n-1]}{\langle 1\;\; n-1 \rangle\;\; N(n)} \left(\prod_{i=1}^{n-3} \prod_{j=i+2}^{n-1} \langle i\, j \rangle \right) \prod_{l=3}^{n-3} \left(-\langle n^- | \not{K}_{l+1,n-1} | l^- \rangle \right) + \text{Perms} \right],$$

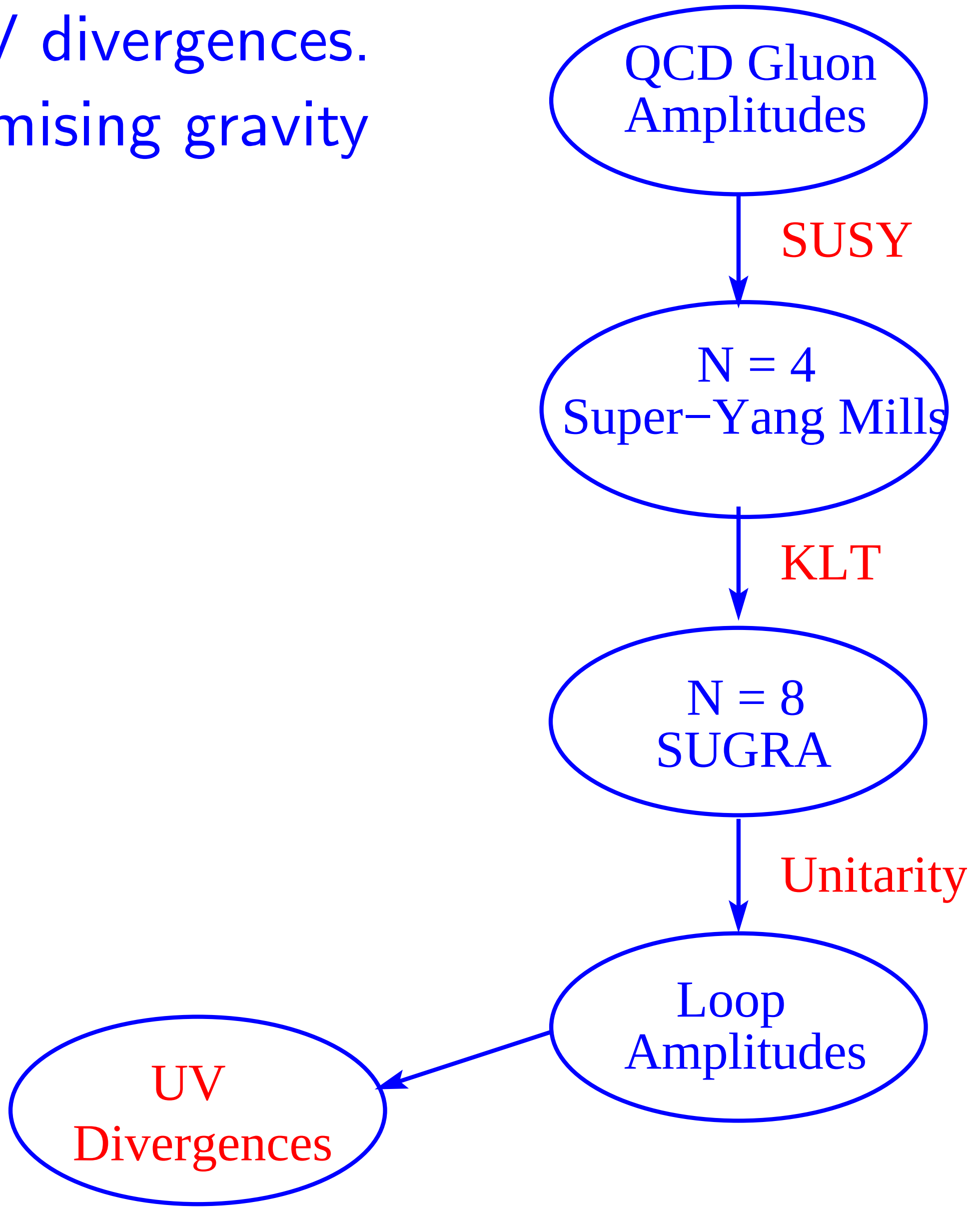
Same idea works for gravity coupled to matter.

Bern, De Freitas, Wong

Key idea: If you know a gauge theory tree amplitude, you immediately know corresponding gravity amplitudes!

Plan of Attack for Gravity UV Divergences

- More supersymmetry $\longrightarrow$ milder UV divergences.
- $N = 8$ supergravity is the most promising gravity theory to investigate for finiteness.
- More susy $\longrightarrow$ simpler calculations (with the right formalism).

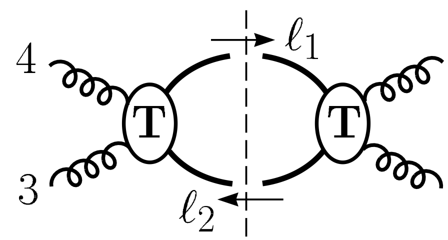


From Trees to Loops

Given tree amplitudes, unitarity can easily be used to determine functions in the loop amplitudes with branch cuts, i.e. $\log(s/t)$ or $\mathrm{Li}_n(s/t)$.

But in general we want rational function pieces, i.e., the divergences. Consider following construction:

$$\int \frac{d^D\ell_1}{(2\pi)^D}\frac{i}{\ell_1^2}\, A_4^{\rm tree}(-\ell_1,1,2,\ell_2)\,\frac{i}{\ell_2^2}\, A_4^{\rm tree}(-\ell_2,3,4,\ell_1)$$



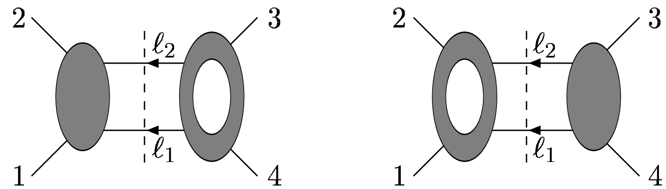
$\ell_1^2 = \ell_2^2 = 0$

- Promote the phase space integrals to unrestricted loop momentum integrals. (Misses functions with no cuts in channel of interest.)
- Systematically combine cuts into a single function whose cuts are correct in all channels.
- Results precisely *identical* to those obtained by Feynman diagrams, *except* no gauge was chosen.

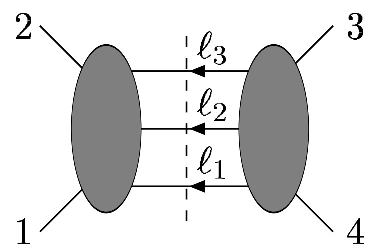
Generalized Cuts

For higher loops it is useful to define a generalized notion of unitarity cuts.

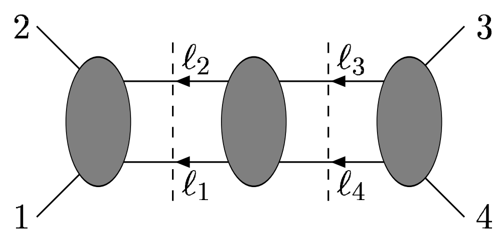
Two-particle cuts:



Three-particle cuts:



Generalized double two-particle cut:



This does not mean "imaginary part of imaginary part". It should be interpreted as demanding that cut propagators do not cancel.

Complete Amplitudes

Basic Claim: In **any** massless quantum field theory, the **on-shell** tree-level S-matrix elements in D-dimensions contain all information for obtaining **all** loop contributions via unitarity. Bern, Dixon, Dunbar and Kosower

Bern and Morgan

Dimensional regularization eliminates subtraction ambiguities commonly present in dispersion relations. van Neerven

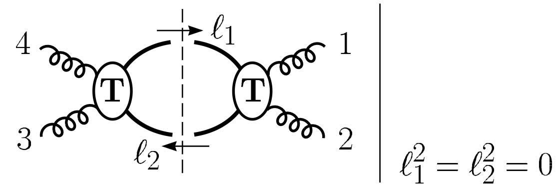
This provides a way to apply the KLT tree-level relations to loop level!

$*$ In massive theories, e.g. $\ln(m^2)$ causes glitches.

$N = 4$ Super-Yang-Mills Loop Amplitudes

ZB, Rozowsky, Yan

Consider $N = 4$ super-Yang-Mills.

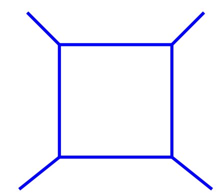


The basic D-dimensional two-particle sewing equation:

$$\sum_{N=4 \text{ states}} A_4^{\text{tree}}(-\ell_1, 1, 2, \ell_2) \times A_4^{\text{tree}}(-\ell_2, 3, 4, \ell_1) = -st A_4^{\text{tree}}(1, 2, 3, 4) \frac{1}{(\ell_1 - k_1)^2} \frac{1}{(\ell_2 - k_3)^2}$$

Applying this equation at one-loop we have

$$\mathcal{A}_4^{\text{1-loop}}(1, 2, 3, 4) = -st A_4^{\text{tree}} \mathcal{I}_4^{\text{1-loop}}(s, t)$$



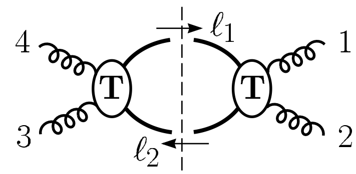
This amplitude has the correct s and t channel cuts in all dimensions. It agrees with the results of Green, Schwarz and Brink.

Since we get back A_4^{tree} we can recycle the two-particle cut algebra to all loop orders!

$N=8$ Supergravity Cuts

How do we calculate $N=8$ supergravity amplitudes?

Two-particle sewing equation:



$$\Big|_{\ell_1^2=\ell_2^2=0}$$

$$\sum_{N=8\text{ states}} M_4^{\text{tree}}(-\ell_1,1,2,\ell_2)\times M_4^{\text{tree}}(-\ell_2,3,4,\ell_1)$$

$$= s^2 \sum_{N=4\text{ states}} \Big(A_4^{\text{tree}}(-\ell_1,1,2,\ell_2)\times A_4^{\text{tree}}(\ell_2,3,4,-\ell_1)\Big)$$

$$\times \sum_{N=4\text{ states}} \Big(A_4^{\text{tree}}(\ell_2,1,2,-\ell_1)\times A_4^{\text{tree}}(\ell_1,3,4,-\ell_2)\Big)$$

Easy to evaluate using known super-Yang-Mills results.

$$\sum_{N=8\text{ states}} M_4^{\text{tree}}(-\ell_1,1,2,\ell_2)\times M_4^{\text{tree}}(-\ell_2,3,4,\ell_1) = stu\, M_4^{\text{tree}}(1,2,3,4)\left[\frac{1}{(\ell_1-k_1)^2}+\frac{1}{(\ell_1-k_2)^2}\right]$$

$$\times\left[\frac{1}{(\ell_2-k_3)^2}+\frac{1}{(\ell_2-k_4)^2}\right]$$

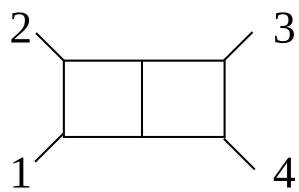
This is all you need to iterate two-particle cuts to *all* loop orders!

Two-Loop $N = 8$ SUGRA

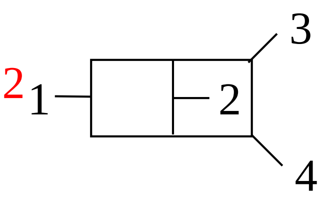
Z.B., Dixon, Dunbar, Perelstein and Rozowsky

From 2 and 3 particle cuts we obtain exact two-loop result:

$(s\,K)^2$



$+ \quad (s\,K)^2$



$+$ perms

where $K = st A_4^{\rm tree}$. The two-loop divergences start at $D = 7$:

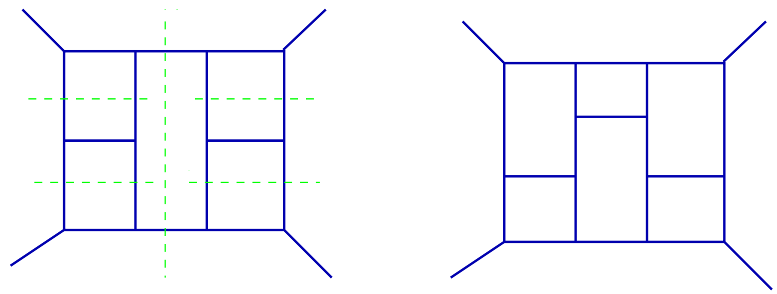
$$\mathcal{M}_4^{\text{2-loop},\ D=7-2\epsilon}\Big|_{\rm pole} = \frac{1}{2\epsilon\ (4\pi)^7}\frac{\pi}{3}(s^2 + t^2 + u^2)\ stu M_4^{\rm tree}$$

For $D = 5, 6$ the amplitude is finite contrary to earlier expectations from Howe and Stelle (1988) superspace power counting arguments.

Concrete example where earlier superspace arguments point to divergence which is actually not present.

Power Counting Beyond Two Loops.

The two-particle cut sewing equation iterate to *all* loop orders!



Power counting this subclass of contributions suggests the following simple finiteness formula

$$L < \frac{10}{D-2}$$

Confirmed by investigating n particle cuts with MHV configurations crossing the cuts.

To complete this investigation, need a compact representation of sum of non-MHV configurations crossing the cuts.

Can there be non-trivial divergence cancellations between integral types?

Finiteness: $$L < \frac{10}{D-2}$$

This formula indicates finiteness when earlier superspace arguments of Howe and Stelle do not, e.g. $D = 4$, $L = 3$ and $D = 5, 6$, $L = 2$.

For $N = 8$ sugra, the first $D = 4$ potential counterterm should occur at 5 loops not 3 loops.

Result confirmed by Howe and Stelle using harmonic superspace hep-th/0211279.

Interestingly, Howe and Stelle suggest that the 5 loop divergence may also be absent!

For this to happen there would have to be non-trivial cancellations between integral types.

Summary

- Gravity $\sim$ (gauge theory) $\times$ (gauge theory) at tree-level.
- D-dimensional unitarity takes us from trees to full loop amplitudes.
- $N = 8$ SUGRA is less divergent than previously thought. Finite through at least 5 loops. QCD MHV gluon amplitudes are a crucial ingredient.
- Open question if $D = 4, N = 8$ SUGRA diverges at 5 loops and beyond.

Gravity $\sim$ (gauge theory)^2 can be exploited to develop a deeper understanding of the perturbative expansions of gravity theories. It would be helpful to develop an even deeper understanding of the gauge theory tree amplitude inputs.

Analytic Properties for $D \neq 4$

Consider: $$|A_4^{\text{1-loop}}(1^+, 2^+, 3^+, 4^+)| = \frac{1}{48\pi^2}$$

Has no imaginary part! How do we construct real rational parts from nothing?

Trick: Continue the amplitude to $D = 4 - 2\epsilon$ dimensions.

From dimensional analysis in massless theories:

$$A^{D=4-2\epsilon} \sim \int d^{4-2\epsilon}p \, \cdots$$
$$\sim \sum_i (s_i)^{-\epsilon} \times \text{rational}_i + \cdots$$
$$\sim \sum_i \text{rational}_i (1 - \epsilon \ln s_i) + \cdots$$

Thus:

$$\text{rational} = \sum_i \text{rational}_i$$

From $\mathcal{O}(\epsilon)$ branch cuts can reconstruct $\mathcal{O}(\epsilon^0)$ rational terms.