Emergent symmetry – a new approach to gapless states and quantum field theories

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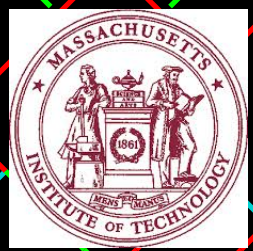
(2023/01/18, UCSB)



Quantum matter (at $T = 0$ temperature)

- **Three kinds of quantum states of matter**

- Gapped $\rightarrow$ no excitations
  Band insulators, FQH states

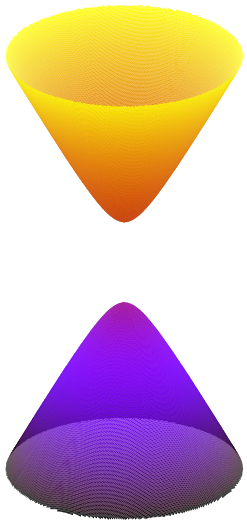
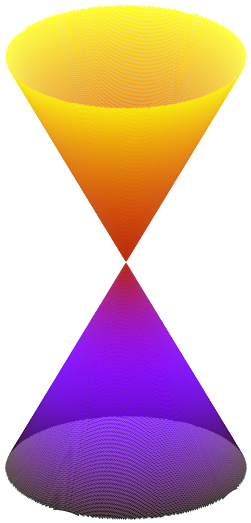
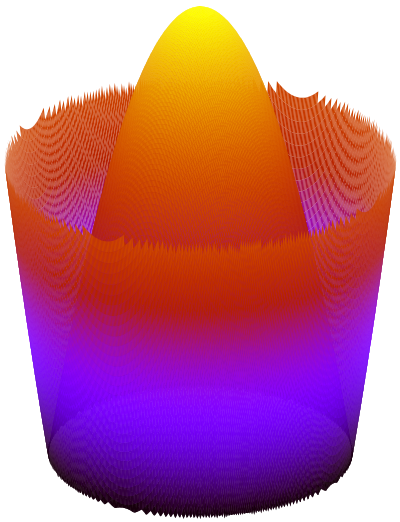
- Gapless (interesting) $\rightarrow$ few excitations, within QFT
  Dirac/Weyl semimetal, superfluid, critical point at continuous phase transition

- Gapless (messy) $\rightarrow$ many excitations, beyond QFT
  Fermi metal, Bose metal, *etc*

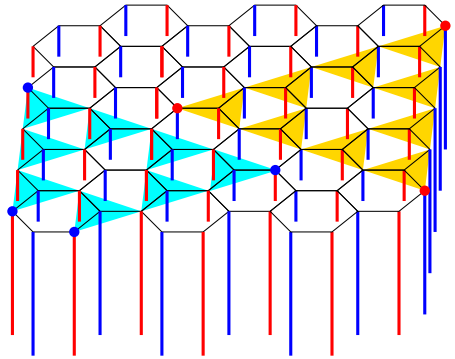
- The above states of matter can be divided into two classes:
  weekly correlated and **strongly correlated**
  (or **weekly interacting** and **strongly interacting**)

Weakly correlated quantum matter

- Weakly correlated **gapped states**:
  General theory: Band theory, K-theory for topological insulators
- Weakly correlated **"interesting" gapless states**:
  General theory: Band theory with Dirac points for fermion systems. Boson condensation for boson systems. Quantum field theory.
- Weakly correlated **"messy" gapless states**:
  General theory: Fermi liquid theory for fermion systems. The low energy effective field theory contain infinity fields (beyond QFT).

Strongly correlated quantum matter



- Strongly correlated **gapped states**:
  General theory: Group theory and Ginzburg-Landau theory for symmetry breaking states. (Higher) category theory and topological quantum field theory for highly entangled states (*ie* **topological order**, such as FQH states, spin liquids). Network formed by topological orders give rise to highly entangled fracton states
- Strongly correlated **"interesting" gapless states**:
  General theory: 1+1D conformal field theory (CFT) for critical points at continuous phase transitions. No general theory for higher dimensions.
- Strongly correlated **"messy" gapless states**:
  General theory: No general theory (? Beyond quantum field theory). Some example: strongly correlated boson systems may have emergent fermionic quasiparticles with Fermi surface.

"Interesting" gapless state & emergent symmetry

How to develop a general theory for **"interesting" gapless states**?
Find labels (*ie* invariants) that fully characterize the "interesting" gapless states, but not the labels like phase-A, phases-B, *etc* .

- A gapless state has more symmetry at low energies than the original lattice Hamiltonian, which is called **emergent symmetry**. *Maybe the emergent symmetry is a good label to characterize gapless state.*
- We find that the emergent symmetry can be very rich: symmetry, anomalous symmetry, higher symmetry, anomalous higher symmetry, algebraic higher symmetry, anomalous algebraic higher symmetry, ...
  algebraic (higher) symmetry = non-invertible (higher) symmetry = fusion (higher) category symmetry = ...

Generalized symmetries and gapless states

- Maybe the emergent generalized symmetry can fully characterize gapless states. **Each gapless state may has its own distinct characteristic emergent (generalized) symmetry.**
  Use emergent (generalized) symmetry as a starting point to develop a general theory of gapless states
- The ordinary symmetry is described group theory. What is the unified theory that describes generalized symmetry, which include symmetry, anomalous symmetry, higher symmetry, anomalous higher symmetry, algebraic higher symmetry, anomalous algebraic higher symmetry, ...

  We find that **generalized symmetry $\sim$ non-invertible gravitational anomaly = topological order in one higher dimension**
- **A general theory for "interesting" gapless state as a boundary theory of topo. order in one higher dimension.**

Symmetry, selection rule, and fusion category

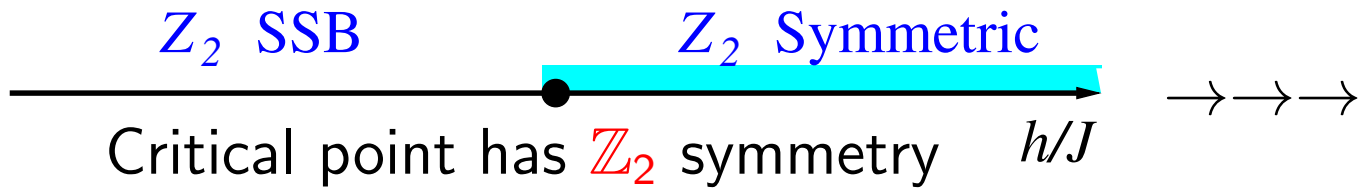
Consider 1+1D transverse Ising model $H = -\sum_J Z_i Z_{i+1} + hX_i$ with $\mathbb{Z}_2$ symmetry generated by $U = \prod_i X_i$.

- The local symmetric operators allowed by the $\mathbb{Z}_2$ symmetry are generated by $Z_i Z_{i+1}$, X_i
- Choosing the left-right basis $\rightarrow$, $\leftarrow$:
  Under the arbitrary actions of the local symmetric operators, the mod-2 conservation of left-spins $\rightarrow\rightarrow \leftarrow \rightarrow\rightarrow\rightarrow \leftarrow \rightarrow\rightarrow$
- Use $\mathbf{1}$ to represent ground state $|\cdots \rightarrow\rightarrow \cdots\rangle$, $e = \leftarrow$ an excitation

- $e \otimes e = \mathbf{1}$ ($\mathbb{Z}_2$ conservation law, fusion rule $\rightarrow$ fusion category)
- $\mathbb{Z}_2$ symmetry

- A $\mathbb{Z}_2$ symmetry can be described by
- a group $\mathbb{Z}_2 = \{\mathrm{id}, \prod_i X_i\}$ (we will avoid)
- a fusion category $\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2} = \{e \otimes \mathbf{1} = e, e \otimes e = \mathbf{1}\}$ (we will adapt)

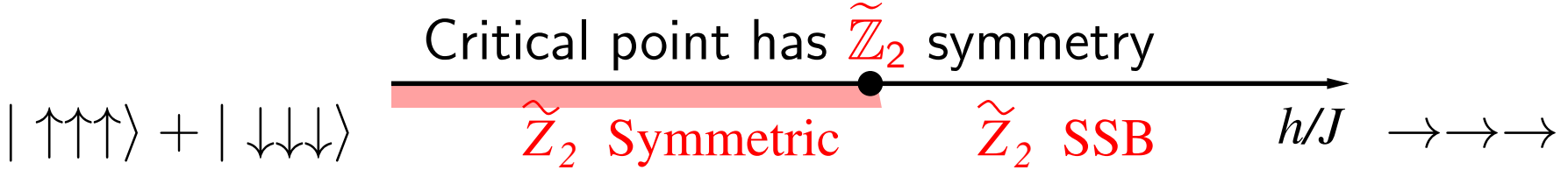
- $\uparrow\uparrow\uparrow$ or $\downarrow\downarrow\downarrow$



A hidden symmetry – dual $\widetilde{\mathbb{Z}}_2$ symmetry

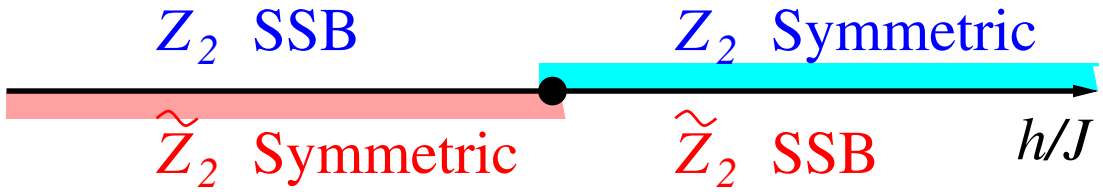
- Choosing the up-down basis $\uparrow$, $\downarrow$:
  Under the arbitrary actions of the local symmetric operators, the mod-2 conservation of domain walls (m's) $\uparrow\uparrow \mid \downarrow\downarrow\downarrow \mid \uparrow\uparrow\uparrow$
- Use $\mathbf{1}$ for ground state $|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle$, m for a domain wall

- $m \otimes m = \mathbf{1}$ ($\widetilde{\mathbb{Z}}_2$ conservation law, fusion rule $\to$ fusion category)
- $\widetilde{\mathbb{Z}}_2$ dual symmetry, by a fusion category $\mathcal{V}\text{ec}_{\mathbb{Z}_2} = \{m \otimes m = \mathbf{1}\}$

- 

Categorical symmetry = symmetry + dual symmetry

- The critical point has a larger combined symmetry $\mathbb{Z}_2 \vee \widetilde{\mathbb{Z}}_2$, which is called **categorical symmetry**.



Use the larger categorical symmetry to understand the critical point

- The 1+1D transverse-field Ising model has both $\mathbb{Z}_2$-symmetry ($e \otimes e = \mathbf{1}$) and dual $\widetilde{\mathbb{Z}}_2$-symmetry ($m \otimes m = \mathbf{1}$). But one of them must spontaneously broken.
- The critical point has both the symmetries, $\mathbb{Z}_2$ and $\widetilde{\mathbb{Z}}_2$, unbroken.
- The combined symmetry of the critical point is not given by the combined symmetry group $\mathbb{Z}_2 \times \widetilde{\mathbb{Z}}_2$.

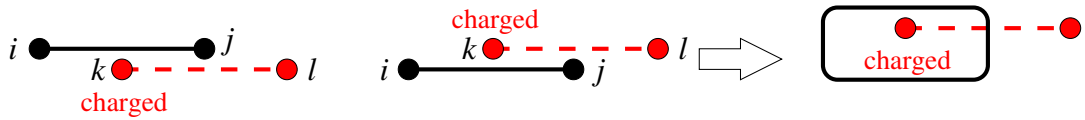
What theory describes the combined symmetry, which is referred to as **categorical symmetry** $\mathbb{Z}_2 \vee \widetilde{\mathbb{Z}}_2$?

Categorical symmtry: algebra of patch operator

We consider charge pair-creation operators:

- A pair of $\mathbb{Z}_2$ symmetry charges, $\rightarrow\rightarrow \leftarrow \rightarrow\rightarrow\rightarrow \leftarrow \rightarrow\rightarrow$, is created by an open-string (patch) operator $W^e_{ij} = Z_i Z_j$ (a string operator with empty bulk)
- A pair of $\widetilde{\mathbb{Z}}_2$ dual-symmetry charges, $\uparrow\uparrow \mid \downarrow\downarrow\downarrow\downarrow \mid \uparrow\uparrow\uparrow$, is created by another open-string (patch) operator $W^m_{ij} = \prod_{i<k<j} X_k$
- The $\mathbb{Z}_2$ and $\widetilde{\mathbb{Z}}_2$ conservations do not combine into a $\mathbb{Z}_2 \times \widetilde{\mathbb{Z}}_2$ symmetry, since the $\mathbb{Z}_2$-charges and $\widetilde{\mathbb{Z}}_2$-charges have a "π-mutual statistics":

$$W^e_{ij} W^m_{kl} = -W^m_{kl} W^e_{ij}$$

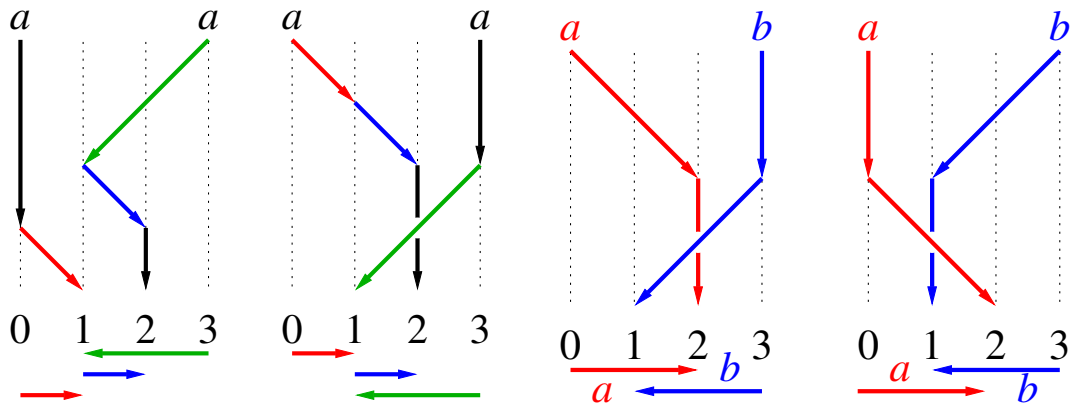


as revealed by the algebra of the string operators that create those charge pairs. So we denote the combined symmetry as $\mathbb{Z}_2 \vee \widetilde{\mathbb{Z}}_2$.

Algebra of extended operators is characterized by braidings in one higher dimension

Algebra of patch operators = self/mutual statistics

- The patch charge operator $W^a_{ij} = [a_i \bar{a}_j]$ can be viewed as a hopping operator of the a-particle. Using the statistics algebra of hopping operator, we can obtain the **self statistics** of the particles a:



Levin Wen cond-mat/0302460

$$W^a_{31} W^a_{12} W^a_{01} = \mathrm{e}^{\mathrm{i}\theta_a} W^a_{01} W^a_{12} W^a_{31}, \quad a \in \{e, m\} = \text{bosons}.$$

and the **mutual statistics** between the particles a, b:

$$W^a_{02} W^b_{31} = \mathrm{e}^{\mathrm{i}\theta_{ab}} W^b_{31} W^a_{02}, \quad a, b \in \{e, m\}$$

π-mutual statistics between e and m.

Chatterjee Wen arXiv:2203.03596

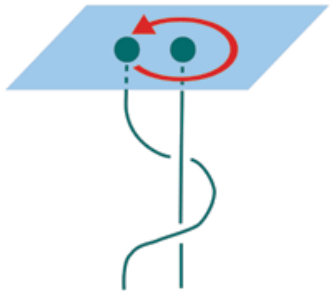
- **Fusion rule**: $W^e_{ij} W^e_{ij} = \mathrm{id} \to e \otimes e = \mathbf{1}$.
  $W^m_{ij} W^m_{ij} = \mathrm{id} \to m \otimes m = \mathbf{1}$.
- **The algebra of 1+1D patch operators = A braided fusion category = A set of 2+1D anyons (a 2+1D topo. order)**

Categorical symmtry as a topological order in one higher dimension

- Consider a 2+1D $\mathbb{Z}_2$-topological order, which can be viewed as (1) a $\mathbb{Z}_2$-gauge theory, or (2) a toric code model, or (3) 2+1D superconductor (SC) with dynamical electromagnetic field but no spin-rotation symmetry. It has four types of excitations:

- e: a boson, $\mathbb{Z}_2$-gauge-charge, an electron-vortex bound state in SC
- m: a boson, $\mathbb{Z}_2$-gauge-vortex, a vortex state in SC
- f: a fermion, $\mathbb{Z}_2$-charge-vortex bound state, an electron in SC
- $\mathbf{1}$: a boson, trivial excitation or null excitation (*ie* ground state)

- Those excitations have the following fusion rule:



$$f = e \otimes m, \quad m = f \otimes e, \quad e = m \otimes f,$$
$$\mathbf{1} = e \otimes e, \quad \mathbf{1} = m \otimes m, \quad \mathbf{1} = f \otimes f,$$
$$e = e \otimes \mathbf{1}, \quad m = m \otimes \mathbf{1}, \quad f = f \otimes \mathbf{1},$$

- e, m, f have π-mutual statistics between them.

Holographic theory of symmetry & dual symmetry

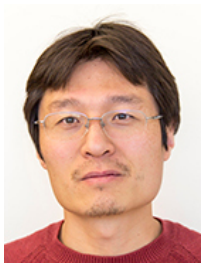
- **Categorical symm. = topol. order in one higher dimension**

  1+1D $\mathbb{Z}_2 \vee \widetilde{\mathbb{Z}}_2$ categorical symmetry = 2+1D $\mathbb{Z}_2$ gauge theory.

- In the bulk, $e \otimes e = \mathbf{1} \to \mathbb{Z}_2$ symm. $m \otimes m = \mathbf{1} \to \widetilde{\mathbb{Z}}_2$ dual symm.
- The $\mathbb{Z}_2$-charge e and the $\widetilde{\mathbb{Z}}_2$-charge m are bosons, and have a π mutal statistics between them.

- Symmetry constrains the low energy dynamics. Topological order constrains the low energy dynamics of its boundary. Hamiltonians H_{symm} with a symmetry and boundary Hamiltonians H_{bdry} of the corresponding topological order in one higher dimension, have a one-to-one correspondence, such that the corresponding Hamitionians have the identical **local low energy properties**.

  $\neq$ AdS/CFT Ji Wen arXiv:1905.13279, 1912.13492; Thorngren Wang arXiv:1912.02817
  Kong Lan Wen Zhang Zheng arXiv:2003.08898, 2005.14178



Holographic theory of symmetry & dual symmetry

How 2+1D bulk is related to the 1+1D boundary?

- e-particle condensation in $\mathbb{Z}_2$-gauge theory (Higgs phase): In the e-condensed phase $\phi_e \neq 0$, m, f-particles are confined, since they generate a branch cut $\phi_e \to -\phi_e$: $\times$——. e-particle becomes trivial, since it is absorbed by the e-condensation. **The e-condensed phase has a trivial topological order.**

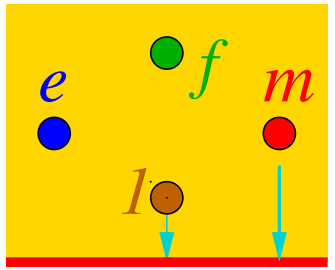
- The domain wall between the $\mathbb{Z}_2$ gauge theory and the e-condensed phase, is a gapped boudary of $\mathbb{Z}_2$ gauge theory induced by e-condensation.

- m-particle condensation in $\mathbb{Z}_2$-gauge theory (confined phase) has a trivial topological order.

- The domain wall between the $\mathbb{Z}_2$ gauge theory and the m-condensed phase, is a gapped boudary of $\mathbb{Z}_2$ gauge theory induced by m-condensation.

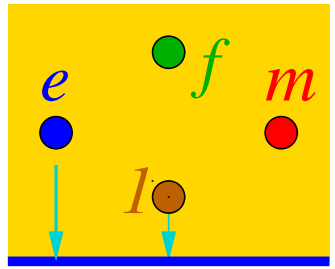
Holographic theory of symmetry & dual symmetry

- The $\mathbb{Z}_2$ gauge theory has a gapped boundary induced by m condensation on the boundary.
  The boundary excitations are described by even number of e particles with fusion $e \otimes e = \mathbf{1}$.
  $\rightarrow$ the $\mathbb{Z}_2$-symmetric phase ($\widetilde{\mathbb{Z}}_2$ symm. broken).
  $\rightarrow$ The boundary describes a $\mathbb{Z}_2$-symmetric sub-Hilbert space.



$\mathrm{Z}_2 \vee \cancel{\tilde{\mathrm{Z}}_2}$ symm

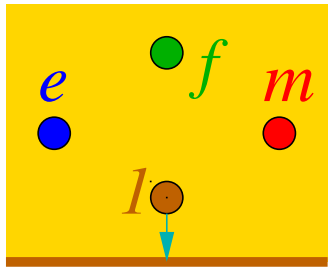
- The $\mathbb{Z}_2$ gauge theory has another gapped boundary induced by e condensation. The boundary excitations are described by even number of m particles with fusion $m \otimes m = \mathbf{1}$.
  $\rightarrow$ the $\mathbb{Z}_2$ symm. broken phase ($\widetilde{\mathbb{Z}}_2$ symmetric).
  $\rightarrow$ The boundary describes a $\widetilde{\mathbb{Z}}_2$-symmetric sub-Hilbert space.



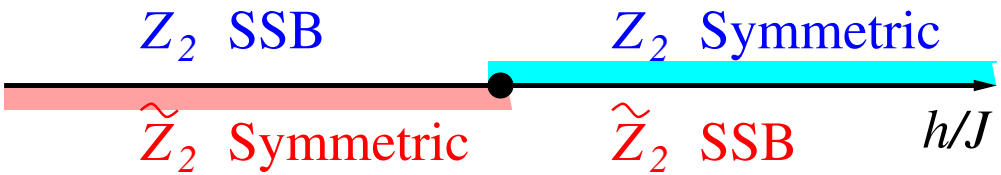
$\cancel{\mathrm{Z}_2} \vee \tilde{\mathrm{Z}}_2$ symm

Holographic theory of symmetry & dual symmetry

- The $\mathbb{Z}_2$ gauge theory has boundary with no condensation (*ie* **1**-condensed).
  $\rightarrow$ the gapless critical point with the full $\mathbb{Z}_2 \vee \widetilde{\mathbb{Z}}_2$ categorical symmetry
  $\rightarrow$ The boundary describes a $\mathbb{Z}_2$-symmetric and $\widetilde{\mathbb{Z}}_2$-symmetric sub-Hilbert space.



$Z_2 \vee \tilde{Z}_2$ symm



- We cannot break the symmetry $\mathbb{Z}_2$ and the dual symmetry $\widetilde{\mathbb{Z}}_2$ at the same time (*ie* we cannot condense all the bulk anyons at same time).

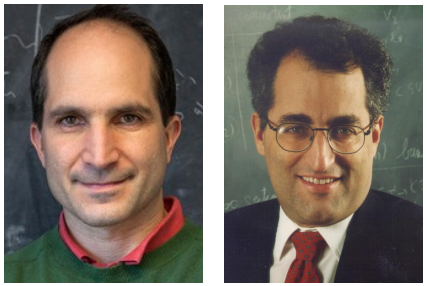
Levin, arXiv:1903.09028

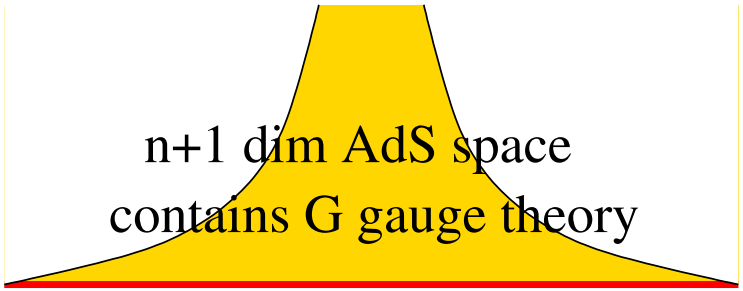
- A gapped state must spontaneously break part of the categorical symmetry $\mathbb{Z}_2 \vee \widetilde{\mathbb{Z}}_2$ (for example, $\mathbb{Z}_2$ or $\widetilde{\mathbb{Z}}_2$).
- A state with unbroken categorical symmetry (without condensating any anyons) must be gapless.

Compare to AdS/CFT picture

Maldacena arXiv:hep-th/9711200; Witten arXiv:hep-th/9802150

- **Witten**: "for gauge theory, suppose the AdS theory has a gauge group G, [...] Then in the scenario of [13], the group G is a global symmetry group of the conformal field theory on the boundary."





CFT in n dim space contains G symm

The symmetry is conituous and group-like

The symmetry is finite and can go beyond group

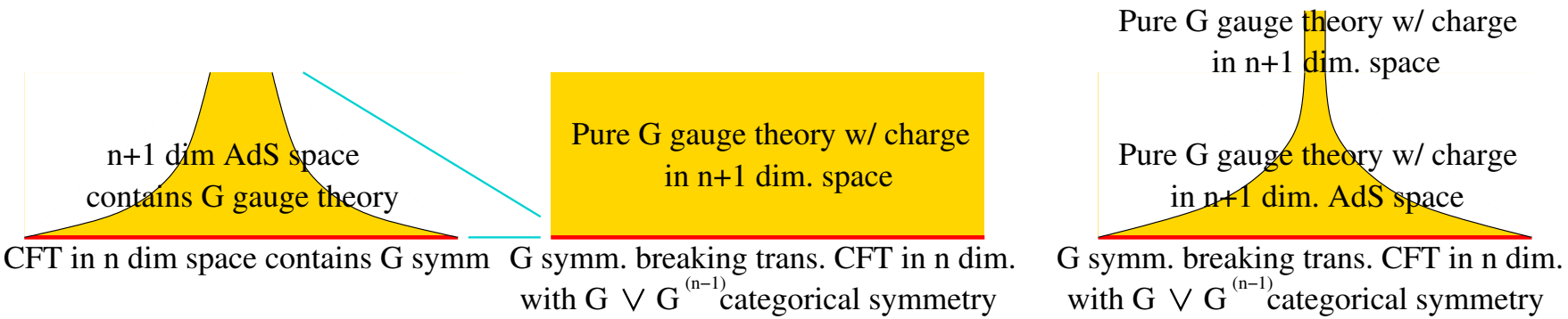
Bulk has an AdS geometry

Bulk has no geometry (independent of metrics)

Compare to AdS/CFT picture

The boundary condition at $r = \infty$ in AdS space is given by the bulk topological order (*ie* by the emergent categorical symmetry of the CFT).

Ji Wen arXiv:1912.13492



CFT in n dim space contains G symm — G symm. breaking trans. CFT in n dim. with G $\vee$ G$^{(n-1)}$ categorical symmetry

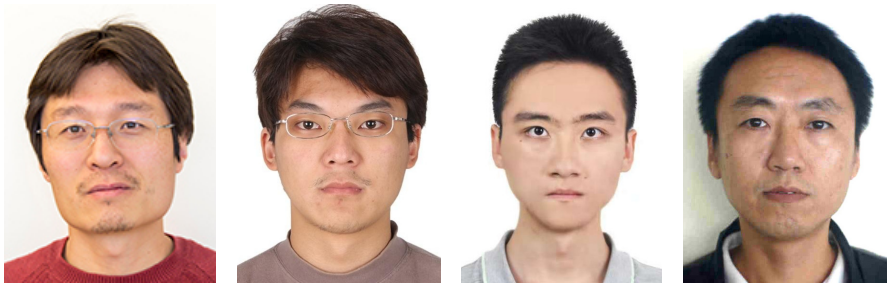
G symm. breaking trans. CFT in n dim. with G $\vee$ G$^{(n-1)}$ categorical symmetry

- How does bulk topological order (*ie* categorical symmetry) determine the CFT at the boundary?

  How does the boundary condition (*ie* the bulk topological order or categorical symmetry) at $r = \infty$ determine the AdS geometry?

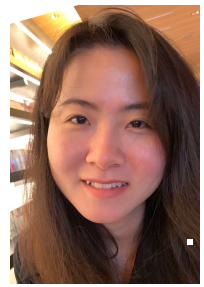
From categorical symmetry to CFT

- **What is the data to describe categorical symmetry (*ie* to describe the bulk topological order)?**
  Emergent symmetries may not be described by groups. Emergent symmetry is always described by topological orders in one higher dimension.



Kong Lan Wen Zhang Zheng arXiv:2003.08898, 2005.14178

- **What is the data to describe 1+1D boundary CFT?**
  Multi-component modular covariant partition function

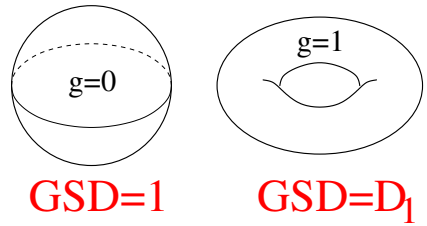


Ji Wen arXiv:1905.13279

- **How are the two data related?**

Data to measure/define topological order

- **Topo. ground state degeneracy** that depends on topology of space, but robust against any small changes of the Hamiltonian



Wen, PRB **40** 7387 (1989), Wen Niu, PRB **41** 9377 (1990)

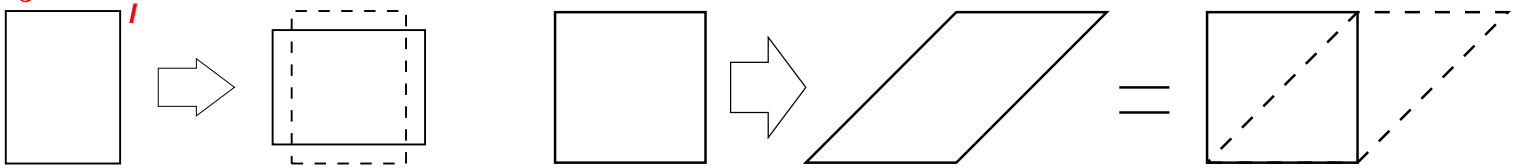
- **Mapping class group** (map a space to itself) representations is a data to describe topo. order.

- A $SL(2,\mathbb{Z})$ representation S, T can be computed from **wave function overlap** of the degenerate ground states Ψ_α:

Heidar-Wen arXiv:1401.0518, He-Heidar-Wen arXiv:1401.5557

$$\int \prod_i \mathrm{d}x_i \mathrm{d}y_i \ \Psi^*_\alpha(\{x_i, y_i\})\Psi_\beta(\{y_i, -x_i\}) = \mathrm{e}^{-\mathrm{Area}/\xi_S^2} S_{\alpha,\beta}$$

$$\int \prod_i \mathrm{d}x_i \mathrm{d}y_i \ \Psi^*_\alpha(\{x_i, y_i\})\Psi_\beta(\{x_i + y_i, y_i\}) = \mathrm{e}^{-\mathrm{Area}/\xi_T^2} T_{\alpha,\beta}$$

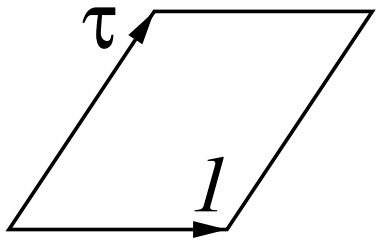


Boundary partition function is a vector!

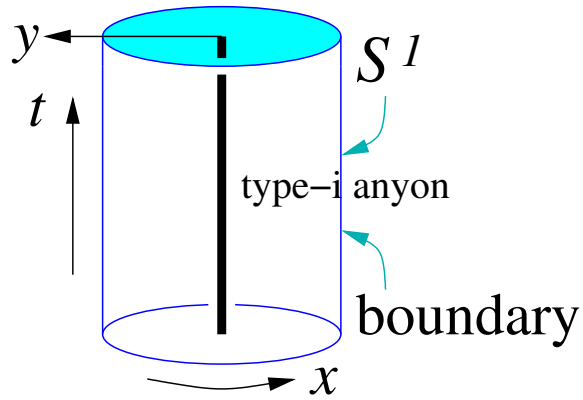
- 1+1D gapless state realizable by lattice model has a **modular invariant partition function**:

$$Z(\tau+1)=Z(\tau),\quad Z(-1/\tau)=Z(\tau)$$

where $\tau \in \mathbb{C}_+$ describes the shape of spacetime torus.

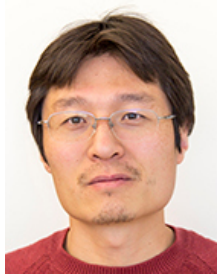


- Boundary of 2+1D topological orders has a **vector-like partition function**, whose component $Z_i(\tau)$ is labeled by the anyon types i of the 2+1D bulk topological order.



Ji Wen arXiv:1905.13279, 1912.13492 Kong Zheng arXiv:1905.04924, arXiv:1912.01760

$$Z(\tau)\big|_{\text{size } L}=\mathrm{Tr}\, e^{-\mathrm{Im}(\tau)Lv^{-1}\hat{H}+\mathrm{i}\,\mathrm{Re}(\tau)L\hat{P}}$$



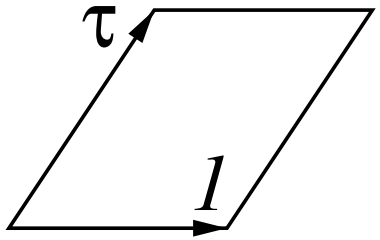
From bulk topological order to gapless boundary – a number theoretical approach

- $Z_i(\tau)$ is not modular invariant but **modular covariant**:
  $T^{\mathcal{M}}:\ Z_i(\tau+1) = T^{\mathcal{M}}_{ij} Z_j(\tau), \quad S^{\mathcal{M}}:\ Z_i(-1/\tau) = S^{\mathcal{M}}_{ij} Z_j(\tau).$
  where $S^{\mathcal{M}}$, $T^{\mathcal{M}}$-matrix characterize the 2+1D bulk topo. order $\mathcal{M}$.

  Ji Wen arXiv:1905.13279, 1912.13492; Lin Shao arXiv:2101.08343

- The multi-component partition function

  $$Z_i(\tau)\big|_{\text{size } L} = \mathrm{Tr}_{i\text{-sector}}\, \mathrm{e}^{-\mathrm{Im}(\tau) L v^{-1} \hat{H} + \mathrm{i}\,\mathrm{Re}(\tau) L \hat{P}}$$

  must also satisfies **non-negative-int condition**:

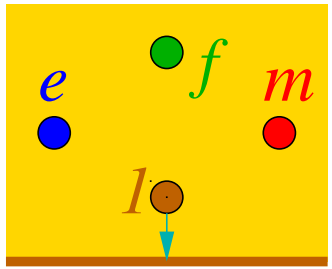
  $$Z_i(\tau) = q^{h_i - c/24} \bar{q}^{\bar{h}_i - \bar{c}/24} \mathrm{Poly}^{\text{non-neg-int}}_{h_i,\bar{h}_i}(q,\bar{q}), \quad \text{where} \quad q = \mathrm{e}^{2\pi \mathrm{i}\,\tau}$$

  where $h_i, \bar{h}_i$ are rational numbers. $\mathrm{Poly}^{\text{non-neg-int}}_{h,\bar{h}}$ is a polynomial with non-negative integral coefficients.

  $\rightarrow$ **Generalized modular boostrap**. For a special case of chiral modular boostrap, see Ashrafi arXiv:1909.13308

An $\mathbb{Z}_2$ example: bulk $\mathcal{M}$ (= categorical symmetry)

- For a 1+1D $\mathbb{Z}_2$ symmetry, its categorical symmetry is the 2+1D $\mathbb{Z}_2$-topological order $\mathcal{M} = \mathcal{G}\text{au}_{\mathbb{Z}_2}$ ($\mathbb{Z}_2$-gauge theory with gauge charge), with anyons $\mathbf{1}, e, m, f$. Its S, T matrix are given by



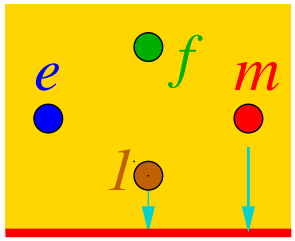
$\mathrm{Z}_2 \vee \tilde{\mathrm{Z}}_2$ symm

$$T^{\mathcal{G}\text{au}_{\mathbb{Z}_2}} = \begin{pmatrix} 1 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 1 & 0 & 0 & e \\ 0 & 0 & 1 & 0 & m \\ 0 & 0 & 0 & -1 & f \end{pmatrix}, \quad S^{\mathcal{G}\text{au}_{\mathbb{Z}_2}} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

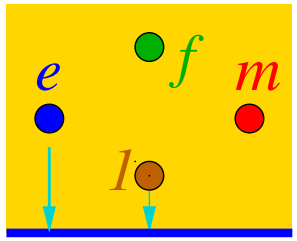
$\mathcal{M}$

CFT_{ano}

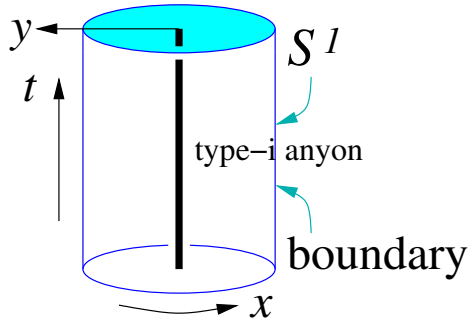
An $\mathbb{Z}_2$ example: gapped boundaries $\widetilde{\mathcal{R}}$



$\mathrm{Z}_2 \vee \tilde{\mathrm{Z}}_2$ symm $\to \mathcal{R}\mathrm{ep}_{\mathbb{Z}_2}$



$\mathrm{Z}_2 \vee \tilde{\mathrm{Z}}_2$ symm $\to \mathcal{V}\mathrm{ec}_{\mathbb{Z}_2}$



- Two gapped boundaries induced by m-condensation and by e-condensation. They are described by two modular covariant constant partition functions

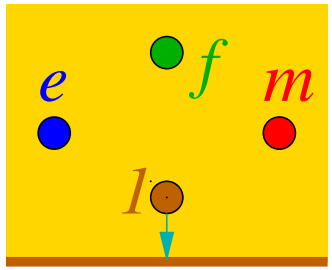
$$\begin{pmatrix} Z_{m\text{-cnd};\mathbf{1}} \\ Z_{m\text{-cnd};e} \\ Z_{m\text{-cnd};m} \\ Z_{m\text{-cnd};f} \end{pmatrix} = \overset{\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2}}{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}, \quad \begin{pmatrix} Z_{e\text{-cnd};\mathbf{1}} \\ Z_{e\text{-cnd};e} \\ Z_{e\text{-cnd};m} \\ Z_{e\text{-cnd};f} \end{pmatrix} = \overset{\mathcal{V}\mathrm{ec}_{\mathbb{Z}_2}}{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}.$$

satisfying $Z_i(\tau+1) = T_{ij}^{\mathcal{G}\mathrm{au}_{\mathbb{Z}_2}} Z_j(\tau)$, $Z_i(-1/\tau) = S_{ij}^{\mathcal{G}\mathrm{au}_{\mathbb{Z}_2}} Z_j(\tau)$.

- Z_i's are non-negative integers since they describe ground state degeneracy in the i-sector.

An $\mathbb{Z}_2$ example: gapless boundary CFT_{ano}

- One gapless boundary induced by **1**-condensation (*ie* no condensation). It is described by the following modular covariant partition function



$Z_2 \vee \tilde{Z}_2$ symm

$$\begin{pmatrix} Z_{\mathbf{1}\text{-cnd};\mathbf{1}}(\tau,\bar\tau) \\ Z_{\mathbf{1}\text{-cnd};e}(\tau,\bar\tau) \\ Z_{\mathbf{1}\text{-cnd};m}(\tau,\bar\tau) \\ Z_{\mathbf{1}\text{-cnd};f}(\tau,\bar\tau) \end{pmatrix} = \begin{pmatrix} |\chi_0^{\text{Is}}(\tau)|^2 + |\chi_{\frac12}^{\text{Is}}(\tau)|^2 \\ |\chi_{\frac{1}{16}}^{\text{Is}}(\tau)|^2 \\ |\chi_{\frac{1}{16}}^{\text{Is}}(\tau)|^2 \\ \chi_0^{\text{Is}}(\tau)\bar\chi_{\frac12}^{\text{Is}}(\tau) + \chi_{\frac12}^{\text{Is}}(\tau)\bar\chi_0^{\text{Is}}(\tau) \end{pmatrix},$$

where $\chi_h^{\text{Is}}(\tau)$, $h = 0, \frac12, \frac{1}{16}$ are conformal characters of Ising CFT.

- The above partition function describes the **1**-condensed boundary since the conformal character for the trivial primary field, $|\chi_0^{\text{Is}}(\tau)|^2$, appears only in the **1**-component $Z_{\mathbf{1}}$ of the multi-component partition function.

A CFT with S_3 and dual $\widetilde{S}_3$ symmetry

The categorical symmetry for 1+1D S_3 symmetry is the 2+1D S_3-gauge theory. The $\mathbf{1}$-condensed boundary is given by $(c, \bar{c}) = (\frac{4}{5}, \frac{4}{5})$ CFT $m6 \times \overline{m6}$, where $m6$ is the $(6, 5)$ minimal model:

$$
\begin{aligned}
Z^{\mathcal{G}au_{S_3}}_{\mathbf{1}\text{-cnd};\mathbf{1}} &= |\chi^{m6}_0|^2 + |\chi^{m6}_3|^2 + |\chi^{m6}_{\frac{2}{5}}|^2 + |\chi^{m6}_{\frac{7}{5}}|^2 \\
Z^{\mathcal{G}au_{S_3}}_{\mathbf{1}\text{-cnd};a_1} &= \chi^{m6}_0 \bar{\chi}^{m6}_3 + \chi^{m6}_3 \bar{\chi}^{m6}_0 + \chi^{m6}_{\frac{2}{5}} \bar{\chi}^{m6}_{\frac{7}{5}} + \chi^{m6}_{\frac{7}{5}} \bar{\chi}^{m6}_{\frac{2}{5}} \\
Z^{\mathcal{G}au_{S_3}}_{\mathbf{1}\text{-cnd};a_2} &= |\chi^{m6}_{\frac{2}{3}}|^2 + |\chi^{m6}_{\frac{1}{15}}|^2 \\
Z^{\mathcal{G}au_{S_3}}_{\mathbf{1}\text{-cnd};b} &= |\chi^{m6}_{\frac{2}{3}}|^2 + |\chi^{m6}_{\frac{1}{15}}|^2 \\
Z^{\mathcal{G}au_{S_3}}_{\mathbf{1}\text{-cnd};b_1} &= \chi^{m6}_0 \bar{\chi}^{m6}_{\frac{2}{3}} + \chi^{m6}_3 \bar{\chi}^{m6}_{\frac{2}{3}} + \chi^{m6}_{\frac{2}{5}} \bar{\chi}^{m6}_{\frac{1}{15}} + \chi^{m6}_{\frac{7}{5}} \bar{\chi}^{m6}_{\frac{1}{15}} \\
Z^{\mathcal{G}au_{S_3}}_{\mathbf{1}\text{-cnd};b_2} &= \chi^{m6}_{\frac{2}{3}} \bar{\chi}^{m6}_0 + \chi^{m6}_{\frac{2}{3}} \bar{\chi}^{m6}_3 + \chi^{m6}_{\frac{1}{15}} \bar{\chi}^{m6}_{\frac{2}{5}} + \chi^{m6}_{\frac{1}{15}} \bar{\chi}^{m6}_{\frac{7}{5}} \\
Z^{\mathcal{G}au_{S_3}}_{\mathbf{1}\text{-cnd};c} &= |\chi^{m6}_{\frac{1}{8}}|^2 + |\chi^{m6}_{\frac{13}{8}}|^2 + |\chi^{m6}_{\frac{1}{40}}|^2 + |\chi^{m6}_{\frac{21}{40}}|^2 \\
Z^{\mathcal{G}au_{S_3}}_{\mathbf{1}\text{-cnd};c_1} &= \chi^{m6}_{\frac{1}{8}} \bar{\chi}^{m6}_{\frac{13}{8}} + \chi^{m6}_{\frac{13}{8}} \bar{\chi}^{m6}_{\frac{1}{8}} + \chi^{m6}_{\frac{1}{40}} \bar{\chi}^{m6}_{\frac{21}{40}} + \chi^{m6}_{\frac{21}{40}} \bar{\chi}^{m6}_{\frac{1}{40}}.
\end{aligned}
$$

A CFT with anomalous $S_3^{(2)}$ symmetry

The anomalies for 1+1D S_3 symmetry are classified by $H^3(S_3;\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_6$. A CFT with the full categorical symmetry is given by $E(8)_1 \times \overline{so}(9)_2$ with $(c, \bar{c}) = (8, 8)$

$$Z^{\mathcal{G}au^{(2)}_{S_3}}_{\mathbf{1}\text{-cnd};\mathbf{1}} = \chi^{E(8)_1\times\overline{so}(9)_2}_{0;0}, \qquad Z^{\mathcal{G}au^{(2)}_{S_3}}_{\mathbf{1}\text{-cnd};a_1} = \chi^{E(8)_1\times\overline{so}(9)_2}_{0;-1}$$

$$Z^{\mathcal{G}au^{(2)}_{S_3}}_{\mathbf{1}\text{-cnd};a_2} = \chi^{E(8)_1\times\overline{so}(9)_2}_{0;-1}, \qquad Z^{\mathcal{G}au^{(2)}_{S_3}}_{\mathbf{1}\text{-cnd};b} = \chi^{E(8)_1\times\overline{so}(9)_2}_{0;-\frac{7}{9}}$$

$$Z^{\mathcal{G}au^{(2)}_{S_3}}_{\mathbf{1}\text{-cnd};b_1} = \chi^{E(8)_1\times\overline{so}(9)_2}_{0;-\frac{4}{9}}, \qquad Z^{\mathcal{G}au^{(2)}_{S_3}}_{\mathbf{1}\text{-cnd};b_2} = \chi^{E(8)_1\times\overline{so}(9)_2}_{0;-\frac{10}{9}}$$

$$Z^{\mathcal{G}au^{(2)}_{S_3}}_{\mathbf{1}\text{-cnd};c} = \chi^{E(8)_1\times\overline{so}(9)_2}_{0;-1}, \qquad Z^{\mathcal{G}au^{(2)}_{S_3}}_{\mathbf{1}\text{-cnd};c_1} = \chi^{E(8)_1\times\overline{so}(9)_2}_{0;-\frac{1}{2}}$$

A CFT with anomalous $S_3^{(1)}$ symmetry

A CFT with the full categorical symmetry is given by $so(9)_2 \times u(1)_2 \times \bar{u}(1)_2 \times \bar{E}(8)_1$ with $(c, \bar{c}) = (9, 9)$

$$Z^{\mathcal{G}au^{(1)}_{S_3}}_{\mathbf{1}\text{-cnd};\mathbf{1}} = \chi^{so(9)_2\times u(1)_2\times \bar{u}(1)_2\times \bar{E}(8)_1}_{0;0;0;0} + \chi^{so(9)_2\times u(1)_2\times \bar{u}(1)_2\times \bar{E}(8)_1}_{1;\frac{1}{4};-\frac{1}{4};0}$$

$$Z^{\mathcal{G}au^{(1)}_{S_3};0}_{\mathbf{1}\text{-cnd};a_1} = \chi^{so(9)_2\times u(1)_2\times \bar{u}(1)_2\times \bar{E}(8)_1}_{0;\frac{1}{4};-\frac{1}{4};0} + \chi^{so(9)_2\times u(1)_2\times \bar{u}(1)_2\times \bar{E}(8)_1}_{1;0;0;0}$$

$$Z^{\mathcal{G}au^{(1)}_{S_3}}_{\mathbf{1}\text{-cnd};a_2} = \chi^{so(9)_2\times u(1)_2\times \bar{u}(1)_2\times \bar{E}(8)_1}_{1;0;0;0} + \chi^{so(9)_2\times u(1)_2\times \bar{u}(1)_2\times \bar{E}(8)_1}_{1;\frac{1}{4};-\frac{1}{4};0}$$

$$Z^{\mathcal{G}au^{(1)}_{S_3}}_{\mathbf{1}\text{-cnd};b} = \chi^{so(9)_2\times u(1)_2\times \bar{u}(1)_2\times \bar{E}(8)_1}_{\frac{10}{9};0;0;0} + \chi^{so(9)_2\times u(1)_2\times \bar{u}(1)_2\times \bar{E}(8)_1}_{\frac{10}{9};\frac{1}{4};-\frac{1}{4};0}$$

$$Z^{\mathcal{G}au^{(1)}_{S_3}}_{\mathbf{1}\text{-cnd};b_1} = \chi^{so(9)_2\times u(1)_2\times \bar{u}(1)_2\times \bar{E}(8)_1}_{\frac{4}{9};0;0;0} + \chi^{so(9)_2\times u(1)_2\times \bar{u}(1)_2\times \bar{E}(8)_1}_{\frac{4}{9};\frac{1}{4};-\frac{1}{4};0}$$

$$Z^{\mathcal{G}au^{(1)}_{S_3}}_{\mathbf{1}\text{-cnd};b_2} = \chi^{so(9)_2\times u(1)_2\times \bar{u}(1)_2\times \bar{E}(8)_1}_{\frac{7}{9};0;0;0} + \chi^{so(9)_2\times u(1)_2\times \bar{u}(1)_2\times \bar{E}(8)_1}_{\frac{7}{9};\frac{1}{4};-\frac{1}{4};0}$$

$$Z^{\mathcal{G}au^{(1)}_{S_3}}_{\mathbf{1}\text{-cnd};c} = \chi^{so(9)_2\times u(1)_2\times \bar{u}(1)_2\times \bar{E}(8)_1}_{\frac{1}{2};0;-\frac{1}{4};0} + \chi^{so(9)_2\times u(1)_2\times \bar{u}(1)_2\times \bar{E}(8)_1}_{1;\frac{1}{4};0;0}$$

$$Z^{\mathcal{G}au^{(1)}_{S_3}}_{\mathbf{1}\text{-cnd};c_1} = \chi^{so(9)_2\times u(1)_2\times \bar{u}(1)_2\times \bar{E}(8)_1}_{\frac{1}{2};\frac{1}{4};0;0} + \chi^{so(9)_2\times u(1)_2\times \bar{u}(1)_2\times \bar{E}(8)_1}_{1;0;-\frac{1}{4}}.$$

A CFT with anomalous $S_3^{(3)}$ symmetry

A CFT with the full categorical symmetry is given by $m6 \times u(1)_2 \times \bar{m}6 \times \bar{u}(1)_2$ with $(c, \bar{c}) = (\frac{9}{5}, \frac{9}{5})$

$$
\begin{aligned}
Z^{\mathcal{G}\mathrm{au}^{(3)}_{S_3}}_{\mathbf{1}\text{-cnd};\mathbf{1}} &= \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{0;0;0;0} + \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{0;\frac{1}{4};-3;-\frac{1}{4}} + \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{3;0;-3;0} + \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{3;\frac{1}{4};0;-\frac{1}{4}} \\
&\quad + \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{\frac{2}{5};0;-\frac{2}{5};0} + \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{\frac{2}{5};\frac{1}{4};10,-\frac{7}{5};-\frac{1}{4}} + \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{10,\frac{7}{5};0;10,-\frac{7}{5};0} + \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{10,\frac{7}{5};\frac{1}{4};-\frac{2}{5};-\frac{1}{4}} \\
Z^{\mathcal{G}\mathrm{au}^{(3)}_{S_3}}_{\mathbf{1}\text{-cnd};a_1} &= \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{0;0;-3;0} + \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{0;\frac{1}{4};0;-\frac{1}{4}} + \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{3;0;0;0} + \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{3;\frac{1}{4};-3;-\frac{1}{4}} \\
&\quad + \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{\frac{2}{5};0;10,-\frac{7}{5};0} + \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{\frac{2}{5};\frac{1}{4};-\frac{2}{5};-\frac{1}{4}} + \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{10,\frac{7}{5};0;-\frac{2}{5};0} + \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{10,\frac{7}{5};\frac{1}{4};10,-\frac{7}{5};-\frac{1}{4}} \\
Z^{\mathcal{G}\mathrm{au}^{(3)}_{S_3}}_{\mathbf{1}\text{-cnd};a_2} &= \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{\frac{2}{3};0;-\frac{2}{3};0} + \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{\frac{2}{3};\frac{1}{4};-\frac{2}{3};-\frac{1}{4}} + \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{\frac{1}{15};0;-\frac{1}{15};0} + \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{\frac{1}{15};\frac{1}{4};-\frac{1}{15};-\frac{1}{4}} \\
Z^{\mathcal{G}\mathrm{au}^{(3)}_{S_3}}_{\mathbf{1}\text{-cnd};b} &= \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{\frac{2}{3};0;-\frac{2}{3};0} + \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{\frac{2}{3};\frac{1}{4};-\frac{2}{3};-\frac{1}{4}} + \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{\frac{1}{15};0;-\frac{1}{15};0} + \chi^{m6\times u(1)_2\times \bar{m}6\times \bar{u}(1)_2}_{\frac{1}{15};\frac{1}{4};-\frac{1}{15};-\frac{1}{4}}
\end{aligned}
$$

A CFT with anomalous $S_3^{(3)}$ symmetry

$$Z^{\mathcal{G}\text{au}\,S_3^{(3)}}_{\mathbf{1}\text{-cnd};b_1} = \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{0;0;-\frac{2}{3};0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{0;\frac{1}{4};-\frac{2}{3};-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{3;0;-\frac{2}{3};0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{3;\frac{1}{4};-\frac{2}{3};-\frac{1}{4}}$$
$$+ \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{2}{5};0;-\frac{1}{15};0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{2}{5};\frac{1}{4};-\frac{1}{15};-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{10,\frac{7}{5};0;-\frac{1}{15};0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{10,\frac{7}{5};\frac{1}{4};-\frac{1}{15};-\frac{1}{4}}$$

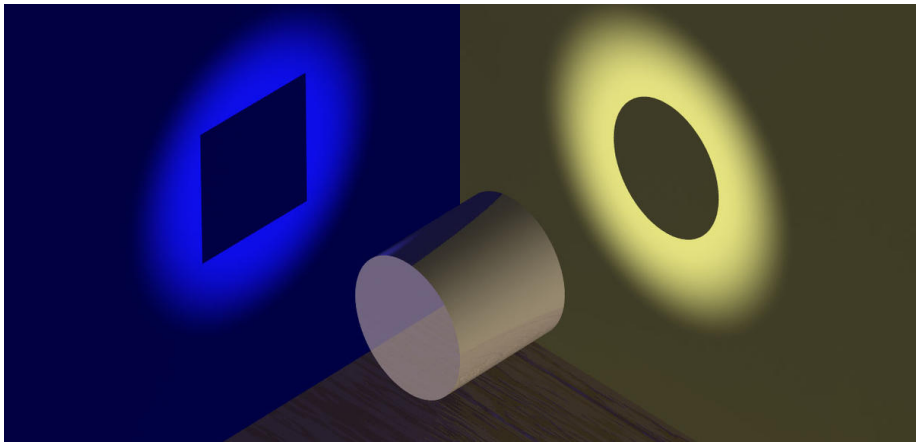
$$Z^{\mathcal{G}\text{au}\,S_3^{(3)}}_{\mathbf{1}\text{-cnd};b_2} = \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{2}{3};0;0;0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{2}{3};0;-3;0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{2}{3};\frac{1}{4};0;-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{2}{3};\frac{1}{4};-3;-\frac{1}{4}}$$
$$+ \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{1}{15};0;-\frac{2}{5};0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{1}{15};0;10,-\frac{7}{5};0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{1}{15};\frac{1}{4};-\frac{2}{5};-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{1}{15};\frac{1}{4};10,-\frac{7}{5};-\frac{1}{4}}$$

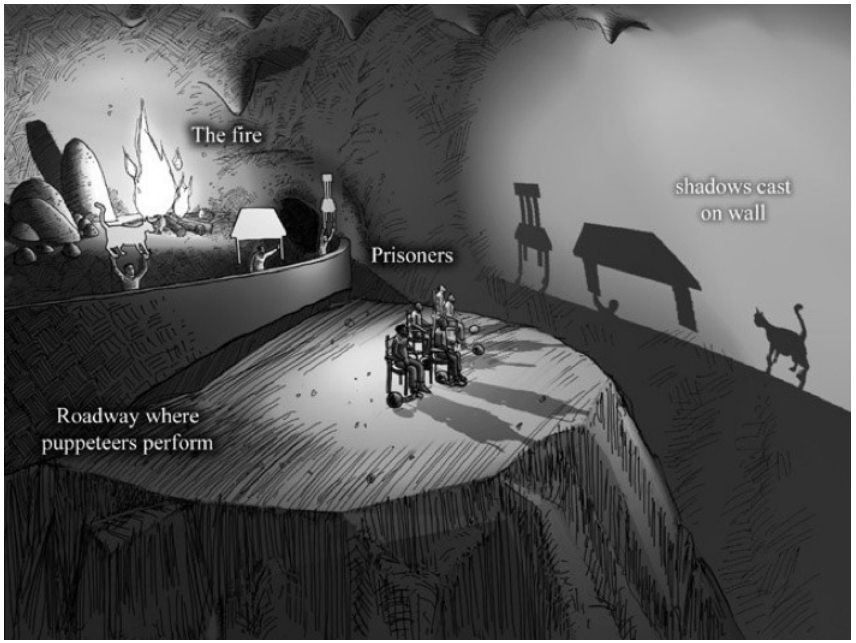
$$Z^{\mathcal{G}\text{au}\,S_3^{(3)}}_{\mathbf{1}\text{-cnd};c} = \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{1}{8};0;-\frac{13}{8};-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{1}{8};\frac{1}{4};-\frac{1}{8};0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{13}{8};0;-\frac{1}{8};-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{13}{8};\frac{1}{4};-\frac{13}{8};0}$$
$$+ \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{1}{40};0;-\frac{21}{40};-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{1}{40};\frac{1}{4};-\frac{1}{40};0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{21}{40};0;-\frac{1}{40};-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{21}{40};\frac{1}{4};-\frac{21}{40};0}$$

$$Z^{\mathcal{G}\text{au}\,S_3^{(3)}}_{\mathbf{1}\text{-cnd};c_1} = \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{1}{8};0;-\frac{1}{8};-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{1}{8};\frac{1}{4};-\frac{13}{8};0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{13}{8};0;-\frac{13}{8};-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{13}{8};\frac{1}{4};-\frac{1}{8};0}$$
$$+ \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{1}{40};0;-\frac{1}{40};-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{1}{40};\frac{1}{4};-\frac{21}{40};0} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{21}{40};0;-\frac{21}{40};-\frac{1}{4}} + \chi^{m6\times u(1)_2\times\bar{m}6\times\bar{u}(1)_2}_{\frac{21}{40};\frac{1}{4};-\frac{1}{40};0}$$

The essence of a symmetry

- Emergent symmetries can go beyond groups and higher groups and anomalies. But their can always be described by
  a topological order in one higher dimension
  $=$ **categorical symmetry**





- The same topological order (in one higher dimensions) can have different shadows $\rightarrow$ **holo-equivalent symmetries**.
- **From the emergent categorical symmetry, we can obtain its corresponding gapless liquid state.**