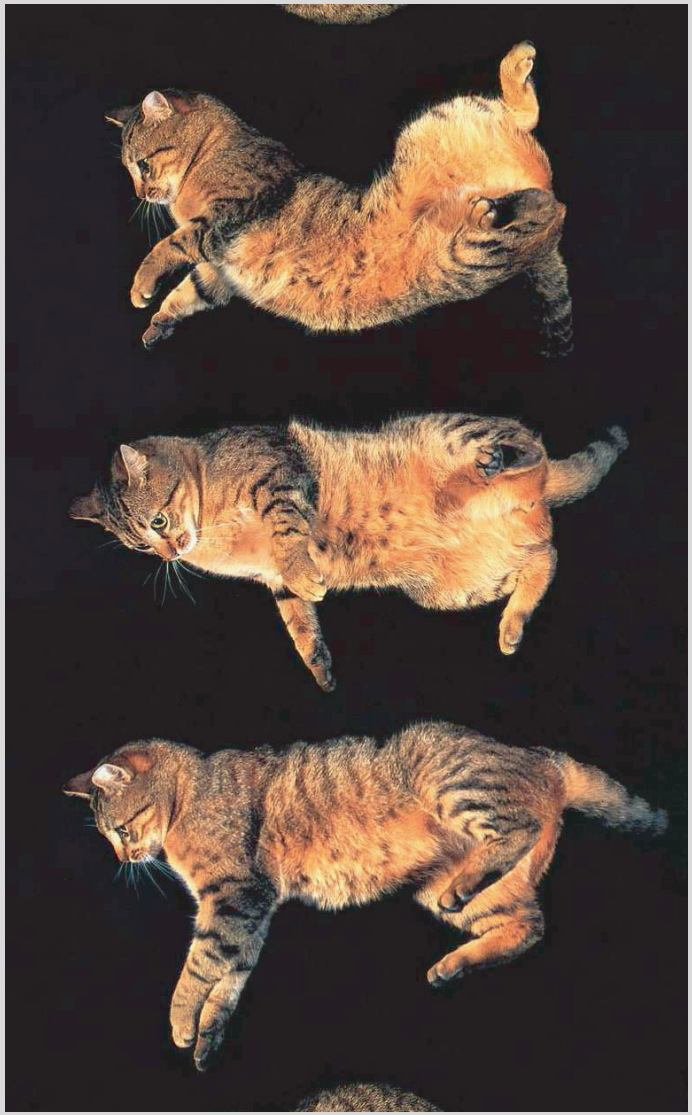
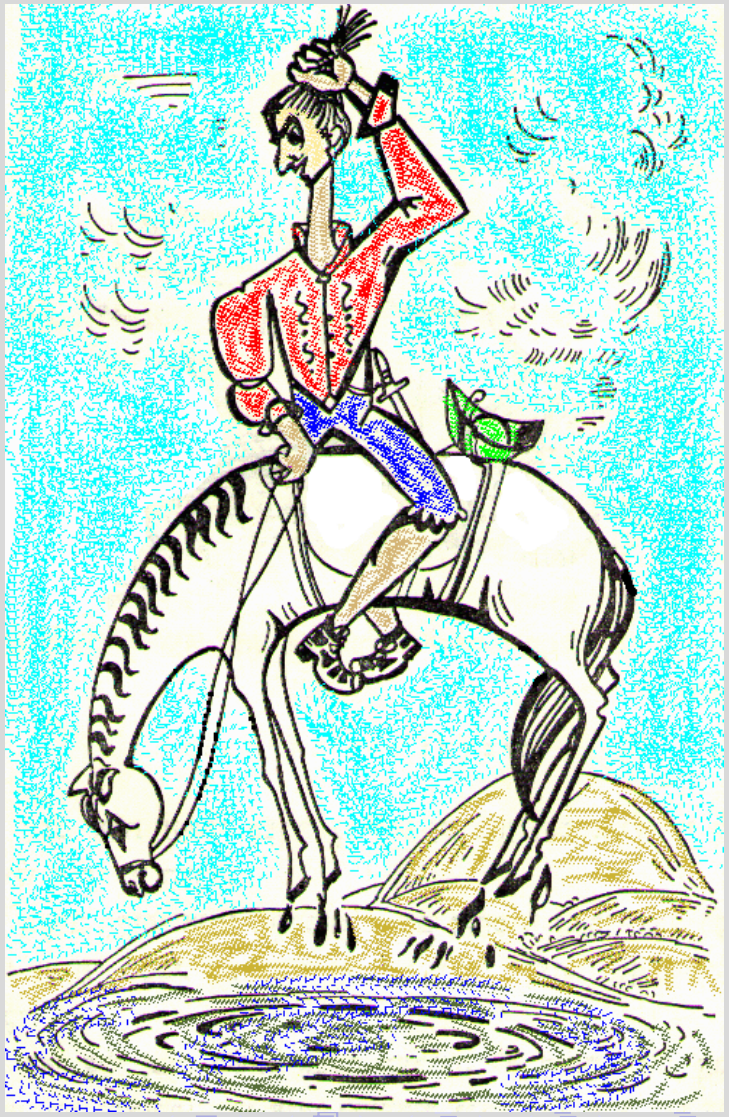
The Baron and the Cat





Baron and Cat

- Cats rotate without angular momentum
- Why can't the Baron displace without momentum?
- Motion as Berry's phase
- Swimming in curved space
- Motion by commutation

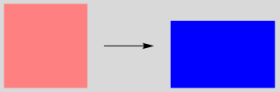
Karl Friedrich Hieronymus, Freiherr von Münchhausen 1720-1797

What I have learned from:

Oded Kenneth, Amos Ori, Jack Wisdom, Frank Wilczek, J.P. Eckmann, O. Raz, O. Gat, B. Gutkin, and D. Oaknin

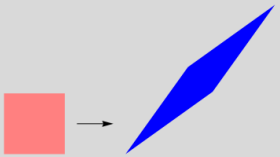
Rotation by commutation

Stretch



$$Z = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{symmetric}$$

Shear



$$X = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{symmetric}$$

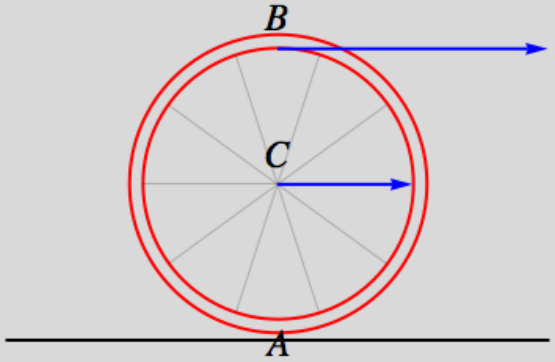
[Stretch,Shear]=Rotation

$$[X, Z] = 2 \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{anti-symmetric} \Longleftarrow Rotation$$

Rigid body motions: Abelian Gauge theory

Rotations: Uniquely defined

Translations: Depend on choice of gauge (=fiducial point)



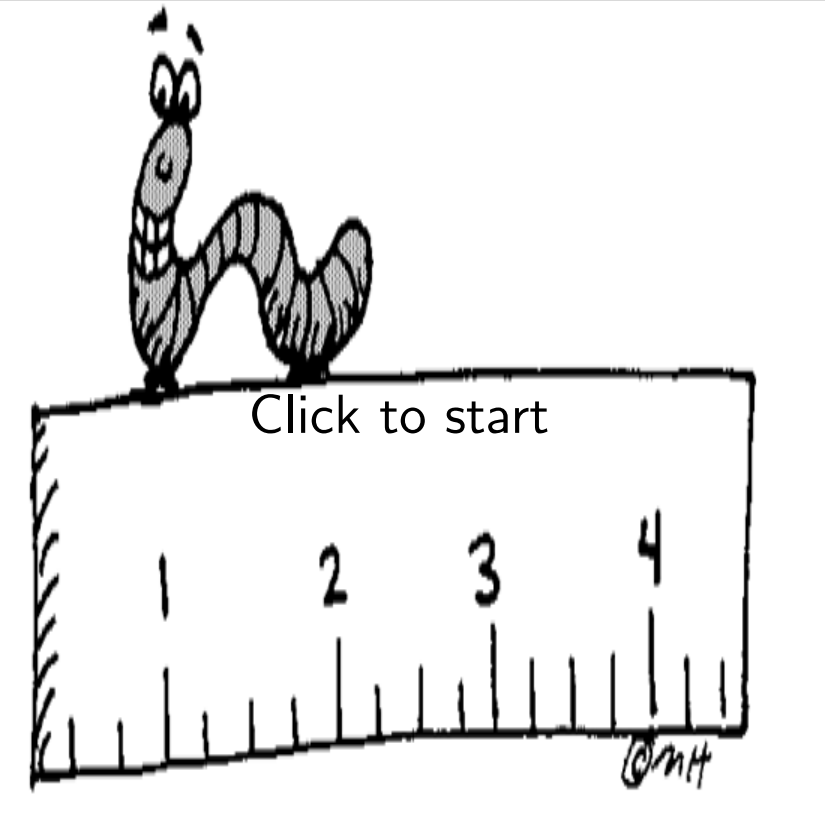
- A: Instantaneously stationary
- C: Center of mass velocity
- B: Maximal speed

You can't avoid getting dirty by going faster

Deformable body motion: Non-Abelian Gauge theory

F. Wilczek and A. Shapere

- Need fiducial frame to fix translations and rotations:
- Translations and rotations: gauge invariant after a full stroke



An analog of: $\oint \underbrace{A}_{Vector\ pot.} \cdot d\ell = \int \underbrace{B \cdot dS}_{flux}$

Newton's Law

Lex I

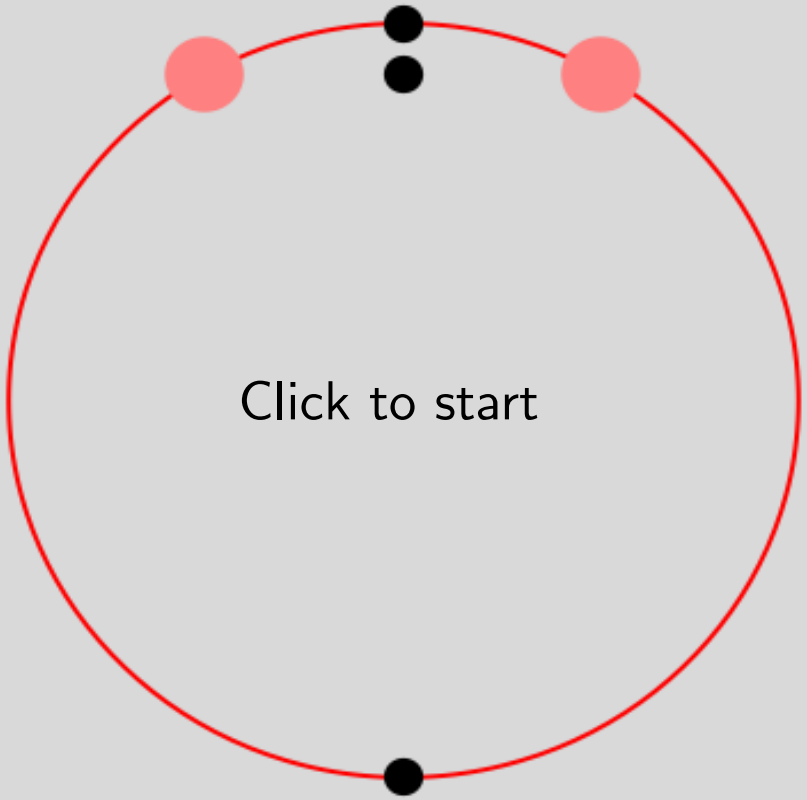
Corpus omne perseverare in statu suo quiescendi vel Movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.

A body perserves being at rest or moving uniformly, except insofar as it is compelled to change its state by force impressed.

Modern formulation

The center-of-mass of the system will remain at rest in the absence of external forces

Ambiguous center of Mass

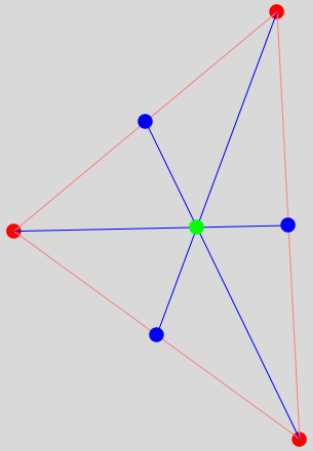


Center of mass: A Euclidean notion

Ambiguous center of Mass: Geometry

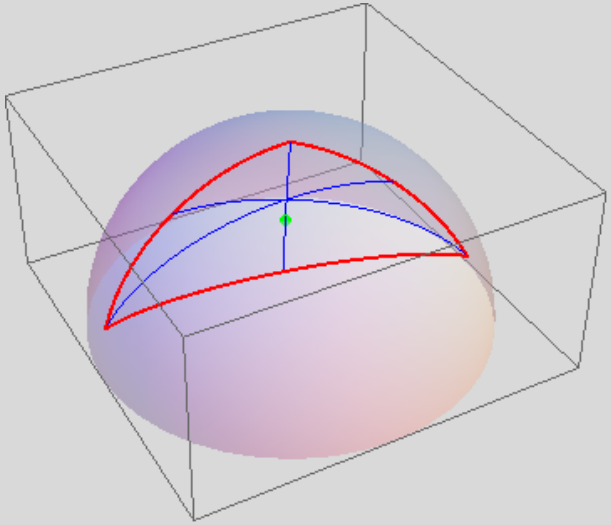
Order irrelevant:

Medians in a (Euclidean) triangle intersect at 2/3

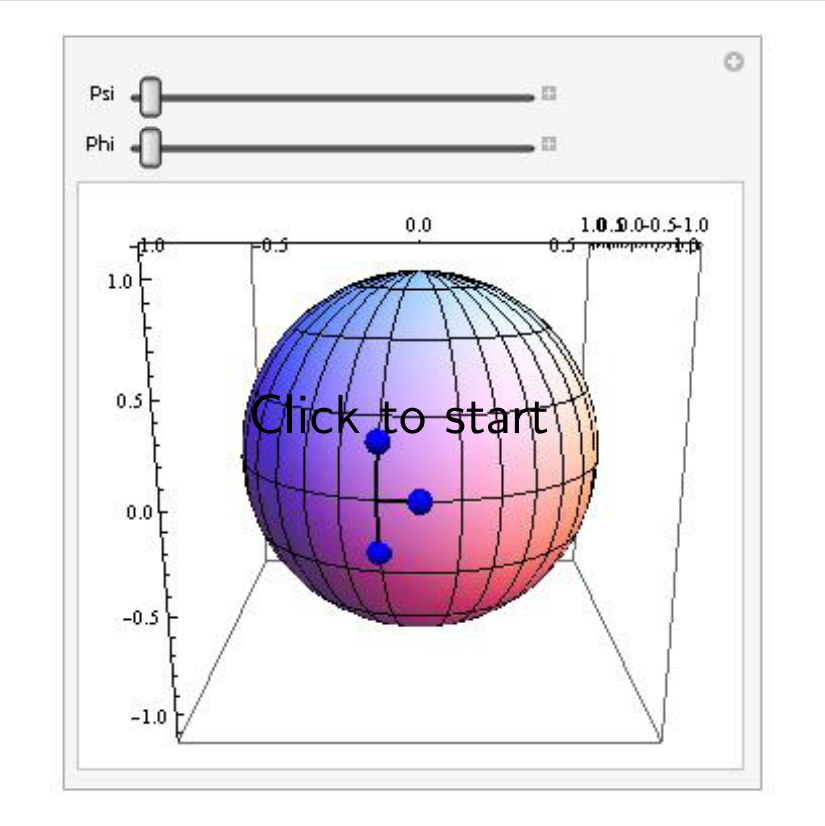


Order matters

Medians in a spherical triangles do not intersect at 2/3 and in general manifolds they even need not intersect



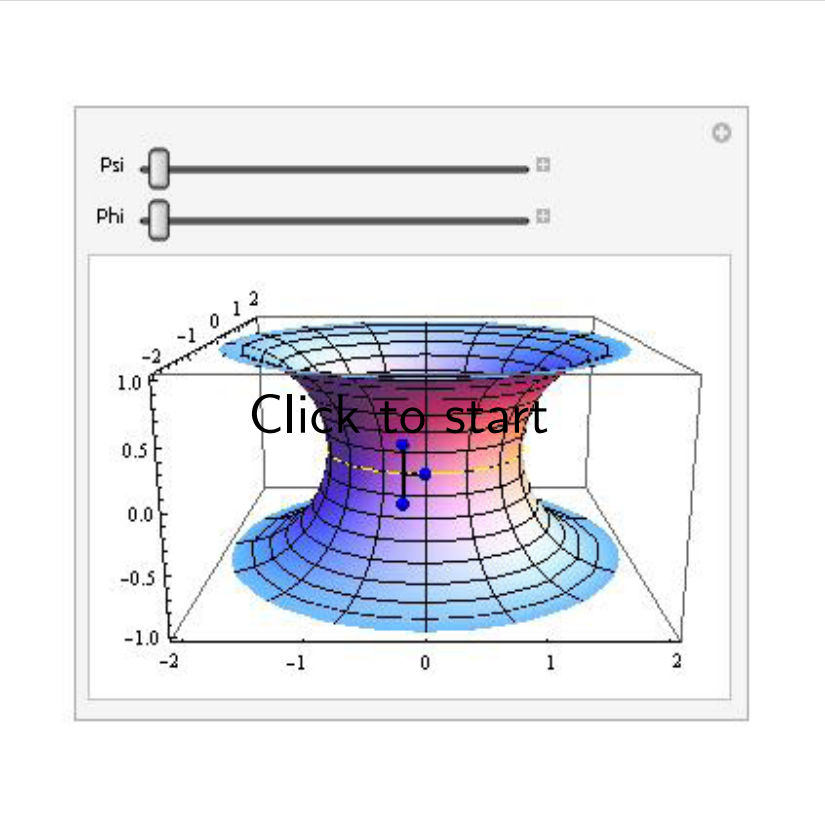
Swimming on a sphere



Swimming triangle on a sphere

J. Wisdom; Animation: O Raz

Swimming on a negatively curved surface

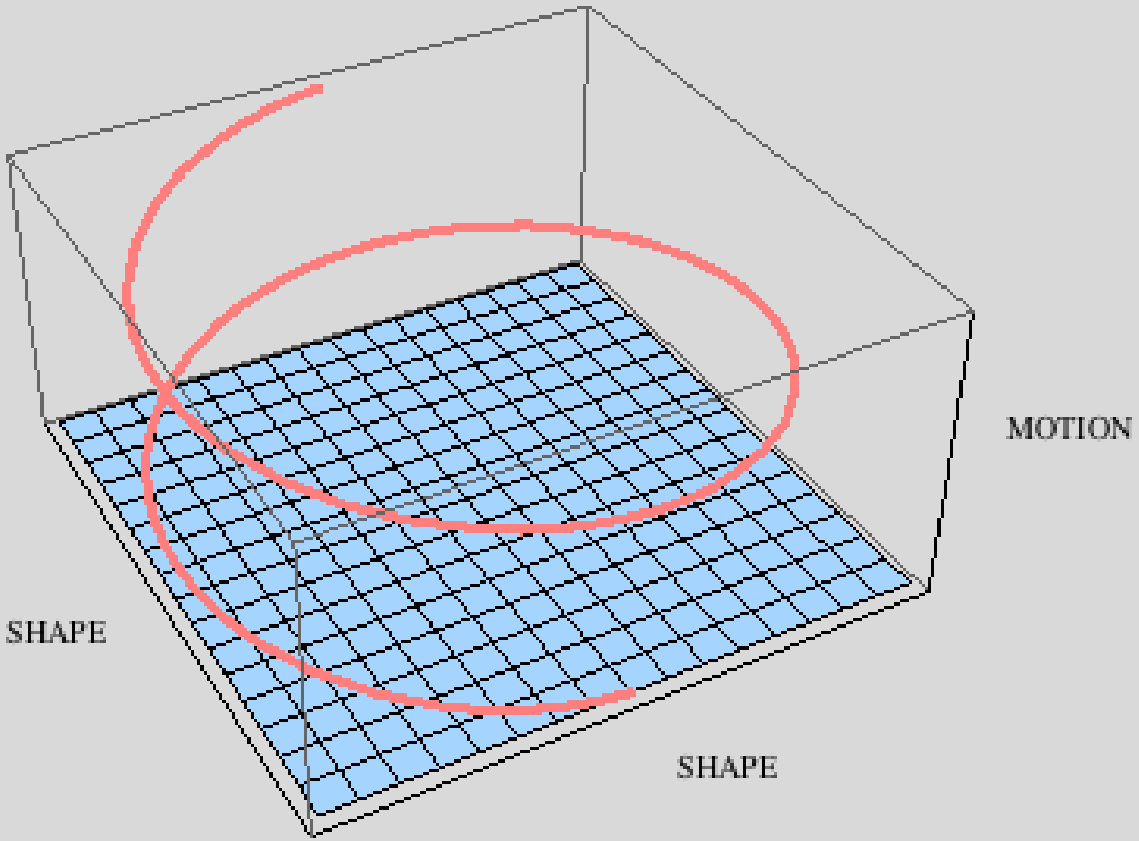


Swimming triangle on a surface with negative curvature

Animation: O Raz

Swimmming as holonomy

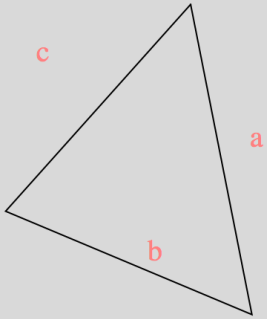
Swimming with periodic strokes



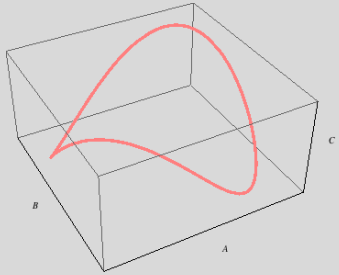
Closed path in shape space

Shape space

- Triangle

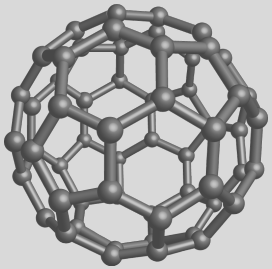


$(a, b, c) \in Cone \subset \mathbb{R}^3$



stroke

- C^{60}



mutual distances

$$\binom{60}{2} = 1770$$

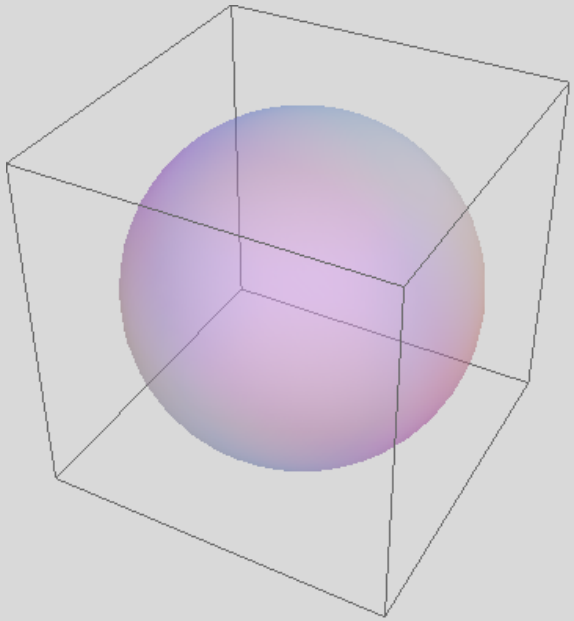
Lucky if u fit anywhere

- Degrees of freedom $180 = 60 \times 3$
- Generic spaces: Shapes locked

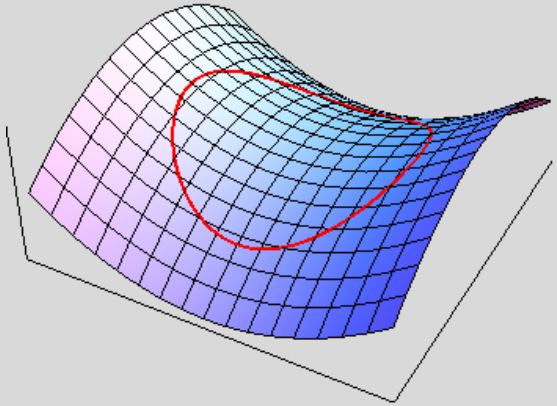
Homogeneous and Isotropic spaces

Swimming with periodic strokes:

Possible only in homogeneous and isotropic spaces



positive

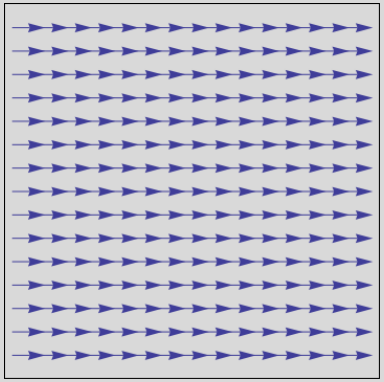


negative

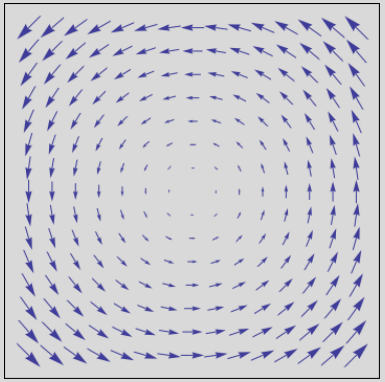
Killing fields

Definition (Killing equation)

Motion (vector field) inducing no strain



translation



rotation

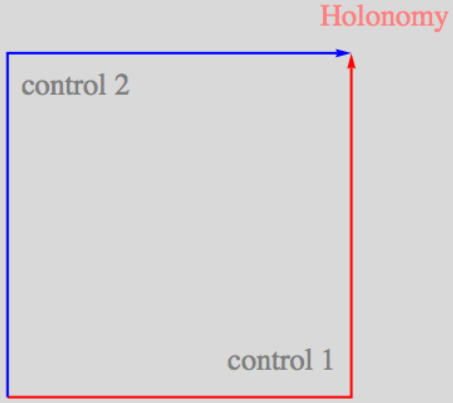
Example (Euclidean plane)

translations: $\xi_x = (1,0)$, $\xi_y = (0,1)$, rotations: $\xi_\theta = (-y, x)$

Conservation laws as Controls

Total (conserved) momentum associated to ξ

$$\underbrace{P_\xi}_{conserved} = \sum \underbrace{m_n \dot{x}_n}_{momentum} \cdot \underbrace{\xi(x_n)}_{direction}$$



$P_\xi = 0$ linear constraint on dx_n

$$0 = \sum m_n \, \xi(x_n) \cdot dx_n$$

$$dx_n = d(rigid\ \ motion) + \underbrace{d(deformation)}_{control}$$

Control equation ODE:

$$\underbrace{d(rigid\ \ body)}_{motion-vector} = \underbrace{M(shape, position)}_{rect-matrix} \underbrace{d(deformation)}_{shape-vector}$$

Cats spin and Euclidean Barons lie

Spatial part of Holonomy: $\nabla \times \xi$

- Euclidean Cats spin:
  $\nabla \times \xi_\theta = \nabla \times (-y, x) = \partial_x x - \partial_y(-y) = 2$
- Euclidean Barons lie: $\nabla \times \xi_x = \nabla \times (1, 0) = 0$
- On curved manifold $\xi_{l;i;j} = -R_{\ell ij}\xi^k$

Swimming in curved space

$$\delta x = \underbrace{8\,R}_{space} \left(\underbrace{\frac{\sum m_n x_n y_n^2}{M}}_{swimmer} \right) \underbrace{dA}_{stroke}$$

Small $\Longrightarrow$ Slow (Kenneth)

Parking

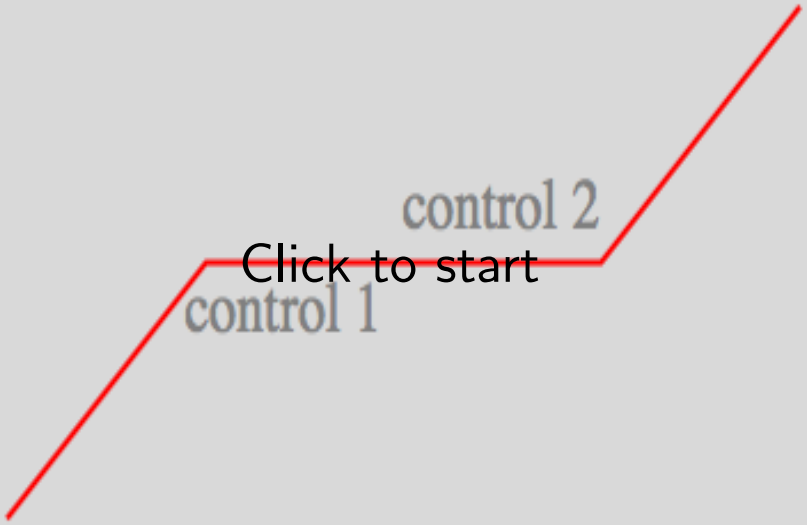
Parking: Excercise in commutation

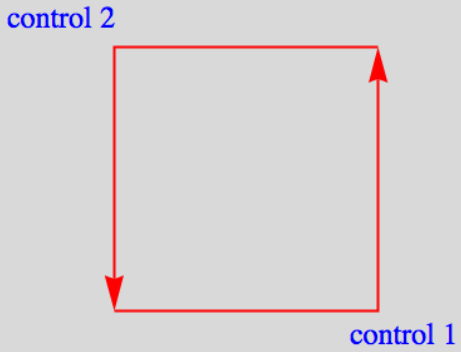
$$[drive, steer] = park$$

With tight space $drive \approx steer \approx \epsilon$: number of maneuvers ϵ^{-2}

Purcell swimmer

- Micro-swimmers: dominated by friction
- No inertia
- Swimming is geometric

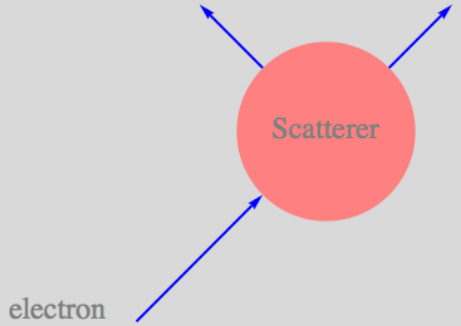




Animation: O. Raz

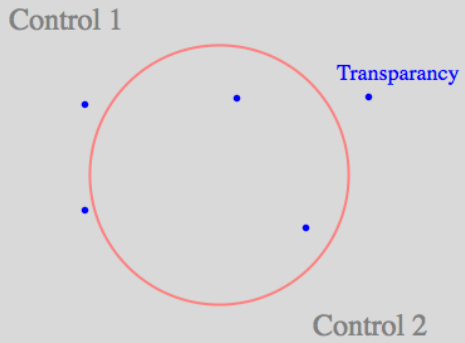
Qauntum Swimmers

- Swimming in Fermi sea
- Swimmer controls scattering matrix S
- Adiabatic swimming is geometric



Quantum swimmer:

At $T = 0$ and in 1-D: $distnace = Integer \times \frac{\lambda_f}{2}$.



Boris Gutkin, David Oaknin

Summary

Swimming by commutation

- Geometric
- Intriguing
- Effective for large strokes