What can we learn about many-body entanglement from holography ?

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Ref:
SR, Takayanagi, PRL (2006),
SR, Takayanagi, JHEP (2006)
Fujita, Li, SR, Takayanagi, JHEP (2009)
Nishioka, SR, Takayanagi, J. Phys. A (2009)

entanglement and entropy of entanglement

entanglement entropy (von-Neumann entropy)

= a measure of entanglement in a given quantum state $|\Psi\rangle$

(i) bipartition the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

(ii) take partial trace $\rho_{\rm tot} = |\Psi\rangle\langle\Psi|$

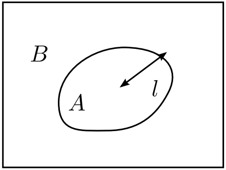
$$\rho_A = {\rm tr}_B |\Psi\rangle\langle\Psi| = \sum_j p_j |\psi_j\rangle_A \langle\psi_j|_A \qquad \left(\sum_j p_j = 1\right)$$

(iii) entanglement entropy

$$S_A = -{\rm tr}_A \left[\rho_A \ln \rho_A\right] = -\sum_j p_j \ln p_j$$

application to many-body systms and field theories:

A, B : submanifold of the total system



motivation for entanglement entropy

EE can be a good "order parameter" for quantum systems (?)

- defined purely in terms of wavefunctions

  (EE measures a response to external gravity)

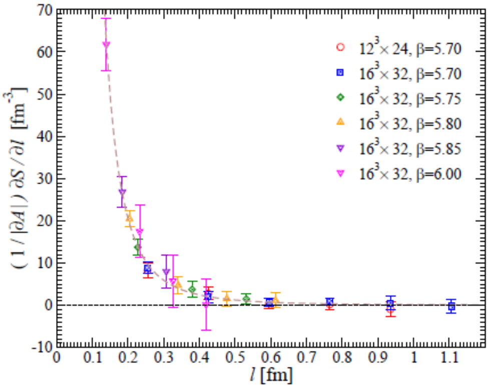
- use computational complexity to classify quantum states ?

- best method to compute central charge in (1+1)D CFT

- EE spectrum: new tool to classify symmetry protected gapped phases

Very difficult to compute !

EE in pure 4D SU(3) Yang-Mills theory



Buividovich, Polikarpov (NPB802, pp458, 2008)

Nakagawa-Nakamura-Motoki-Zakharov (09)

holographic calculations: Nishioka, Takayanagi (2006,2007),
Klebanov, Kutasov, Murugan (2007)

entanglement entropy: some key properties

- when $\rho_{\rm tot} = \text{pure}$ $\quad B = A^{\rm complement}$

$$S_A = S_B$$

- when $\rho_{\rm tot} = \text{mixed}$ $\quad B = A^{\rm complement}$

$$S_A \neq S_B$$

- when $\rho_{\rm tot} = e^{-\beta H}$ $\quad A = \text{total system}$

$$S_A = \text{thermal entropy}$$

- strong subadditivity Lieb-Ruskal (73)

$$S_B + S_{ABC} \leq S_{AB} + S_{BC}$$

scaling of entanglement entropy

Area law (gapped system, CFT in (d+1)D with d>1, etc.)

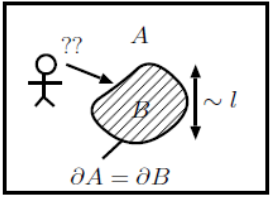
$$S_A = \text{const.} \left(\frac{l}{a}\right)^{d-1} + \cdots$$

Srednicki (93)

↕

Black Hole Entropy (Beckenstein-Hawking)

$$S_{BH} = \frac{\text{Area of horizon}}{4G_N}$$

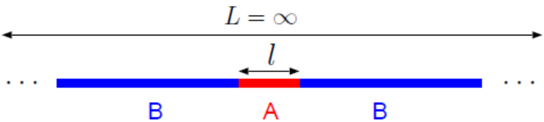


scaling of entanglement entropy

detecting CFT QCP in 1D (Holzhey,Larsen & Wilczek)

$$S_A = \frac{c}{3} \log l/a + c'$$

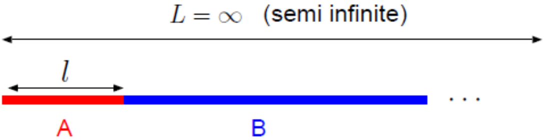
c : central charge
a : cut off



boundary entropy (Zhou, Barthel, Fjaerestad, Schollwock)

$$S_A = \frac{c}{6} \log 2l/a + c'/2 + \log(g)$$

$\log(g)$
:Affleck-Ludwig's boundary entropy



scaling of entanglement entropy

- detecting topological order in (2+1)D

Kitaev & Preskill Levin & Wen (2006)

$$S_A = \gamma \frac{l}{a} - \log(D)$$

$$D = \sqrt{\sum_a d_a^2}$$

quantum dimension

quasi-particle type

$$\log D = \log \sqrt{q}$$ FQHE at nu = 1/q (Chern-Simons theory)

$$\log D = \log 2$$ Z2 lattice gauge theory

- z=2 Lifsitz critical point in (2+1)D

Fradkin & Moore (2006)

$$S_A = \gamma \tfrac{l}{a} + \alpha c \log(l/a) + \cdots$$

- free fermions with Fermi surface

Gioev & Klich, Wolf (2006)

$$S_A = C l^{d-1} \log(l/a)$$

$$C \propto \int_{\partial A} \int_{\mathrm{FS}} |\boldsymbol{n}_r \cdot \boldsymbol{n}_k| dS_r dS_k$$

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holography and AdS/CFT

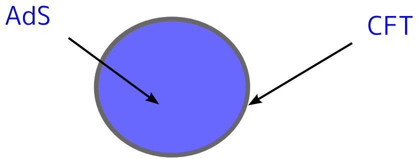
t' Hooft (93'), Susskind (94')   (holographic principle)

Bekenstein-Hawking black hole entropy

$$S_{BH} = \frac{\text{Area of horizon}}{4G_N}$$

Maldacena conjecture (97')   (AdS/CFT correspondence)

(quantum) gravity on d+2 dimensional AdS space
= d+1 dimensional CFT



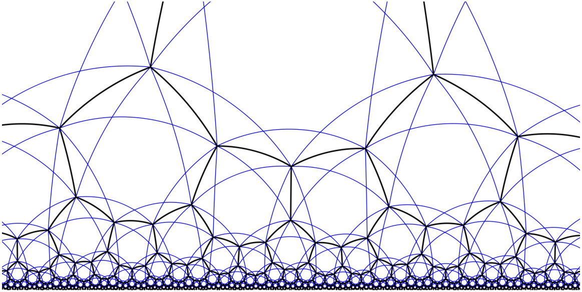
AdS space

- AdS space is a solution to the Einstein equation with a negative cosmologaical constant

$$I_{\rm EH} = \frac{1}{16\pi G_N}\int d^{d+2}x\,\sqrt{-g}\,[R-\Lambda] \qquad \Lambda = -\frac{(d+1)d}{R^2}$$

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 - dt^2 + \sum\nolimits_{i=1}^{d} dx_i^2\right)$$

- AdS space has a boundary
- Isometry of AdS space : SO(2, d+1) = conformal symmetry in d+1 dimensions



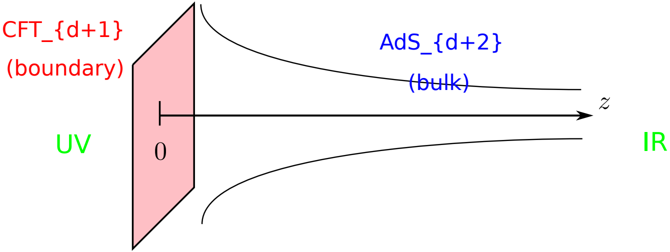
AdS/CFT

string theory in d+2 dimensional AdS space
= d+1 dimensional CFT

- correlation function GKPW relation (98)

$$\left\langle e^{\int d^{d+1}x\, \varphi(x)\mathcal{O}(x)} \right\rangle_{\rm CFT} = Z_{\rm String}\big|_{\phi(x,z)|_{z=0}=\varphi(x)} \simeq e^{-I_{\rm SUGRA}[\phi]_{\phi(x,0)=\varphi(x)}}$$

- geometrical realization of RG $\vec{x} \to \lambda \vec{x}, \quad z \to \lambda z$

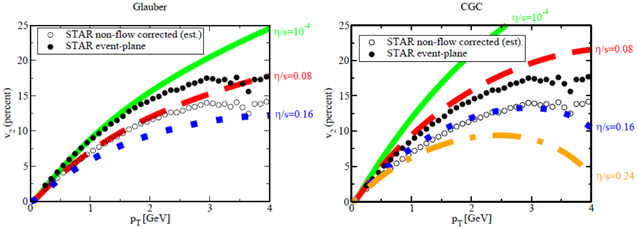


Can we make use of holography ?

Idea: use gravity to study QFT (CFT)

E.g. universal viscosity/entropy ratio   Kovtun-Son-Starinets (03)

viscosity/entropy ratio in RHIC

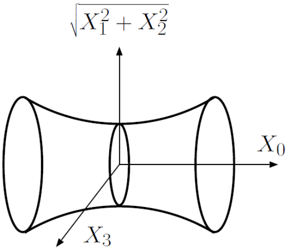


Luzum & Romatschke (08), stolen from Son's slide

Idea: use QFT (CFT) to study gravity

AdS_3 space

- embedding in $\mathbb{R}^{2,2}$



$$-X_0^2 - X_3^2 + X_1^2 + X_2^2 = -R^2$$

$$ds^2 = -dX_0^2 - dX_3^2 + dX_1^2 + dX_2^2$$

SO(2,2) symmetry

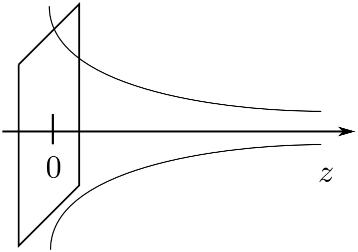
- Poincare coordinates

$$X_0 = \frac{z}{2}\left(1 + \frac{1}{z^2}(R^2 + x^2 - t^2)\right)$$

$$X_3 = \frac{Rt}{z} \qquad X_2 = \frac{Rx}{z}$$

$$X_1 = \frac{z}{2}\left(1 - \frac{1}{z^2}(R^2 - x^2 + t^2)\right)$$

$$ds^2 = R^2 \frac{-dt^2 + dx^2 + dy^2}{y^2} \qquad (y = 1/u)$$



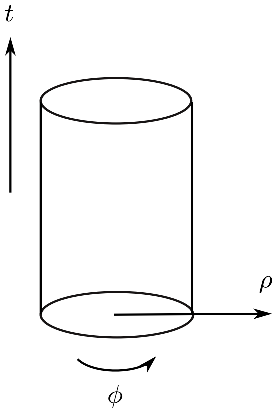
AdS_3 space

- global coordinates

$$
\begin{aligned}
X_0 &= R\cosh\rho\cos t, \\
X_3 &= R\cosh\rho\sin t, \\
X_1 &= R\sinh\rho\cos\phi, \\
X_2 &= R\sinh\rho\sin\phi,
\end{aligned}
$$

$$ds^2 = R^2\left(-\cosh\rho^2 dt^2 + d\rho^2 + \sinh\rho^2 d\phi^2\right)$$

boundary at $\rho = \infty$



Baby version of AdS/CFT

- Einstein-Hilbert action with negative cosmological const.

$$S = \frac{1}{16\pi G_N} \int d^3x \sqrt{g}(R + \Lambda) \qquad \Lambda = -2/R^2 < 0$$

- vielbein and spin connection

$$A^a = -\frac{1}{2}\epsilon^a{}_{bc}\omega^{bc} + \frac{i}{R}e^a \qquad A^a = -\frac{1}{2}\epsilon^a{}_{bc}\omega^{bc} - \frac{i}{R}e^a$$

- $\mathrm{SL}(2,\mathbb{C})$ (doubled) Chern-Simons theory

$$S = iI[A] - iI[\bar{A}] \qquad I[A] = \frac{k}{4\pi}\int \mathrm{tr}\,(AdA + \frac{2}{3}A^3) \qquad k = -\frac{R}{4G_N}$$

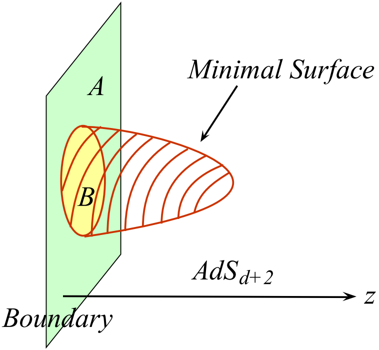
- (1+1)D CFT at boundary

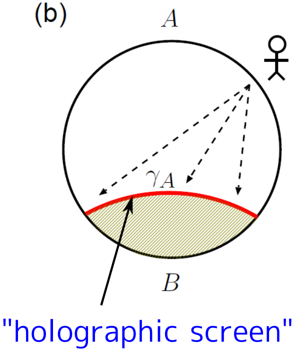
central charge: $c = \dfrac{3R}{2G_N}$

Brown-Henneaux

holographic derivation of entanglement entropy

$$S_A = \frac{\text{Area of minimal surface } \gamma_A}{4G_N}$$





holographic derivation of entanglement entropy

$$d+1=2 \Rightarrow \mathrm{AdS}_3/\mathrm{CFT}_2$$

$$c = \frac{3R}{2G_N^{(3)}}$$

- minimal surface = geodesic

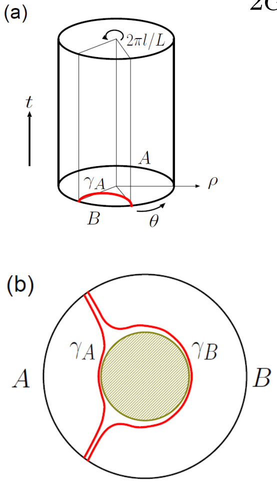
$$S_A = \frac{c}{3}\log(l/a) + O(1)$$

- finite system

$$S_A = \frac{c}{3}\log\left(\frac{L}{\pi a}\sin\frac{\pi l}{L}\right) + O(1)$$

- finite temperature

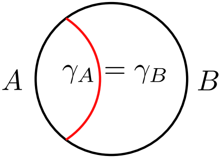
$$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\frac{\pi l}{\beta}\right) + O(1)$$



holographic entanglement entropy: some key properties

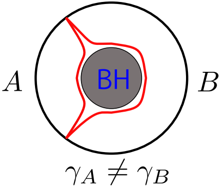
- when $\rho_{\rm tot} = \text{pure} \quad B = A^{\rm complement}$

$$S_A = S_B$$



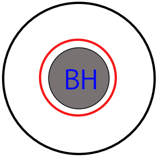
- when $\rho_{\rm tot} = \text{mixed} \quad B = A^{\rm complement}$

$$S_A \neq S_B$$



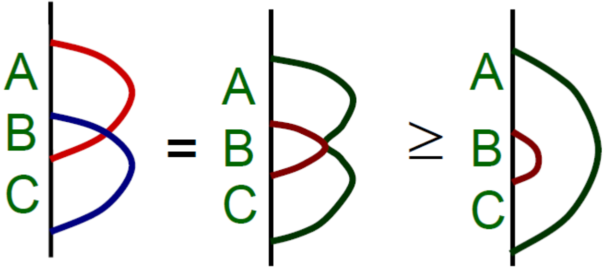
- when $\rho_{\rm tot} = e^{-\beta H} \quad A = \text{total system}$

$$S_A = \text{thermal entropy}$$



- strong subadditivity

$$S_{AB} + S_{BC} \geq S_B + S_{ABC}$$



Headrick-Takayanagi (07)

CFT in general dimensions

d+1 = even:

$$S_A = p_1\left(\frac{l}{a}\right)^{d-1} + p_3\left(\frac{l}{a}\right)^{d-3} + \cdots + p_{d-2}\left(\frac{l}{a}\right)^{2} + q\log l/a + O(1)$$

d+1 = odd:

$$S_A = p_1\left(\frac{l}{a}\right)^{d-1} + p_3\left(\frac{l}{a}\right)^{d-3} + \cdots + p_{d-1}\left(\frac{l}{a}\right)^{1} + p_d + O(a/l)$$

q and p_d : universal and conformal invariant

q: related to central charge in even dim. CFT

p_d: universal althgouh no central charge in odd dim. CFT

Myers & Sinha (2010); c-theorem in genral dimensions

AdS5/CFT4

type II B on AdS_5 x S_5 ⟷ 4D N=4 SU(N) SYM

$$G^{(10)} = 8\pi^6 \alpha'^4 g_s^2 \qquad G_N^{(5)} = \frac{G_N^{(10)}}{\pi^3 R^5} \qquad R = (4\pi g_s \alpha'^2 N)^{1/4}$$

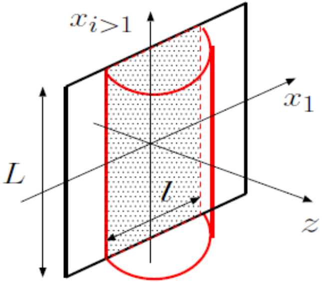
gravity calculation (strongly coupled SYM)

$$S_A = Const'.\frac{N^2 L^2}{a^2} - 0.051\frac{N^2 L^2}{l^2}$$

↕

free field calculation

$$S_A = Const.\frac{N^2 L^2}{a^2} - 0.078\frac{N^2 L^2}{l^2}$$

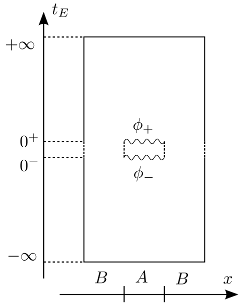


c.f. 4/3 problem in thermal entropy, Gubser-Klevanov-Peet (96)

entanglement entropy in QFTs

ground state wavefunctional:

$$\Psi[\phi_0(x)] = \mathcal{N}^{-1} \int_{-\infty}^{t=0} \mathcal{D}\phi e^{-S} \prod_{x\in A} \delta(\phi(0,x) - \phi_0(x))$$

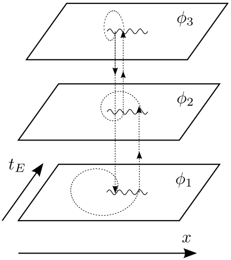


reduced density matrix:

$$[\rho_A]_{\phi_+,\phi_-} = Z^{-1} \int_{-\infty}^{+\infty} \mathcal{D}\phi e^{-S}$$

$$\times \prod_{x\in A} \delta(\phi(+0,x) - \phi_+(x))\delta(\phi(-0,x) - \phi_-(x))$$

$$\mathrm{tr}_A \rho_A^n = (Z_1)^{-n} \int_{\mathcal{R}_n} \mathcal{D}\phi e^{-S} = \frac{Z_n}{(Z_1)^n}$$



QFT on a singular curved space

replica trick --> entanglement entropy

$$S_A = -\frac{\partial}{\partial n} \mathrm{tr}_A \rho_A^n \Big|_{n=1} = -\frac{\partial}{\partial n} \ln \mathrm{tr}_A \rho_A^n \Big|_{n=1}$$

entanglement entropy in QFTs

Weyl rescaling: $g_{\mu\nu} = \delta_{\mu\nu} e^{2\rho} \quad l \sim e^{2\rho}$

$$
\begin{aligned}
l \frac{d}{dl} \ln \mathrm{tr}_A \rho_A^n &= 2 \int d^{d+1}x \, g_{\mu\nu}(x) \frac{\delta}{\delta g_{\mu\nu}(x)} \left(\ln Z_n - n \ln Z_1 \right) \\
&= -\frac{1}{2\pi} \left\langle \int d^{d+1}x \sqrt{g} T_\mu^\mu \right\rangle_{M_n} + \frac{n}{2\pi} \left\langle \int d^{d+1}x \sqrt{g} T_\mu^\mu \right\rangle_{T_1}
\end{aligned}
$$

$$
l \frac{d}{dl} S_A = -l \frac{d}{dl} \frac{\partial}{\partial n} \ln \mathrm{tr}_A \rho_A^n = -\frac{1}{2\pi} \frac{\partial}{\partial n} \left\langle \int d^{d+1}x \sqrt{g} T_\mu^\mu \right\rangle_{M_n} \qquad (n \to 1)
$$

entanglement entropy in CFTs

2D CFT $\left\langle T^\mu_\mu \right\rangle = -\frac{c}{12}R$ → $S_A = \frac{c}{3}\ln\frac{l}{a}$

Holzhey,Larsen, Wilczek (94) Calabrese, Cardy (04)

4D CFT $\left\langle T^\mu_\mu \right\rangle = -\frac{c}{8\pi}W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + \frac{a}{8\pi}\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$

→ $S_A = \gamma_1\frac{l^2}{a^2} + \gamma_2\ln\frac{l}{a} + \cdots$

SR, Takayanagi (06)

e.g. Free scalar field in 4d

$$a = \frac{1}{360}, \quad \gamma_2 = -\frac{1}{90}$$

recently confirmed

Lohmayer-Neuberger-Schwimmer-Theisen (09)

Schwimmer-Theisen (08)

Casini-Huerta (09)

odd dimentional CFTs: no central charges !?

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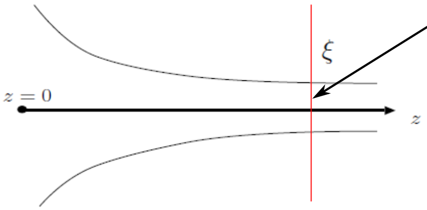
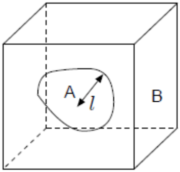
massive deformation



chopping off AdS space, here!

ξ : correlation length $(\ll l)$



a : cut off

$$S_A = p_1\left(\frac{l}{a}\right)^{d-1} + p_3\left(\frac{l}{a}\right)^{d-3} + \cdots + p_{d-2}\left(\frac{l}{a}\right)^{2} + $$
$$+p_1'\left(\frac{\xi}{a}\right)^{d-1} + p_3'\left(\frac{\xi}{a}\right)^{d-3} + \cdots + p_{d-2}'\left(\frac{\xi}{a}\right)^{2} + q\log\xi/a + O(1)$$

d: odd

consistent with d=1 result

$$S_A = \frac{c}{6}\ln\frac{\xi}{a}$$

$$S_A = p_1\left(\frac{l}{a}\right)^{d-1} + p_3\left(\frac{l}{a}\right)^{d-3} + \cdots + p_{d-2}\left(\frac{l}{a}\right)^{2} + $$
$$+p_1'\left(\frac{\xi}{a}\right)^{d-1} + p_3'\left(\frac{\xi}{a}\right)^{d-3} + \cdots + p_{d-1}'\left(\frac{\xi}{a}\right)^{1} + p_d' + O(a/l)$$

d: even

const. part when d=even $p_d' = 0$!

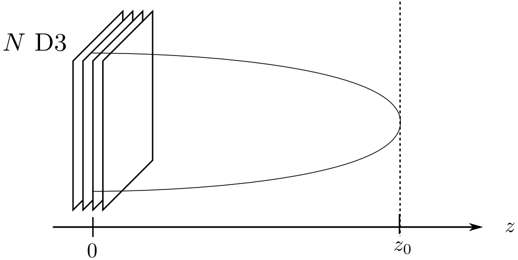
$\longleftrightarrow$

topological EE

$$S_A = \gamma\frac{l}{a} - \log(D)$$

pure Yang-Mills in (2+1)D ⟷ AdS soliton

Witten (98)

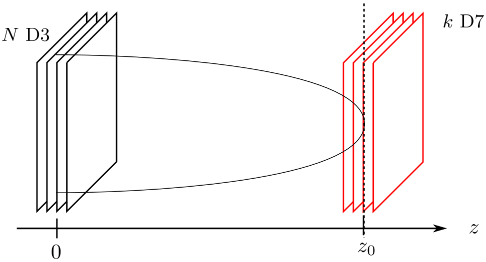


$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + f(z)d\theta^2 + dx^2 + dy^2 + f^{-1}(z)dz^2\right)$$
$$f(z) = 1 - (z/z_0)^4$$

QFT dual:
(i) start from N=4 SYM in d+1 = 4 dim
(ii) compactify one direction, get rid of fermions (susy) by APBC
(iii) scalars get massive by radiative correction

holographic dual of Chern-Simons

Fujita-Li-SR-Takayanagi (2009)



D7-brane
= "Top tensor" ?

Aguado-Vidal (2009):
tensor representation
of topological phases

holographic dual of topological phase

$$S_{\mathrm{D3}} = \frac{k}{4\pi} \int \mathrm{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

$$S_{top} \sim \frac{k^2}{2} \log N$$

Need backreacted geometry

topological phases and D-branes

SR-Takayanagi (2010)

ten-fold classification of topological insulators/SCs

AZ\\d	0	1	2	3	4	5	6	7	8	9
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

Schnyder-SR-Furusaki-Ludwig (08)
Kitaev (09)

1-to-1 correspondence!

K-theory classification of D-branes

	D(−1)	D0	D1	D2	D3	D4	D5	D6	D7	D8	D9
type IIB	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$O9^-$ (type I)	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
$O9^+$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

TABLE III. Dp-brane charges from K-theory, classified by $K(\mathbb{S}^{9-p})$, $KO(\mathbb{S}^{9-p})$ and $KSp(\mathbb{S}^{9-p})$ [24]. A $\mathbb{Z}_2$ charged Dp-brane with p even or p odd represents a non-BPS Dp-brane or a bound state of a Dp and an anti-Dp brane, respectively [26].

Sen, Witten, Horava (98-99)

summary

- holographic calculation of entanglement entropy in CFTs

- entanglement entropy in 4D CFTs from Weyl anomaly

- holographic calculation of topological entanglement entropy ?

- Brane world : EE = black hole entropy Emparan 06

- holography beyond AdS (flatspace) Li-Takayanagi 10

- BH information pradox and quantum quench Ugajin-Takayanagi 10

- Renyi mutal information Headrick 10

- ...

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Keywords:
-- BH entropy, EE in QFTs,
-- Dynamics of quantum information in out-of-equilibrium systems (thermalization, quenches, etc.),
-- Formation and evaporation of black holes,
-- MPS for quantum many-body problems,
-- Fundamental connections between gravity and information (gravity from entropy, etc.),
-- Mathematical analogies between sugra BHs and qubits,
-- Quantum information in cosmology
(entropy, complementarity, and holography for cosmological spacetimes, measures, etc.).