# Semiclassical Path(s) to the Cosmic Web
# Making (Dark Matter) Waves


**Cora Uhlemann**

Newcastle University


Cosmic Web Workshop KITP Santa Barbara, Jan 2023

recent arXiv:2206.11918 [OJA 5 (2022)]
led by PhD Alex Gough

collaborators: Oliver Hahn, Michael Kopp, Cornelius Rampf & Mateja Gosenca

# DARK MATTER MASS

one of the least constrained physical parameters

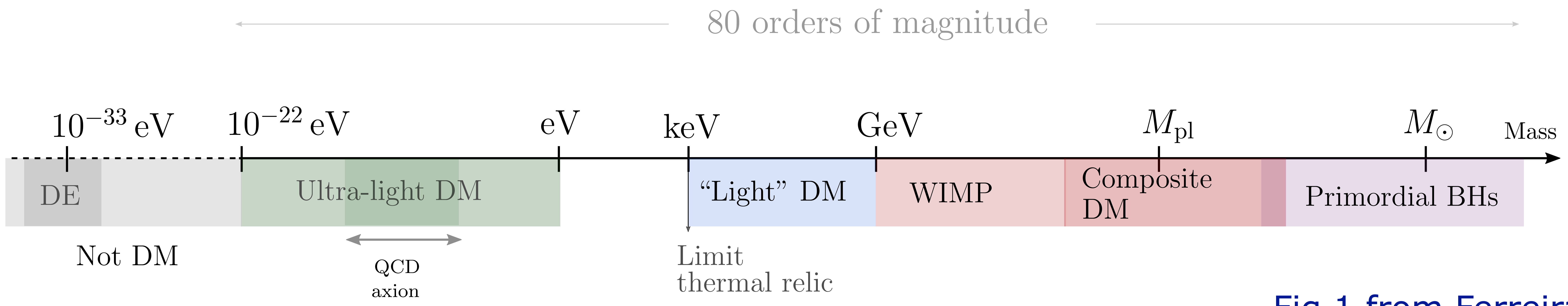



Fig 1 from Ferreira '21

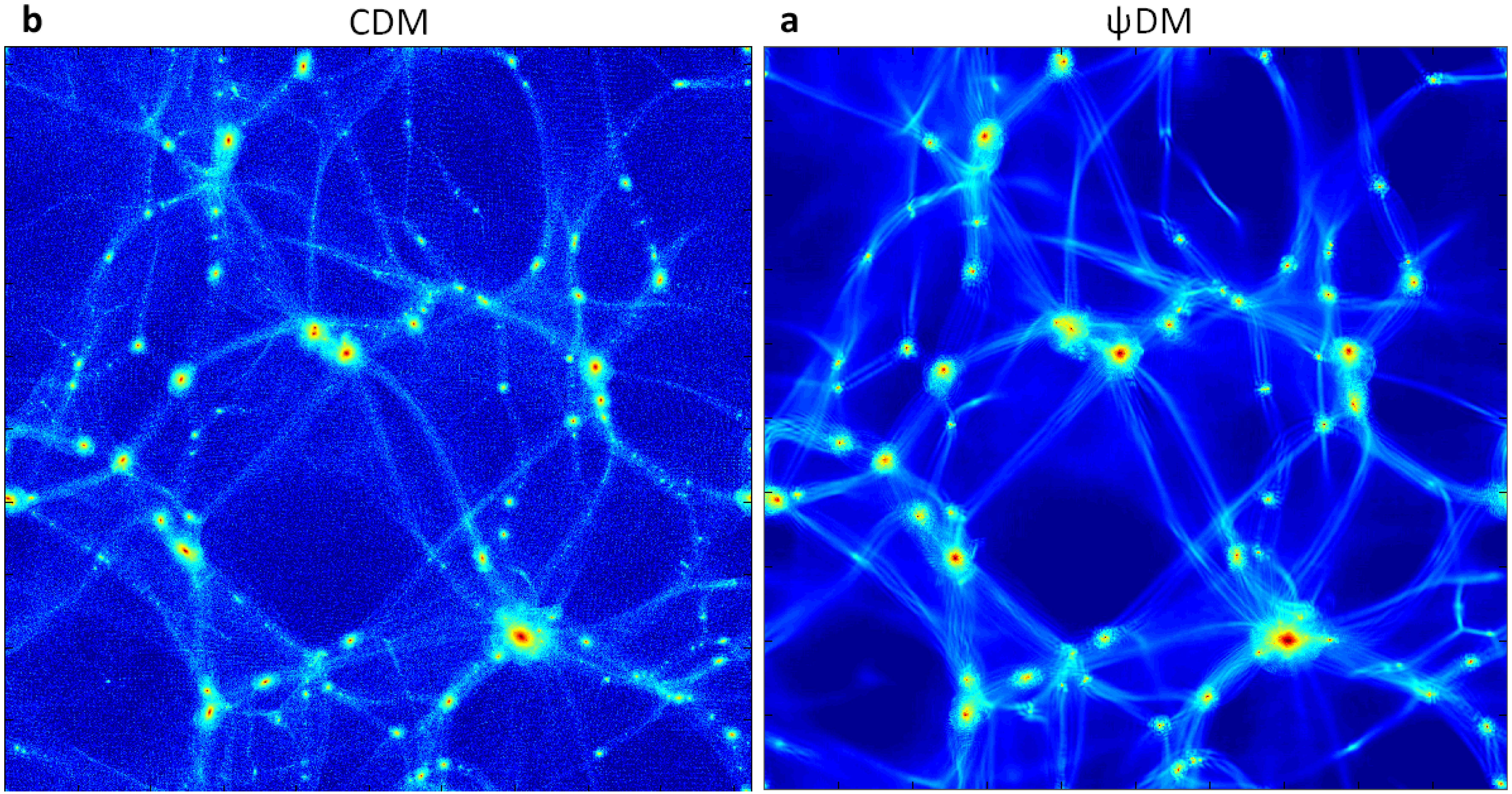
Known particles: ν e⁻ p Higgs

**thermal production:** hot warm cold

wave vs. cold dark matter

# Wave Dark Matter

Schive ++ Nature Phys. Lett '15




astrophysical imprints: Hui, Ostriker, Tremaine & Witten '17, Hui '21

## Phase Space Dynamics

Vlasov-Poisson (collisionless Boltzmann, long range force)

$$\partial_t f(\boldsymbol{x},\boldsymbol{p},t) = \{H, f\} \qquad \Delta V(\boldsymbol{x},t) \propto \int f(\boldsymbol{x},\boldsymbol{p},t)\,\mathrm{d}^3p - 1$$

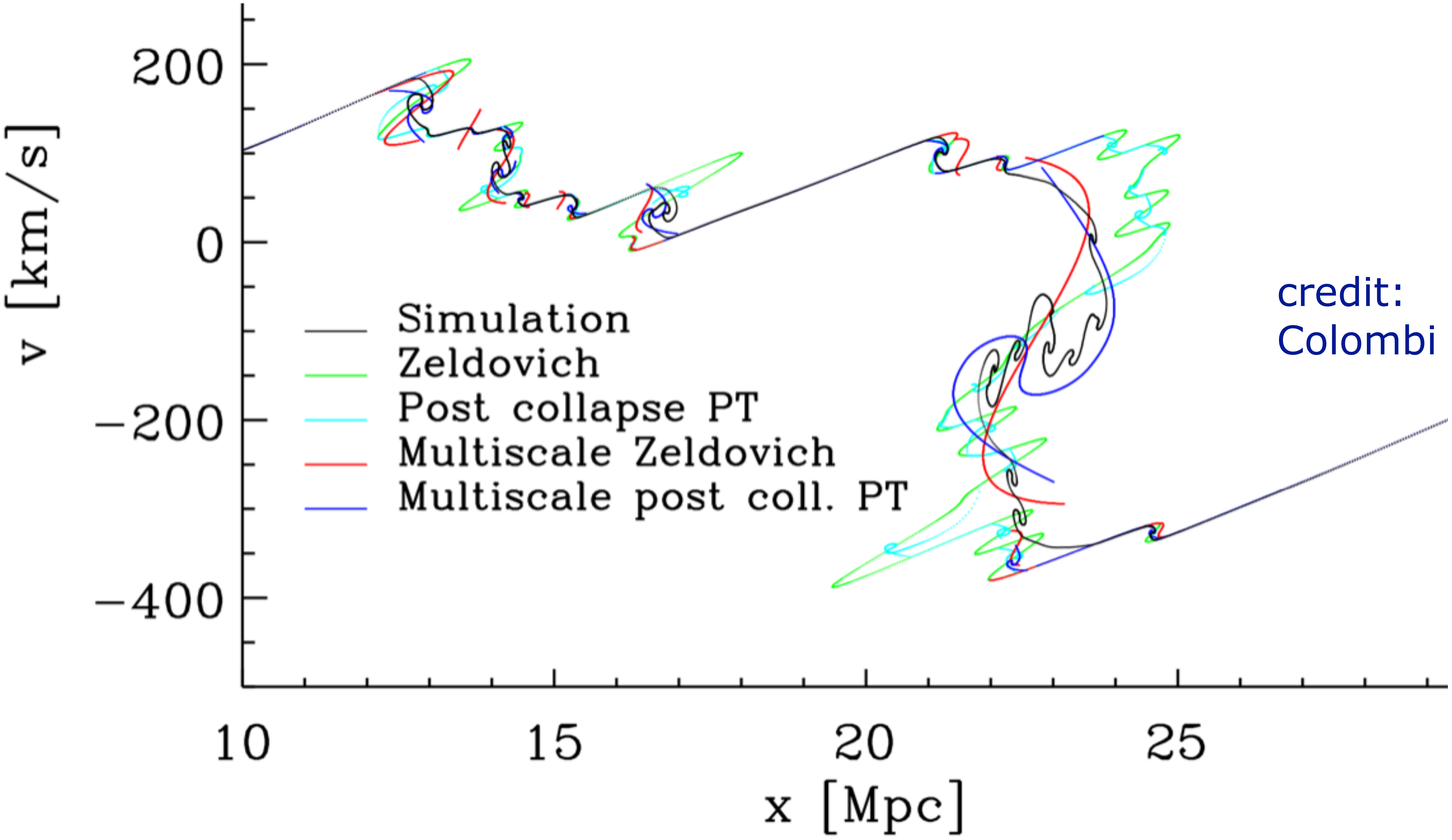
3+3 dim

**nonlinear**

similar in plasma physics (gravity → Coulomb)

simple ‘cold’ initial conditions: flat sheet

# Large-scale View of Phase Space




credit:
Colombi & Taruya

# Cold Dark Matter Approximations

**numerical**

**N particles**

effective CDM particles

*limited sampling*

**1 Complex Wave Function**

wave dark matter

**analytical**

**2 fields**

fluid density & velocity

*limited features*

# Key Idea

## Semiclassical Dynamics

correspondence: classical $\rightleftarrows$ quantum

$$f(\boldsymbol{x}, \boldsymbol{p}, t) \simeq f_\hbar[\psi(\boldsymbol{x}, t)](\boldsymbol{p})$$

3+3 dim

3 dim

numerics idea:
Widrow & Kaiser '93

$$\hbar \simeq \frac{\hbar_{\mathrm{phys}}}{m}$$ small scale

Schrödinger-Poisson equation

$$i\hbar\, \partial_t \psi(\boldsymbol{x}, t) = \hat{H}\psi(\boldsymbol{x}, t) \qquad \Delta V(\boldsymbol{x}, t) \propto |\psi(\boldsymbol{x}, t)|^2 - 1$$

fundamental for (ultra-)light scalar fields

mean field might not be full story: Kopp et al. '22, Eberhardt et al. '22

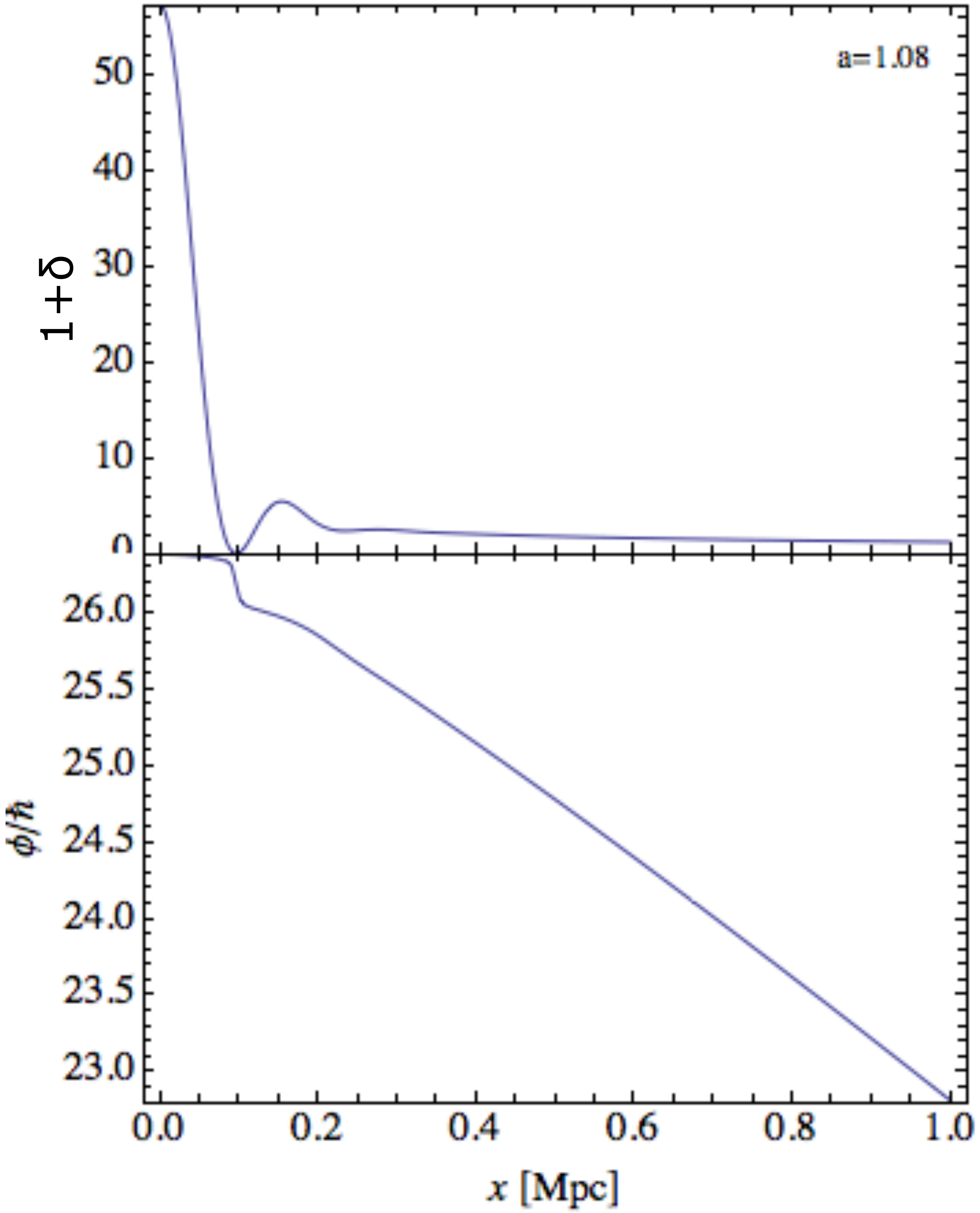
# 1D Wave Numerics

multi-stream translates to

density oscillations

$$\psi \propto \sqrt{1+\delta}\,\exp[i\phi/\hbar]$$

phase jumps



# 1D Wave Numerics

## Semiclassical Dynamics

classical $\rightleftarrows$ quantum

$$f(\boldsymbol{x}, \boldsymbol{p}, t) \simeq f_\hbar[\psi(\boldsymbol{x}, t)](\boldsymbol{p})$$

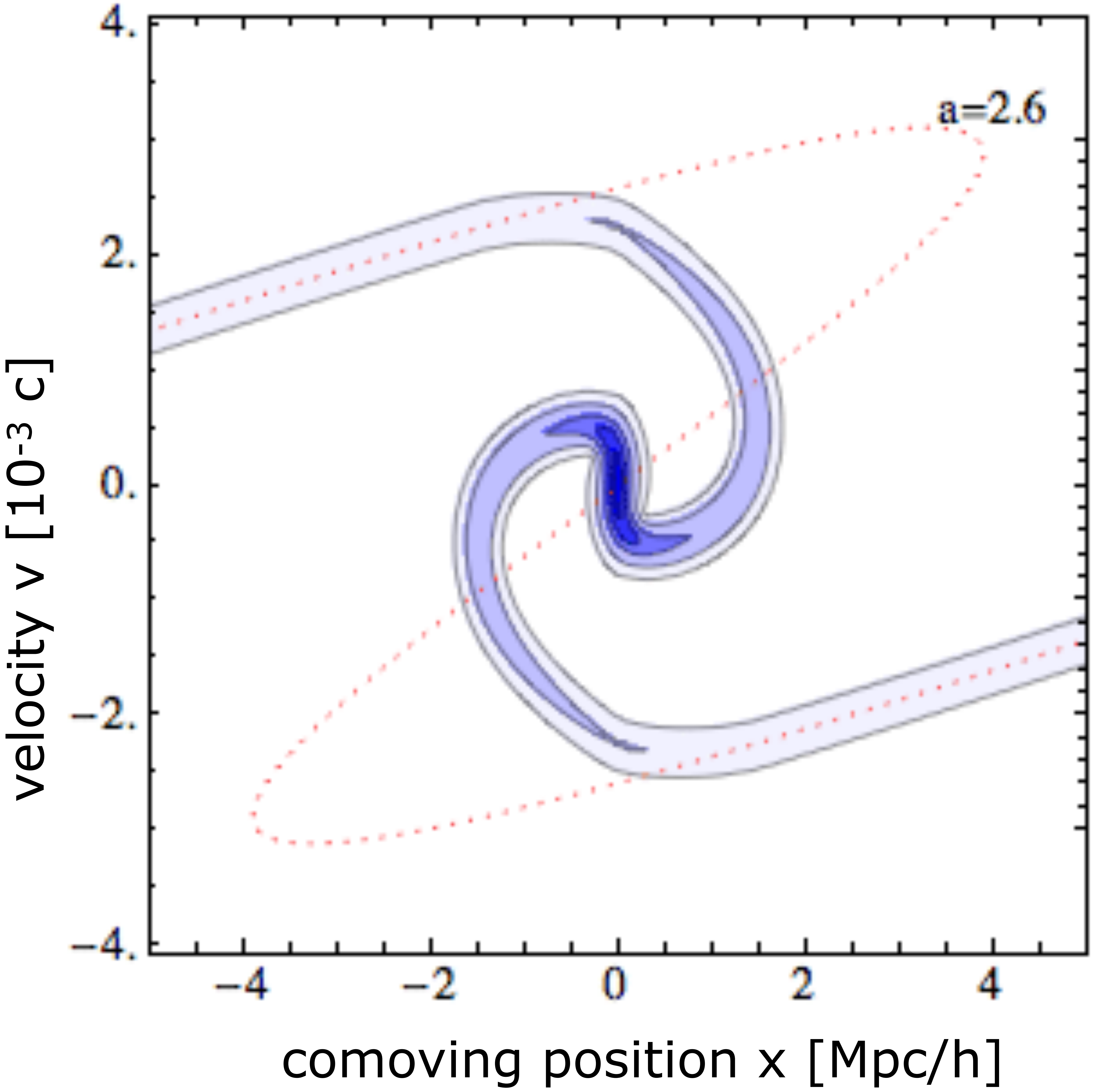
+ coarse-graining $\sigma_x \sigma_p \gtrsim \hbar/2$

multi-stream

→ bound structure

**CU**, Kopp & Haugg PRD '14

2D: Kopp++ PRD '17



# One Wavefunction to rule them all?

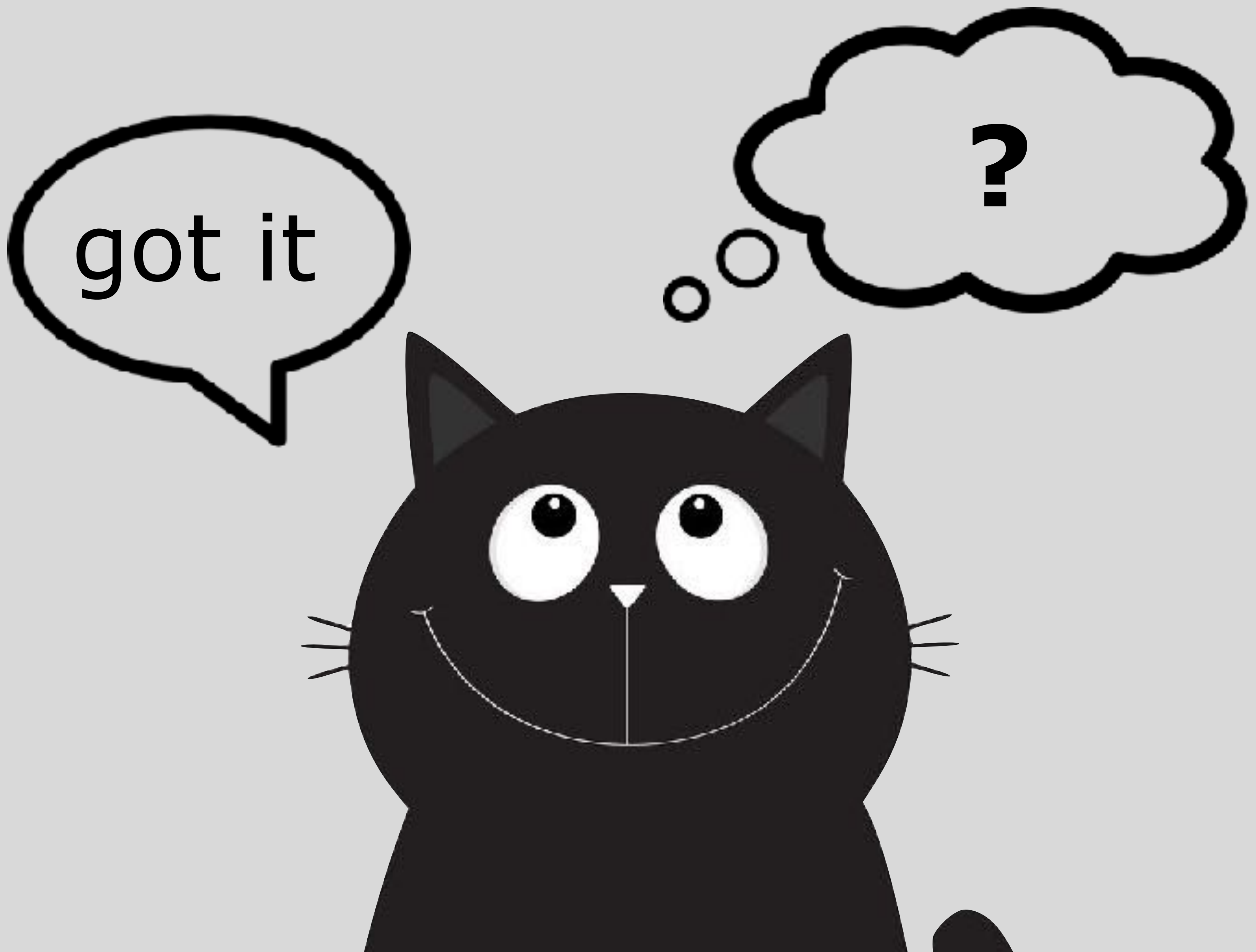
**numerical**

**N particles**

**analytical**

**2 fields**

**1 Wave Function**




Li, Hui & Bryan 18:
naive wave PT
no good

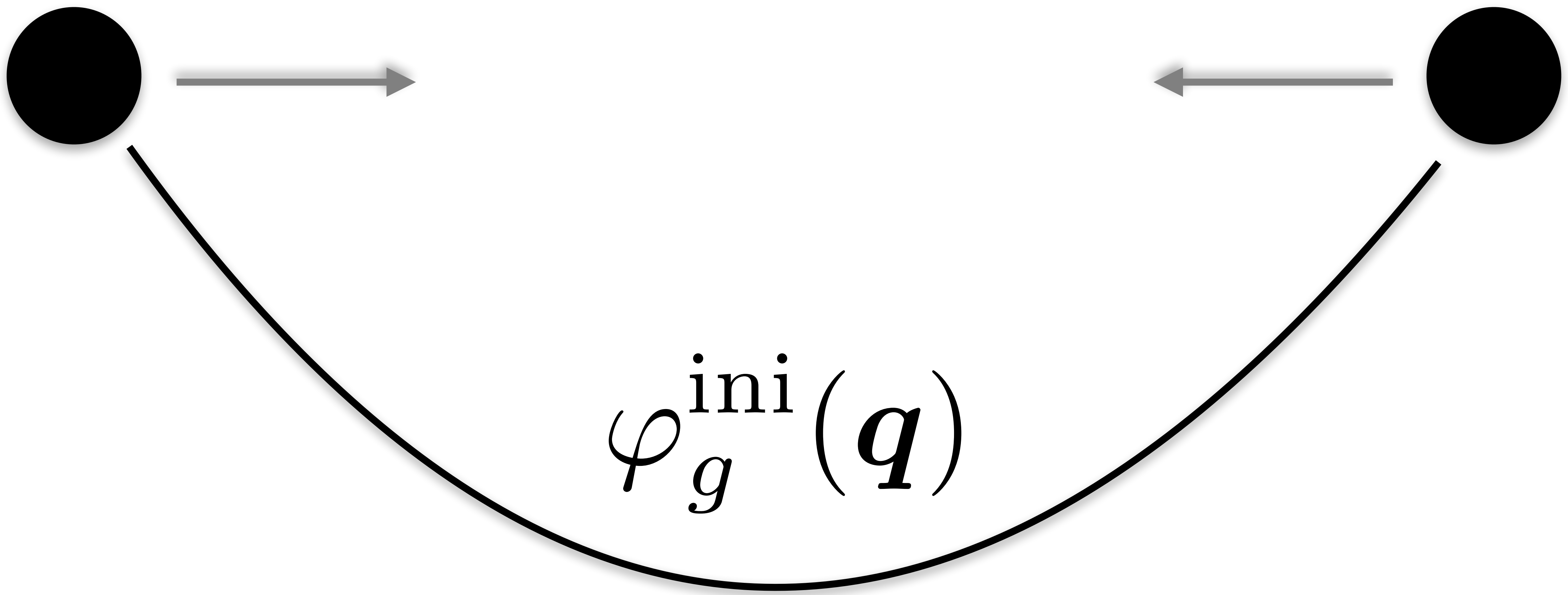
# Classical Dynamics

## Approximate: Shoot Particles

follow initial gravitational potential

$$\boldsymbol{v}(\boldsymbol{q}, a) = -\boldsymbol{\nabla}\varphi_g^{\text{ini}}(\boldsymbol{q})$$

$$\boldsymbol{x}(\boldsymbol{q}, a) = \boldsymbol{q} - a\boldsymbol{\nabla}\varphi_g^{\text{ini}}(\boldsymbol{q})$$



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$$\boldsymbol{x}(\boldsymbol{q}, a) = \boldsymbol{q} - a\boldsymbol{\nabla}\varphi_g^{\mathrm{ini}}(\boldsymbol{q})$$

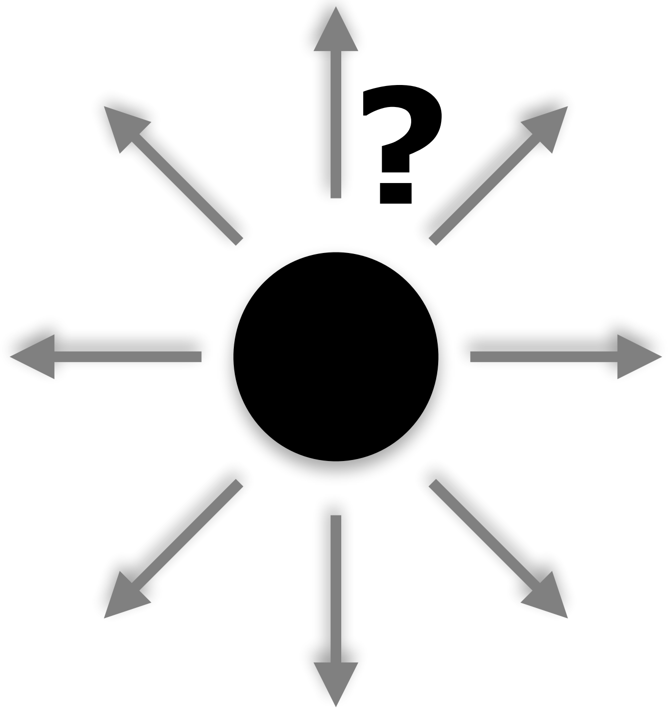
Zel'dovich 1D: exact before shell-crossing

**Coordinates & PT**

**x**: 'standard' Eulerian (SPT)

**q**: Lagrangian (LPT)

2D & 3D:
+ tidal effects



# Free Propagation

## classical action

$$S_0(\boldsymbol{x}, \boldsymbol{q}, a) = \frac{1}{2}(\boldsymbol{x} - \boldsymbol{q}) \cdot \frac{\boldsymbol{x} - \boldsymbol{q}}{a}$$

background expansion

*__CU__, Rampf, Gosenca & Hahn 18*

## Translate Free Propagation

**transition amplitude**

$$\psi_0(\boldsymbol{x}, a) = N \int d^3q \exp\left[\frac{i}{\hbar} S_0(\boldsymbol{x}, \boldsymbol{q}, a)\right] \psi_0^{\mathrm{ini}}(\boldsymbol{q})$$

**Schrödinger equation**

$$i\hbar\partial_a \psi_0 = -\frac{\hbar^2}{2}\nabla^2\psi_0$$

*Coles & Spencer 03*

***CU**, Rampf, Gosenca & Hahn 18*

≊ Zeldovich approximation turned Eulerian

# Classical Observables

## Eulerian

density $\rho(\boldsymbol{x}) = |\psi(\boldsymbol{x})|^2$

$$\psi = \sqrt{\rho}\,\exp[i\phi_v/\hbar]$$

velocity $$\boldsymbol{v}(\boldsymbol{x}) = \frac{i\hbar}{2}\frac{\psi\boldsymbol{\nabla}\bar{\psi} - \bar{\psi}\boldsymbol{\nabla}\psi}{|\psi|^2} = \nabla\phi_v$$

not necessarily potential

+ velocity dispersion, …

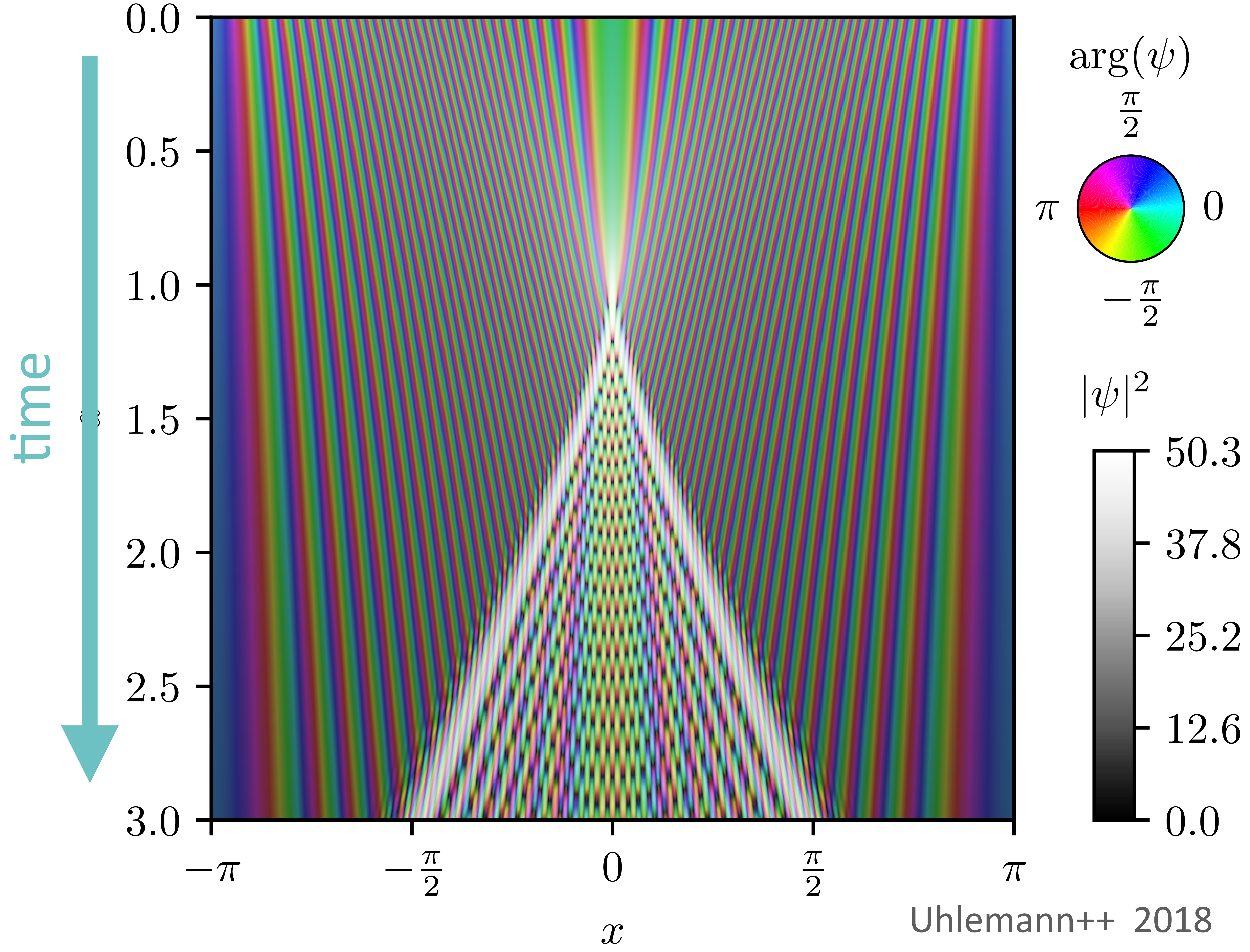
# Free Wave Evolution

**Amplitude**: brightness

**Phase**: colour

## Features

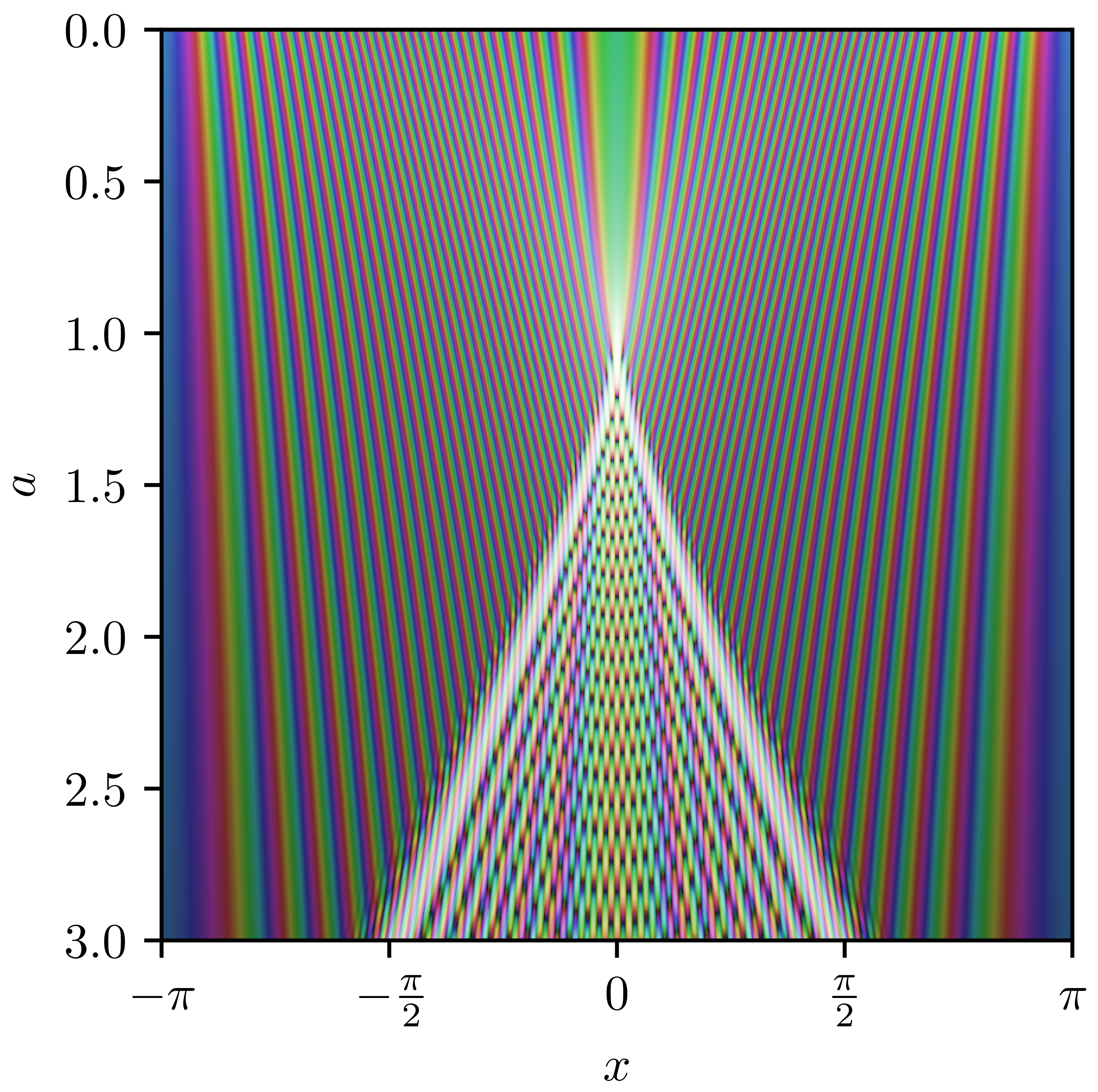
- Interference
- Regularised caustic



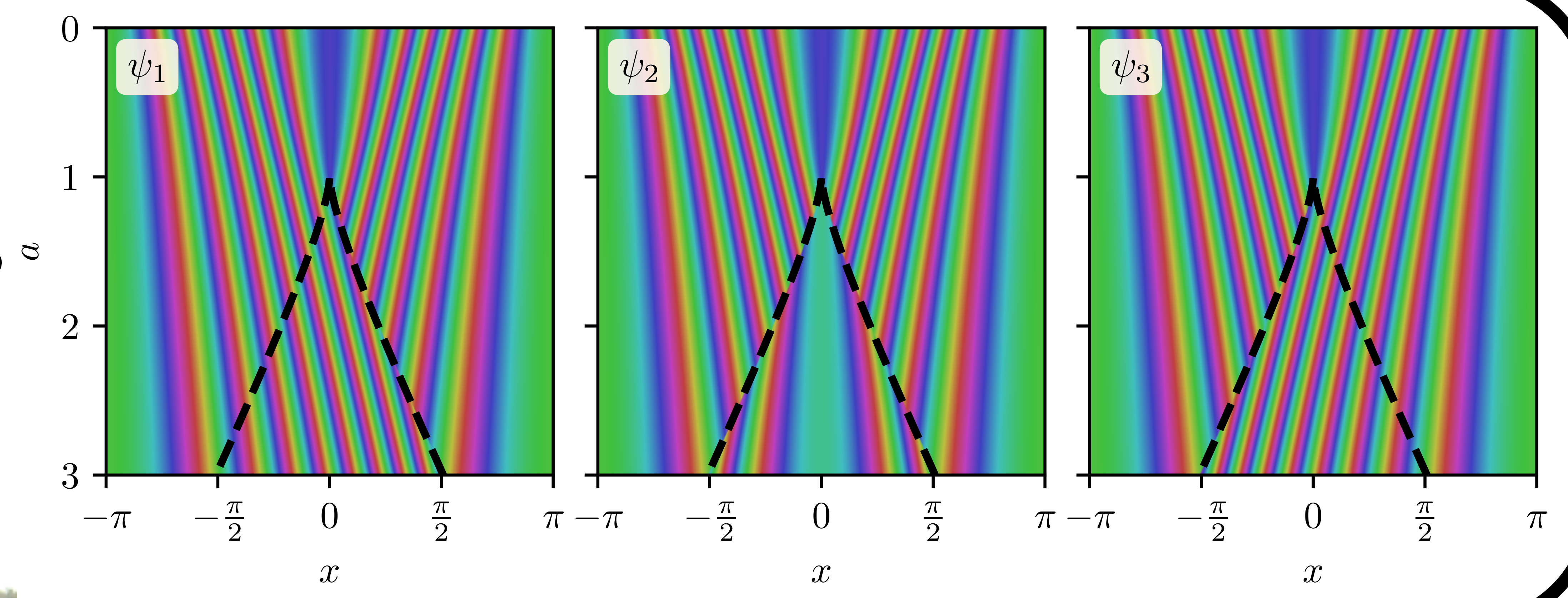

Uhlemann++ 2018

Gough & Uhlemann 2022

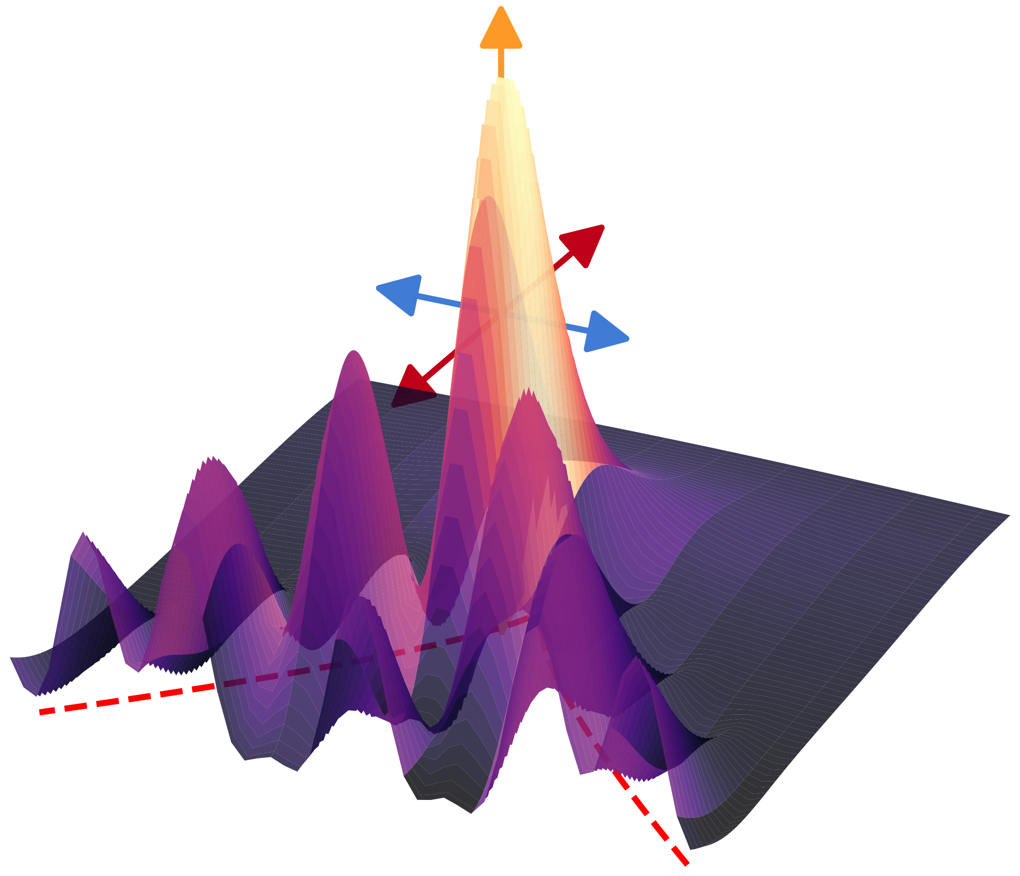
# Unweaving the Wavefunction




stream components



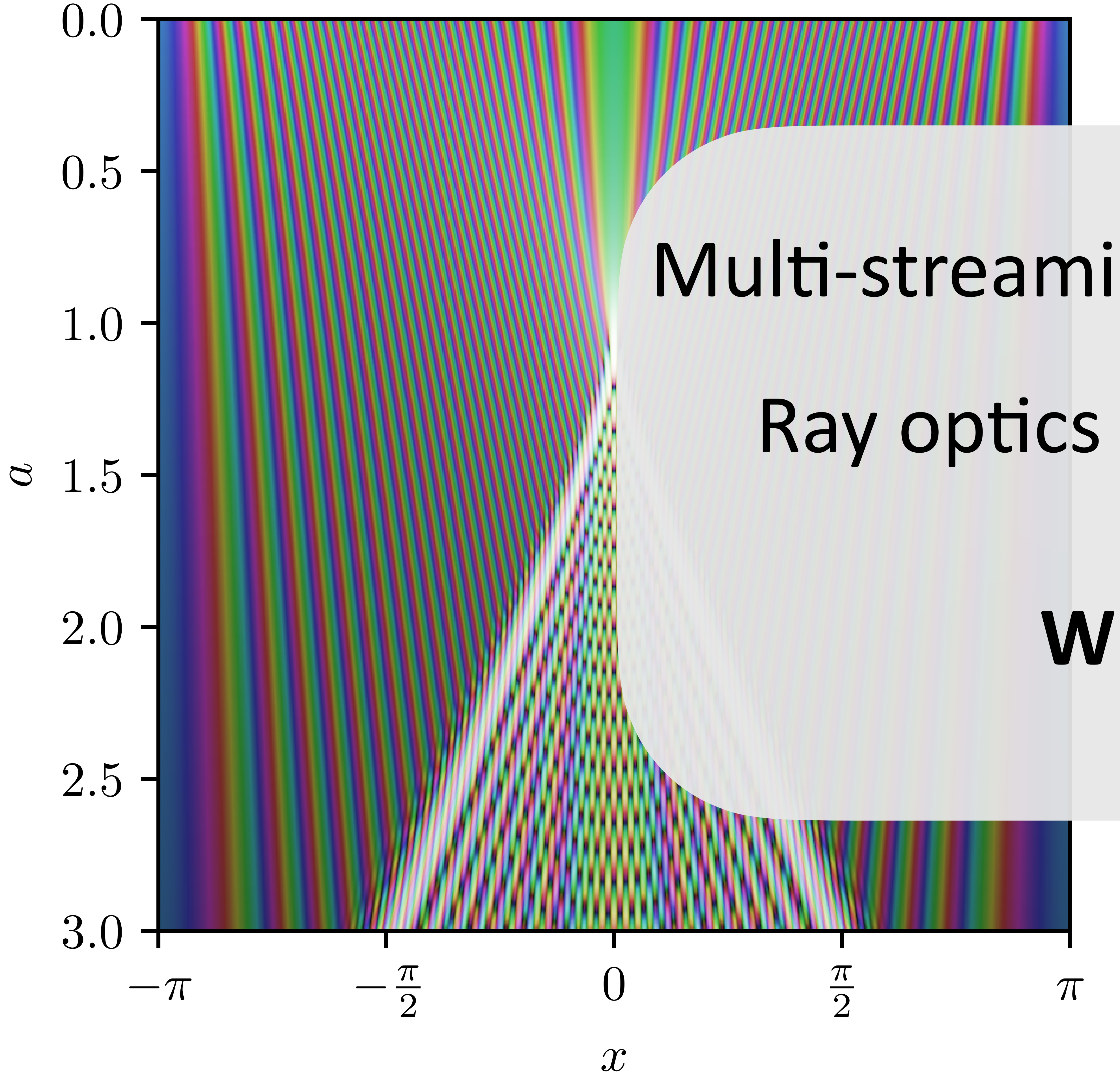

(regularised) caustic classification



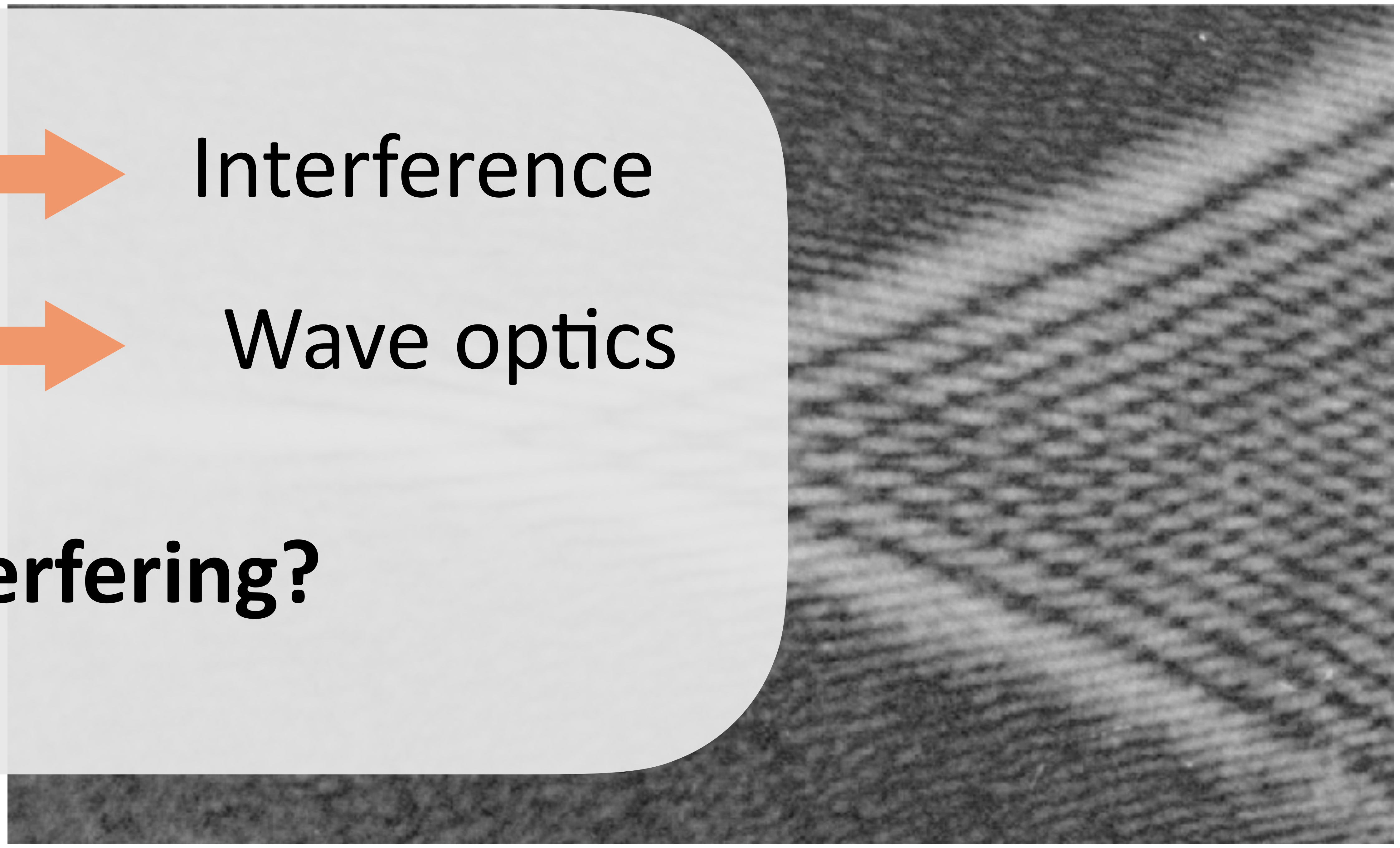
Gough & Uhlemann 2022

# Optics Analogy

Dark matter




Optics



Berry, Nye, Wright `79

Multi-streaming ↔ Interference

Ray optics ↔ Wave optics

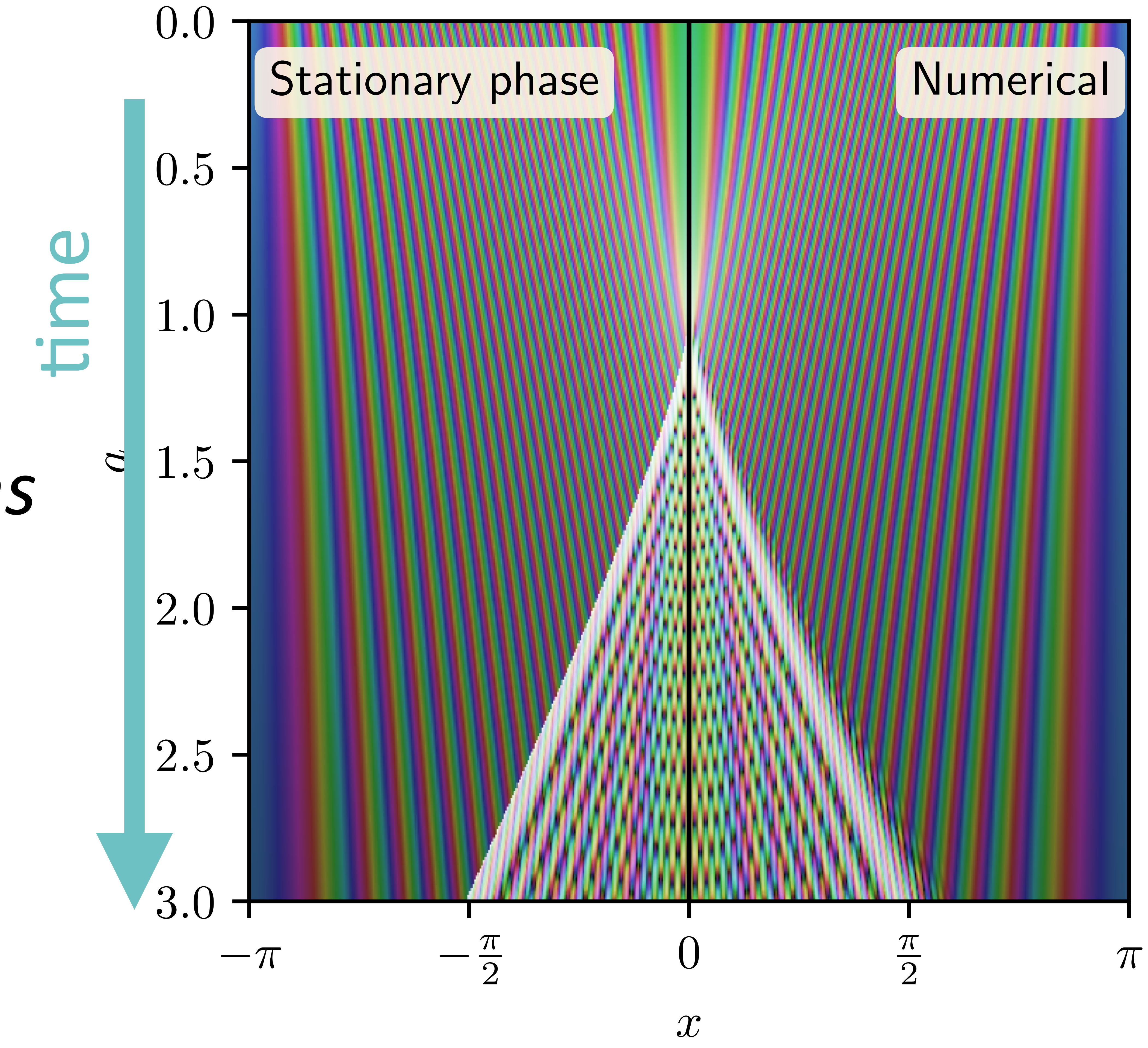
**What is interfering?**

# Unweaving the Wavefunction

Based on propagator

$$\psi(x,a) \sim \int \mathrm{d}q \, \underbrace{K_0(q;x,a)\psi^{(\mathrm{ini})}(q)}_{\exp\left[\frac{i}{\hbar}\zeta(q;x,a)\right]}$$

- $\zeta(q;x,a)$ contains *action* & *initial conditions*
- $K(q;x,a)$ transition amplitude
- $\hbar$ small $\rightarrow$ integrand oscillatory

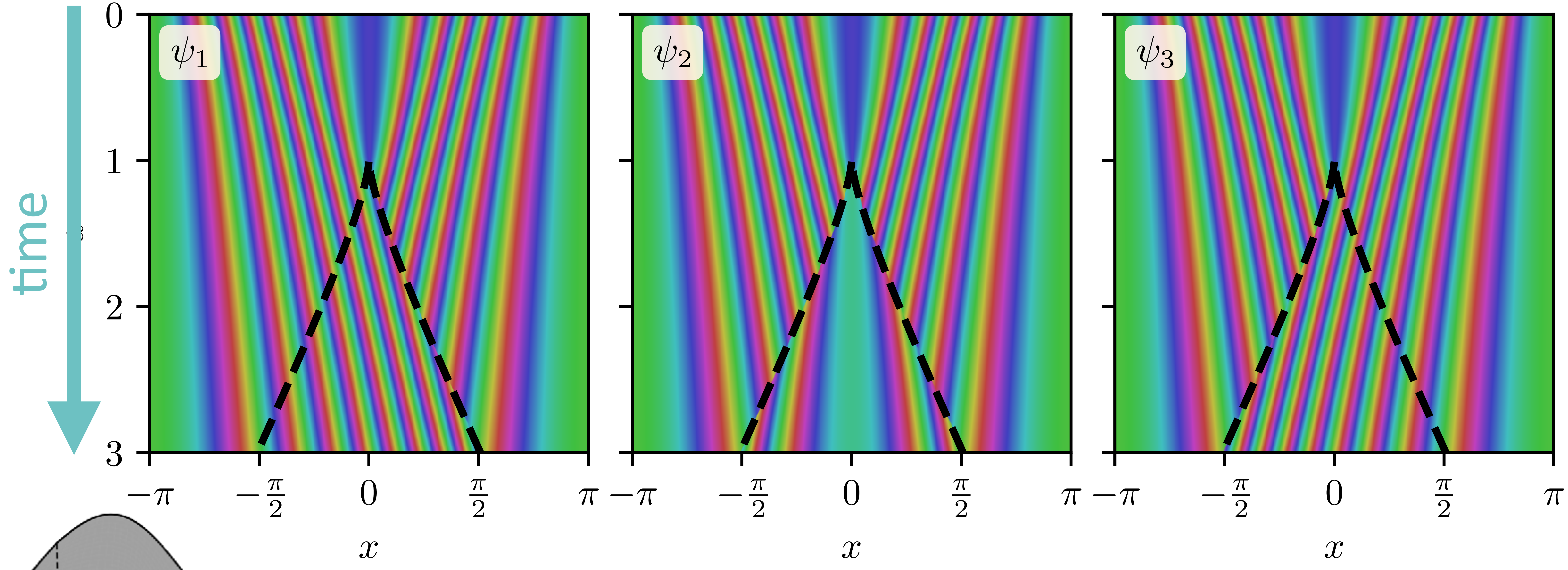


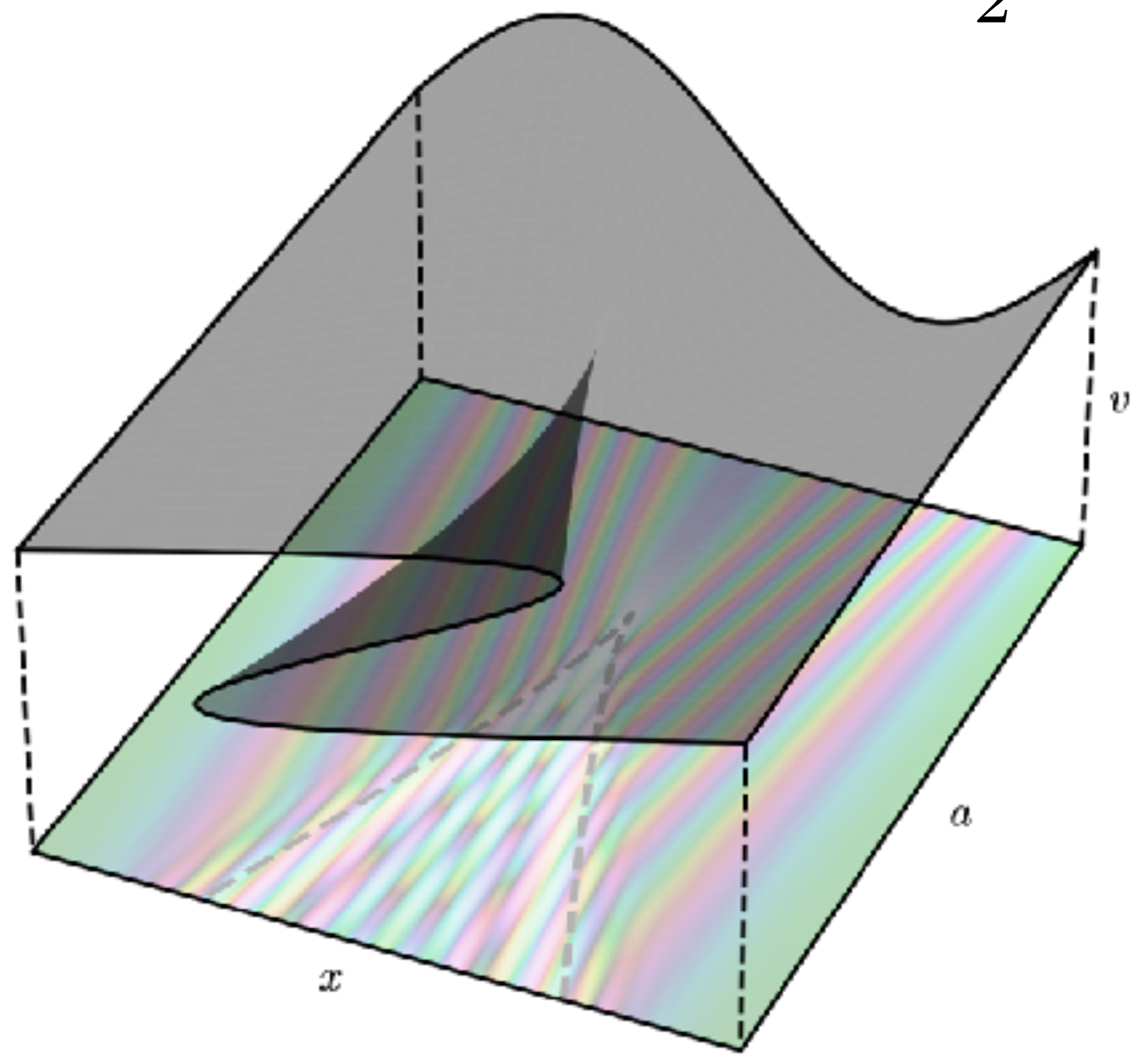

**Stationary Phase Approximation**

$q$ where $\zeta'(q) = 0$ dominate integral

Gough & Uhlemann 2022

# Stream Wavefunctions



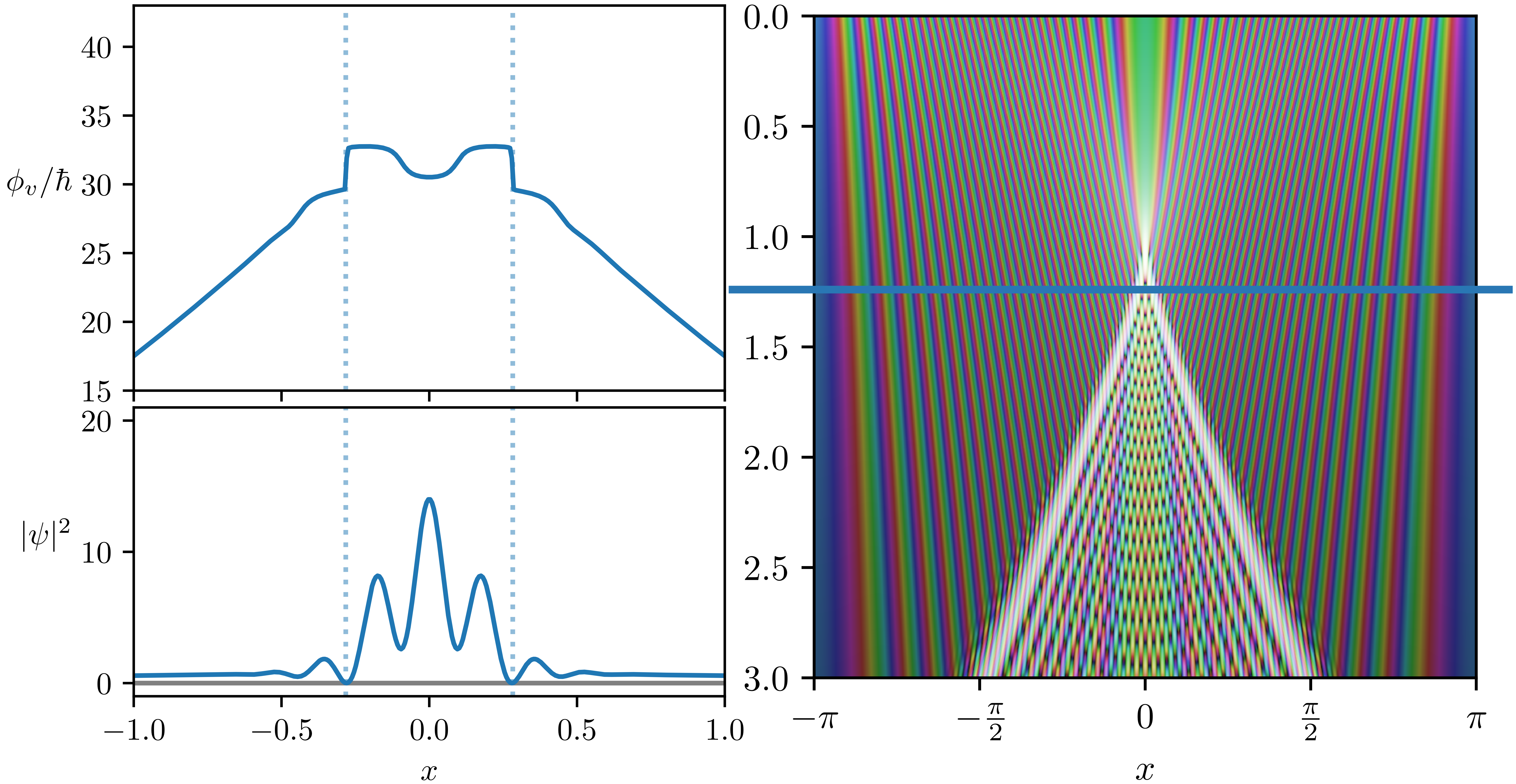




- classical trajectories interfere!
- recover classical multi-stream from interference

Gough & Uhlemann 2022

# Non-Potential Velocity

- Phase jumps at zero density
- $\psi$ captures beyond perfect fluid!




Get effect of stream averaging without explicit dissection of streams!

Gough & Uhlemann 2022

# Velocity Dispersion

- Velocity dispersion

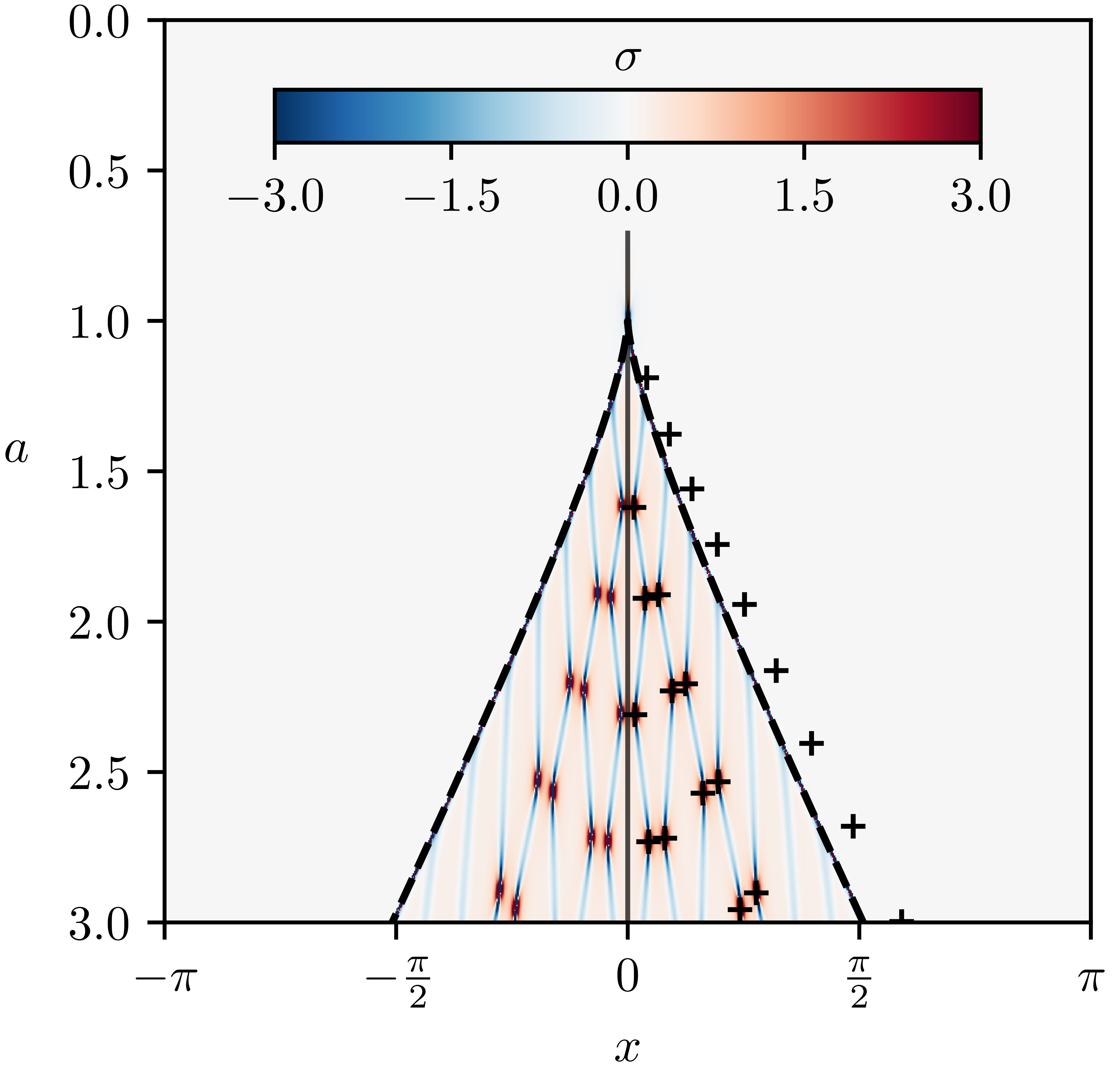
$$\sigma = -\frac{\hbar}{4}\nabla^2 \ln \rho$$

- sourced by density zeros & phase jumps
- beyond perfect fluid in oscillatory $\psi$

$$\psi \approx \psi_{\text{avg}} \times \psi_{\text{hidden}}$$

Fluid part

Oscillatory

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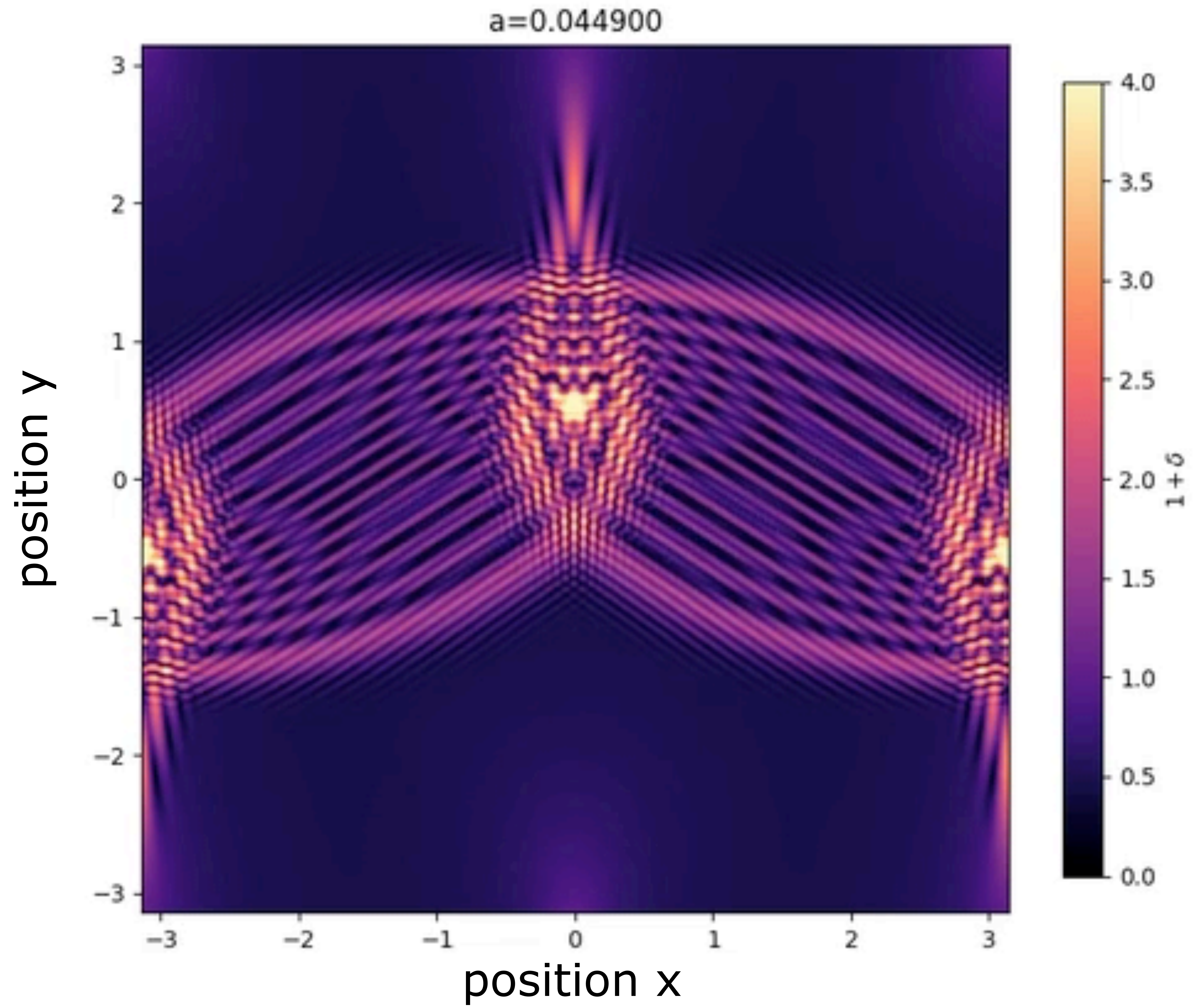
# 2D Phased Wave Example

## Density

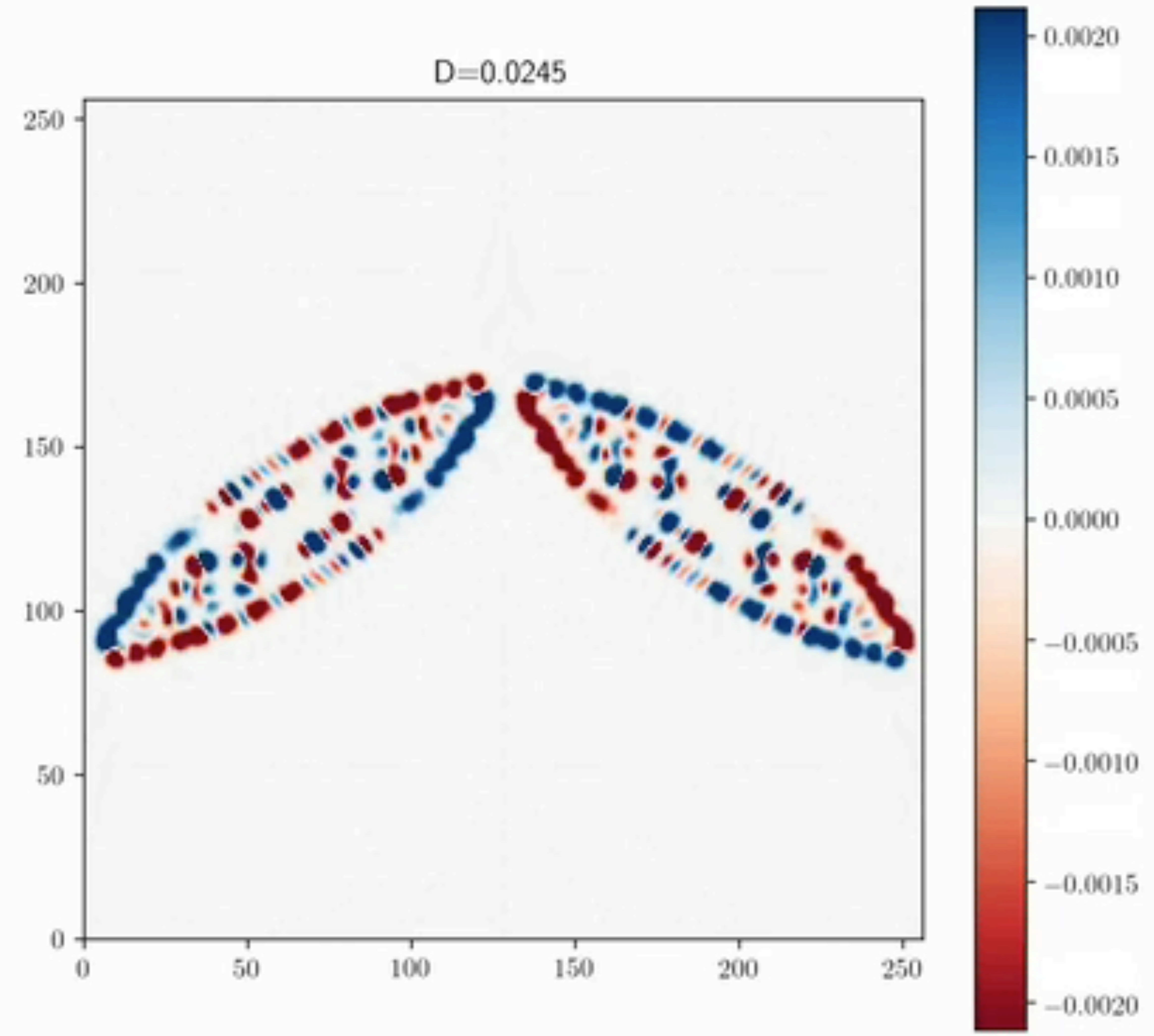
$$1 + \delta(\boldsymbol{x}, a) = |\psi|^2$$




***CU**, Rampf, Gosenca & Hahn 18*

# 2D Phased Wave Example

**Vorticity** from phase jumps $v = \nabla \phi_v$ but $\nabla \times v \neq 0$

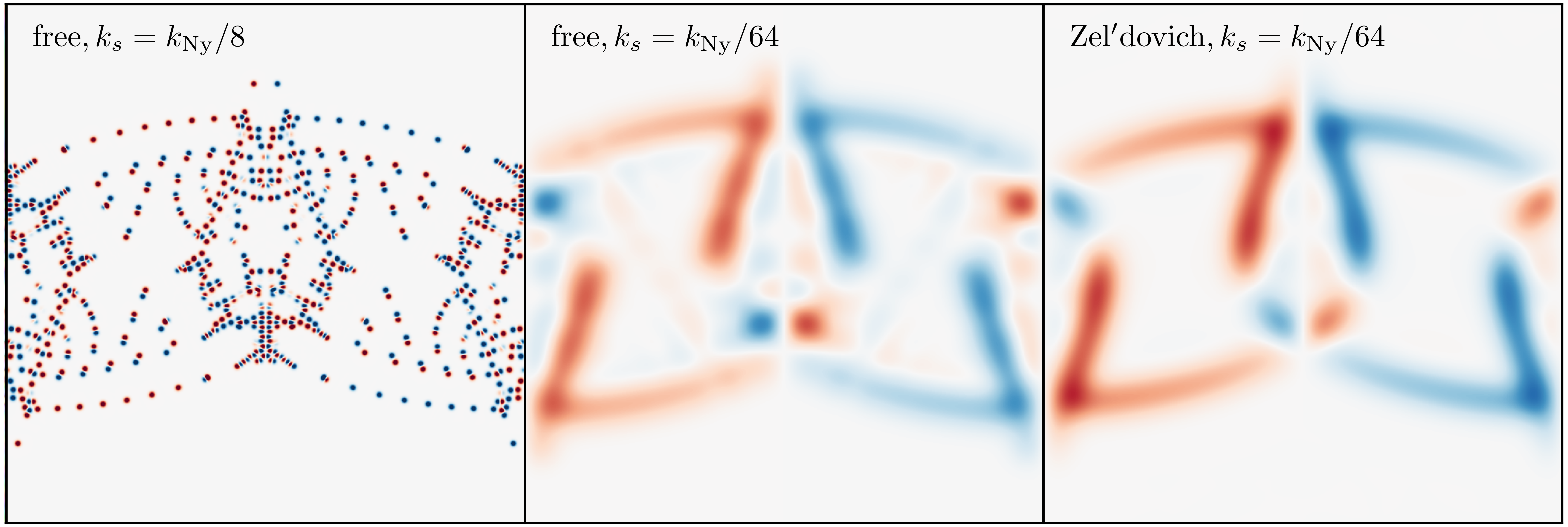



***CU**, Rampf, Gosenca & Hahn 18*

# 2D Phased Wave Example

## Vorticity

small scales large scales




quantised classical appearance

analog to Schrödinger-Poisson vortices
2D: Kopp++ '17, 3D: Hui++ '20

***CU**, Rampf, Gosenca & Hahn 18*

# Conclusion: The Sky from Ψ

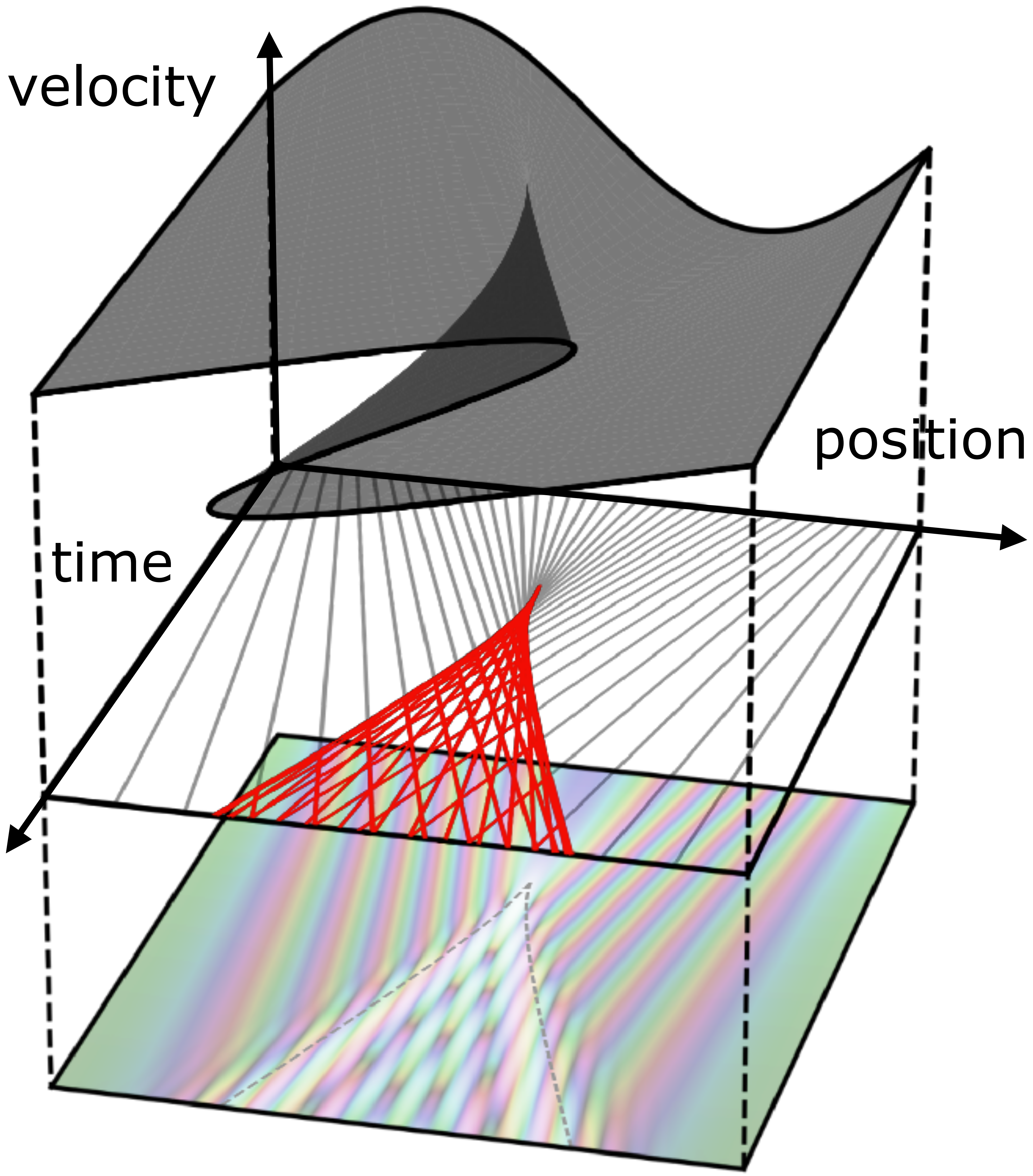
## A new layer of Large-Scale Structure

phase space
high dimensional

particle-based
resolution loss

perturbative fluid
limited physics ❌

**wave space**
full physics
half dimensions




hydro simulation initial conditions:
*Rampf, **CU**, Hahn '20*
*Hahn, Rampf, **CU** '20*

Lyman-$\alpha$ forest: *Porqueres ++ '20*

*map-level* predictions

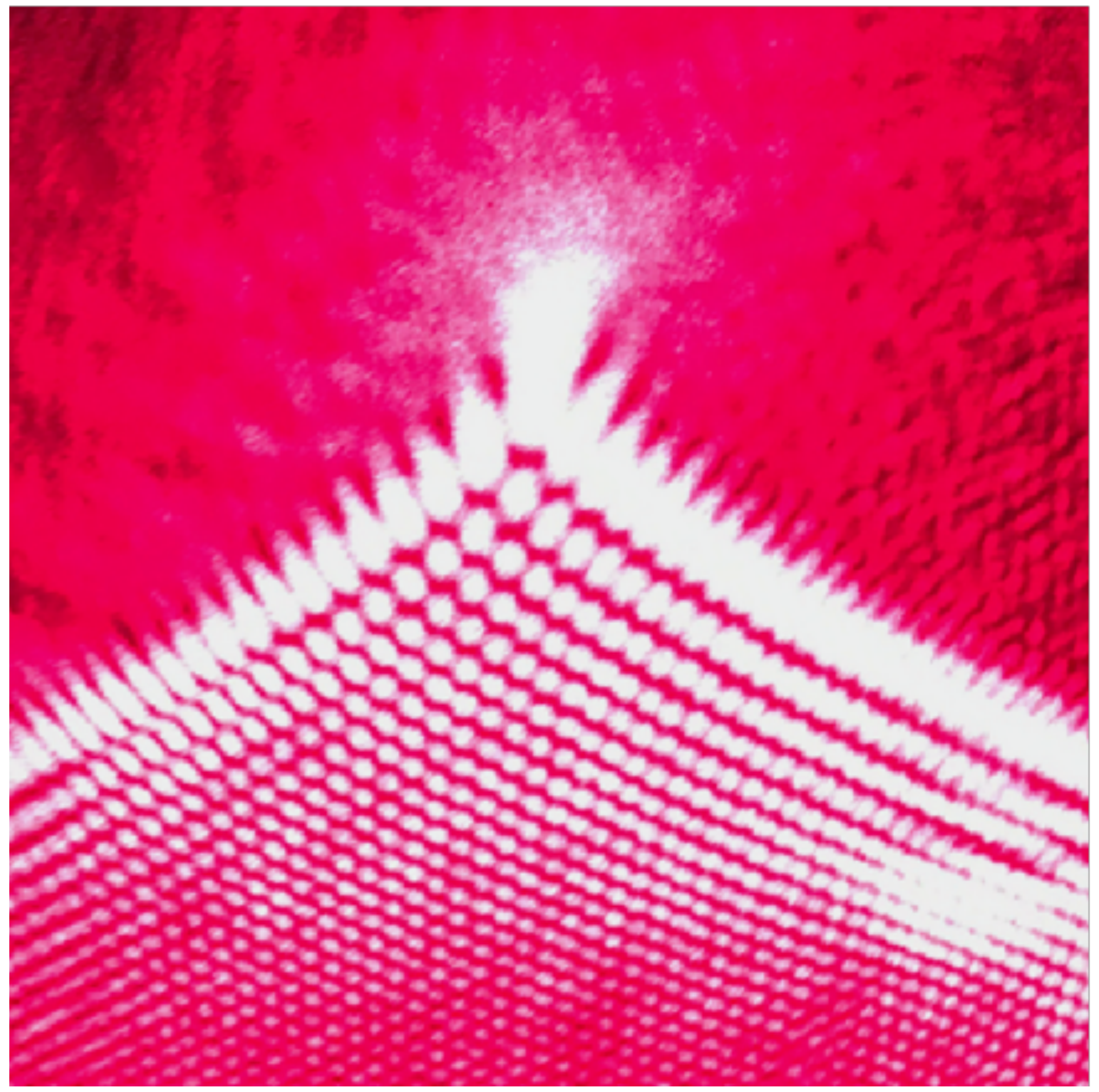
cold dark matter

↑ small $\hbar/m$

wave dark matter

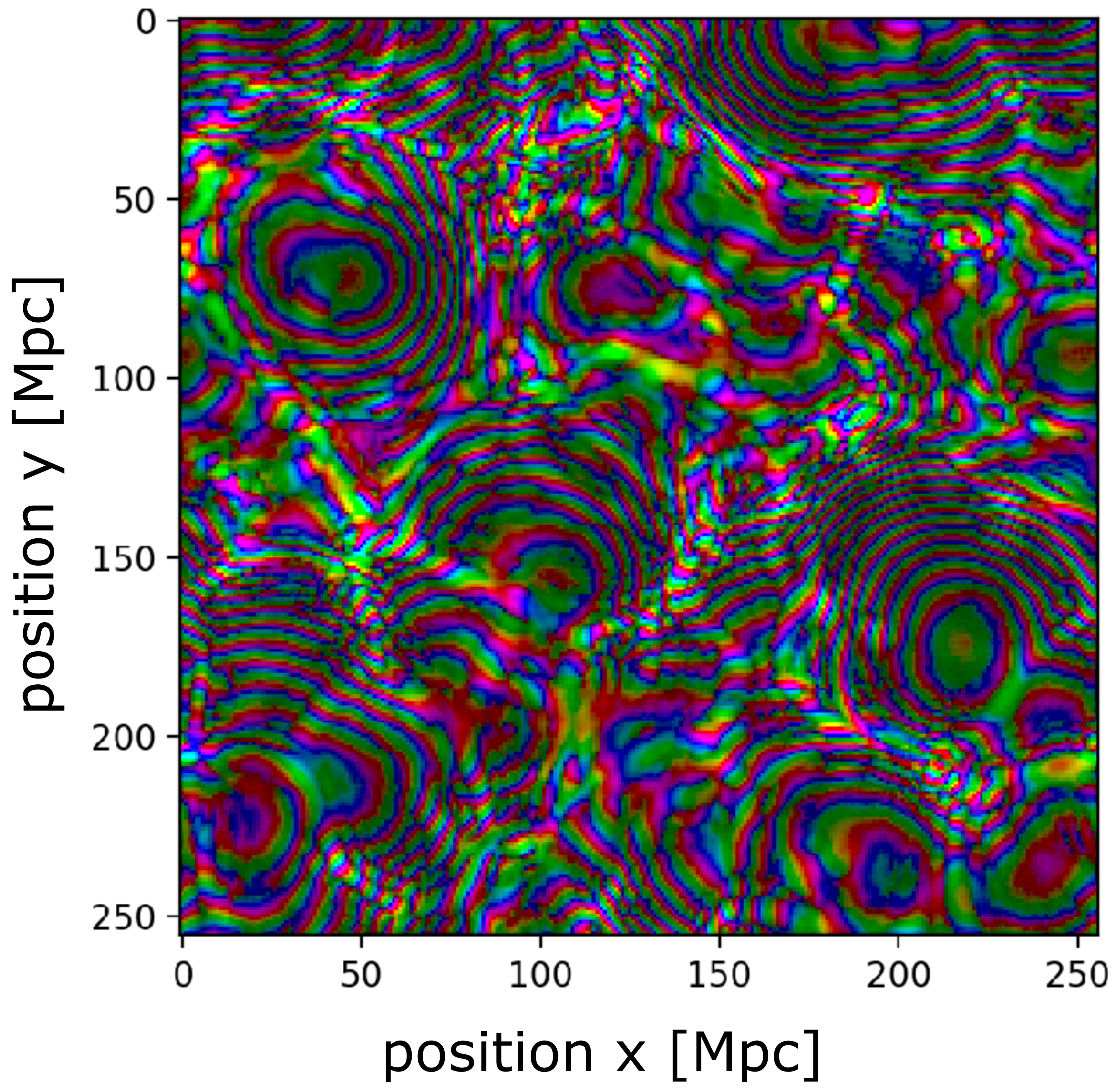
# Complexity in a Wavefunction

## Diffraction Optics



Cusp caustic from
laser droplet diffraction
Wikimedia: Dan Piponi

## Cosmic Web




by Oliver Hahn

## Vorticity Tracking

Schrödinger's Smoke

Albert Chern, Caltech; Felix Knöppel, TU Berlin; Ulrich Pinkall, TU Berlin; Peter Schröder, Caltech; Steffen Weißmann, Google Inc.

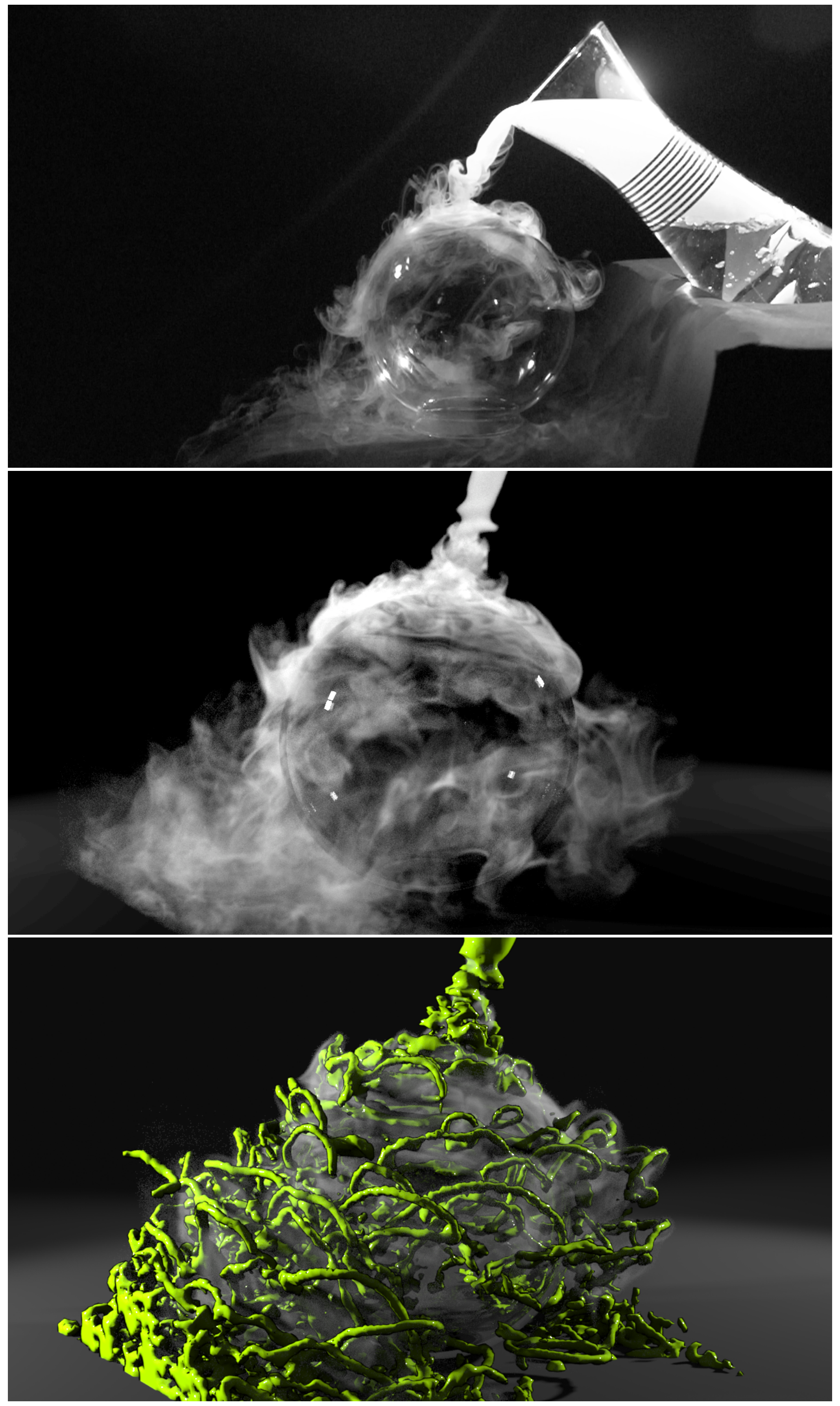


Figure 1: *Comparing experiment (dry ice vapor, top) with ISF simulation (middle), followed by a visualization of the underlying wave function $\psi$. Vorticity is concentrated within the green region.*

# Key Idea

## Semiclassical Dynamics

correspondence: classical $\rightleftarrows$ quantum

$$f(\boldsymbol{x},\boldsymbol{p},t) \simeq f_\hbar[\psi(\boldsymbol{x},t)](\boldsymbol{p})$$

3+3 dim    3 dim

$$\partial_t f_W = \left[\frac{\boldsymbol{p}^2}{2a^2m} + mV\right] \frac{2}{\hbar}\sin\left(\frac{\hbar}{2}(\overleftarrow{\boldsymbol{\nabla}}_x\overrightarrow{\boldsymbol{\nabla}}_p - \overleftarrow{\boldsymbol{\nabla}}_p\overrightarrow{\boldsymbol{\nabla}}_x)\right) f_W$$

$$\simeq \left(\overleftarrow{\boldsymbol{\nabla}}_x\overrightarrow{\boldsymbol{\nabla}}_p - \overleftarrow{\boldsymbol{\nabla}}_p\overrightarrow{\boldsymbol{\nabla}}_x\right)$$

**CU**, Kopp, Haugg PRD '14

## Semiclassical Dynamics

correspondence: classical $\rightleftarrows$ quantum

$$f(\boldsymbol{x}, \boldsymbol{p}, t) \simeq f_\hbar[\psi(\boldsymbol{x}, t)](\boldsymbol{p})$$

3+3 dim

3 dim

add coarse-graining $\sigma_x \sigma_p \gtrsim \hbar/2$

$$\bar{f}_W(\boldsymbol{x}, \boldsymbol{p}) = \int \frac{d^3\tilde{x} d^3\tilde{p}}{(\pi\sigma_x\sigma_p)^3} \exp\left[-\frac{(\boldsymbol{x}-\tilde{\boldsymbol{x}})^2}{2\sigma_x^2} - \frac{(\boldsymbol{p}-\tilde{\boldsymbol{p}})^2}{2\sigma_p^2}\right] f_W(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}})$$

# Key Problem

## Cumulant Hierarchy

1-particle distribution

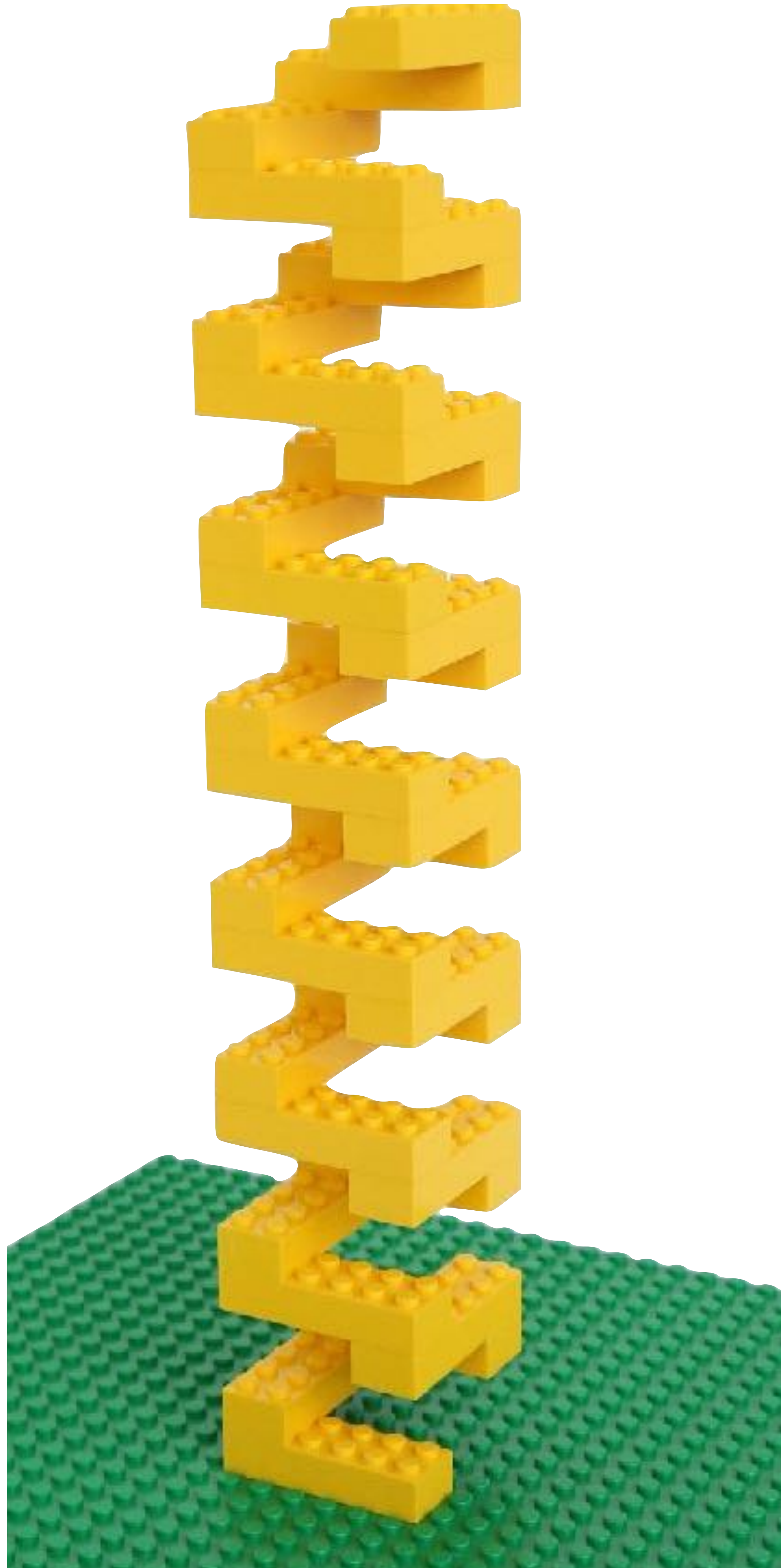
$$C^{(n)}_{i_1 \ldots i_n}(\boldsymbol{x}) \ni \left\{ \int d^3p \, p_{i_1} \cdots p_{i_m} f(\boldsymbol{x}, \boldsymbol{p}) \right\}_{m \leq n}$$

n=0 n=1 n=2

density, velocity, velocity dispersion, …

$$\partial_t C^{(n)} \simeq \nabla \cdot C^{(n+1)} + \sum_{|S|=0}^{n} C^{(n+1-|S|)} \cdot \nabla C^{(|S|)}$$

similar to BBGKY hierarchy of n-particle distributions

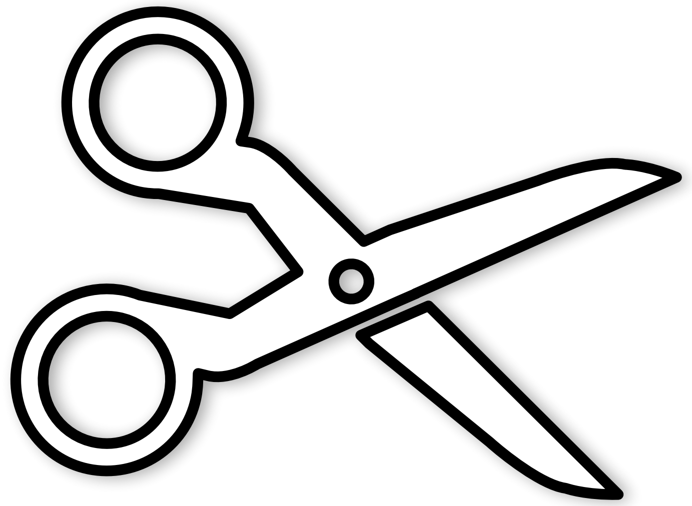


# Key Problem

## Cumulant Hierarchy

infinite & coupled

$$\partial_t C^{(n)} \simeq \nabla \cdot C^{(n+1)} + \sum_{|S|=0}^{n} C^{(n+1-|S|)} \cdot \nabla C^{(|S|)}$$



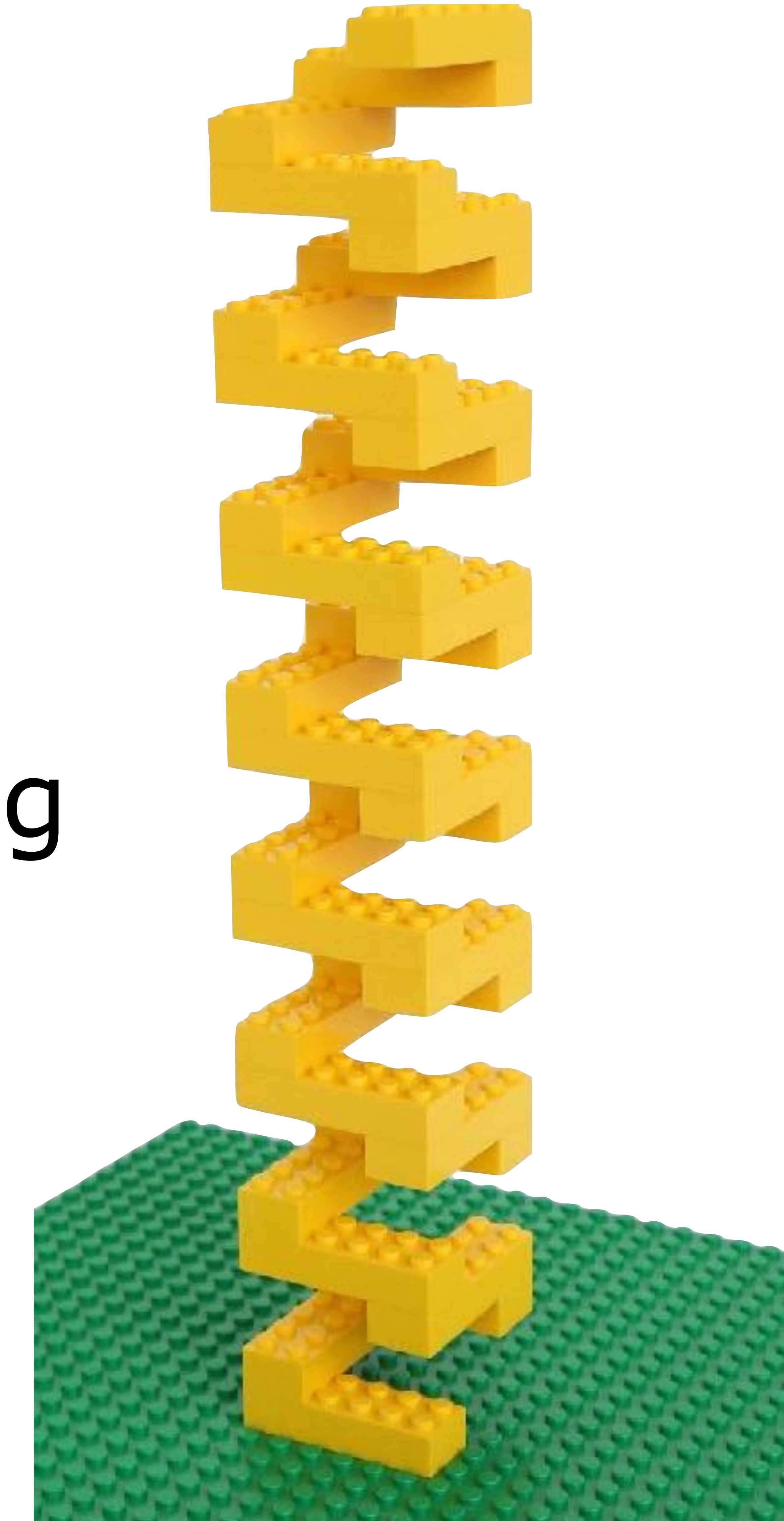
perfect fluid

$$C^{(n\geq 2)} \cancel{=} 0$$

shell-crossing

Pueblas & Scoccimarro `08

analogy: Gaussian is only PDF with a finite set of cumulants



# Key Idea

## Cumulant Hierarchy

infinite & coupled

$$\partial_t C^{(n)} \simeq \nabla \cdot C^{(n+1)} + \sum_{|S|=0}^{n} C^{(n+1-|S|)} \cdot \nabla C^{(|S|)}$$

approximate closure

$$C^{(n\geq 2)} = F[C^{(0)}, C^{(1)}]$$

**CU** JCAP '18

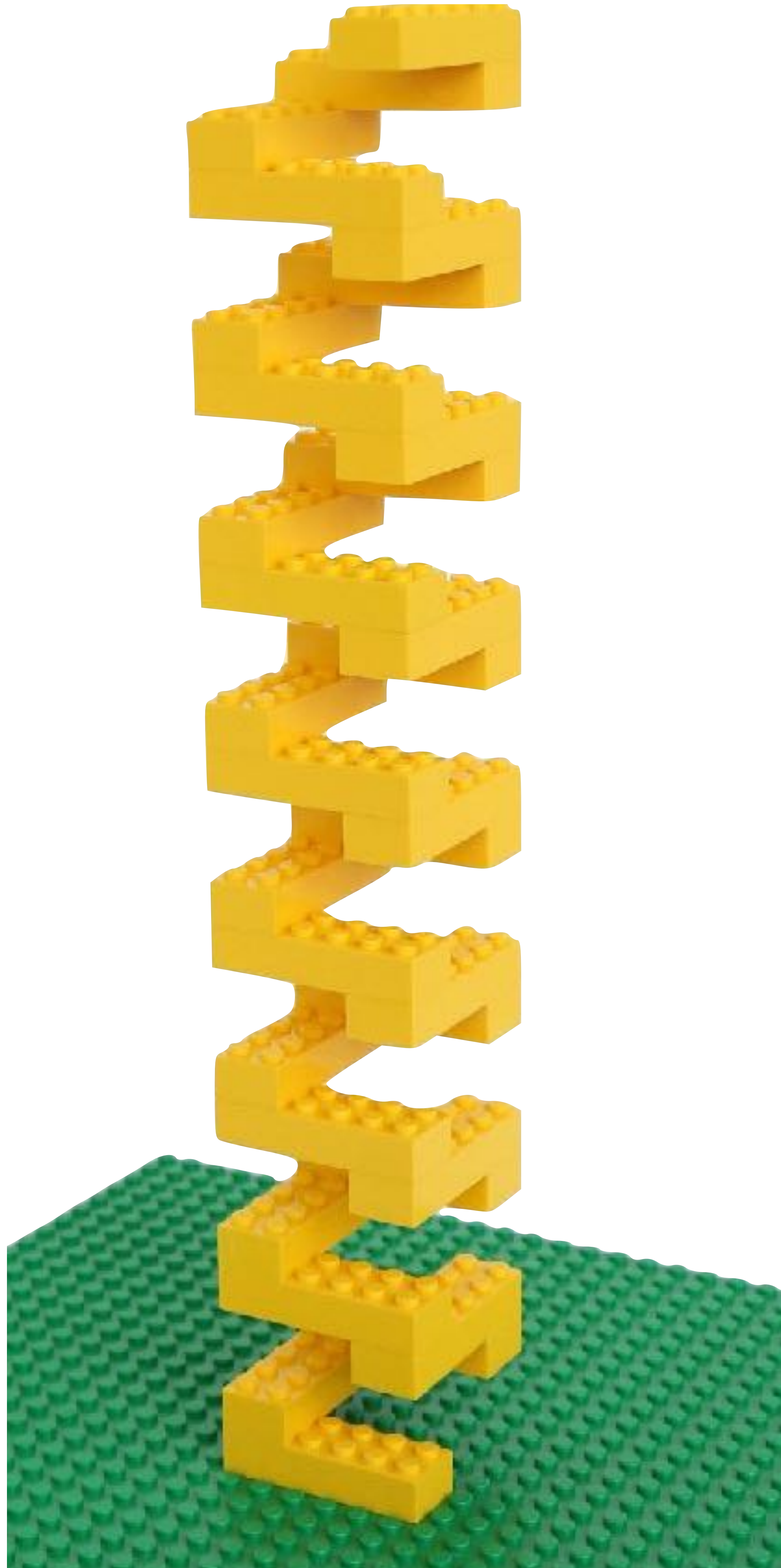
linear functional F

deformation quantisation?

$$C^{(n+2)} = -\frac{\hbar^2}{4}\boldsymbol{\nabla}\boldsymbol{\nabla}C^{(n)}$$

small parameter

analogy: lognormal PDF higher cumulants given by lower ones



# Semiclassical Dynamics

## Interactive Propagation

PT or numerics

$$i\hbar\partial_a\psi = -\frac{\hbar^2}{2}\nabla^2\psi + V_{\text{eff}}(\boldsymbol{x},a)\psi$$

**propagator**

$$\begin{tikzcd}
\psi(\boldsymbol{x},a) & & \psi^{\text{ini}}(\boldsymbol{q}) \arrow[ll, "V_{\text{eff}}(\boldsymbol{x})"'] \arrow[ll, bend left, shift left=3] \\
\end{tikzcd}$$

$$\psi(\boldsymbol{x},a) \xrightleftharpoons[\;]{V_{\text{eff}}(\boldsymbol{x},a)} \psi^{\text{ini}}(\boldsymbol{q})$$

**phase space**

$$\bar{f}_{\text{W}}(\boldsymbol{x},\boldsymbol{p},a) \Longrightarrow$$

**classical observables**

# Classical Observables

## Phase-Space Distribution

coarse-grained Wigner $\bar{f}_W[\psi, \hbar \to 0]$

$$f_{\mathrm{W}}(\boldsymbol{x}, \boldsymbol{p}) = \int \frac{\mathrm{d}^3 x'}{(2\pi)^3} \exp\left[\frac{-\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}'}{a^{3/2}}\right] \psi(\boldsymbol{x} + \tfrac{\hbar}{2}\boldsymbol{x}')\, \bar{\psi}(\boldsymbol{x} - \tfrac{\hbar}{2}\boldsymbol{x}')$$

phase-space info in wave function

# Classical Observables

## Lagrangian Fluid

compare $\bar{f}_W[\psi, \hbar \to 0]$ to

$$f_{\mathrm{fl}}(\boldsymbol{x}, \boldsymbol{p}) = \int \mathrm{d}^3 q \, \delta_{\mathrm{D}}^{(3)} \left[\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{\xi}(\boldsymbol{q})\right] \delta_{\mathrm{D}}^{(3)} \left[\frac{\boldsymbol{p}}{a^{3/2}} - \boldsymbol{v}^{\mathrm{L}}(\boldsymbol{q})\right]$$

displacement velocity

→ usual Lagrangian PT $\boldsymbol{v}^L(\boldsymbol{q}) = \dot{\boldsymbol{\xi}}(\boldsymbol{q})$

# Classical Observables

## Lagrangian Fluid

velocity beyond $v^L(\boldsymbol{q}) = \dot{\boldsymbol{\xi}}(\boldsymbol{q})$

$$\boldsymbol{v}(\boldsymbol{q}) = -\boldsymbol{\nabla}\varphi_g^{(\mathrm{ini})} - a\boldsymbol{\nabla}V_{\mathrm{eff}}^{(2)}$$

$$\boxed{+\frac{a^2}{2}\boldsymbol{\nabla}\nabla V_{\mathrm{eff}}^{(2)} \cdot \nabla\varphi_g^{(\mathrm{ini})}}$$

**vorticity conserver**

## Vorticity Conservation

Eulerian: $\nabla_x \times \boldsymbol{v} = 0$

before shell-crossing

# Multi-Stream Regime

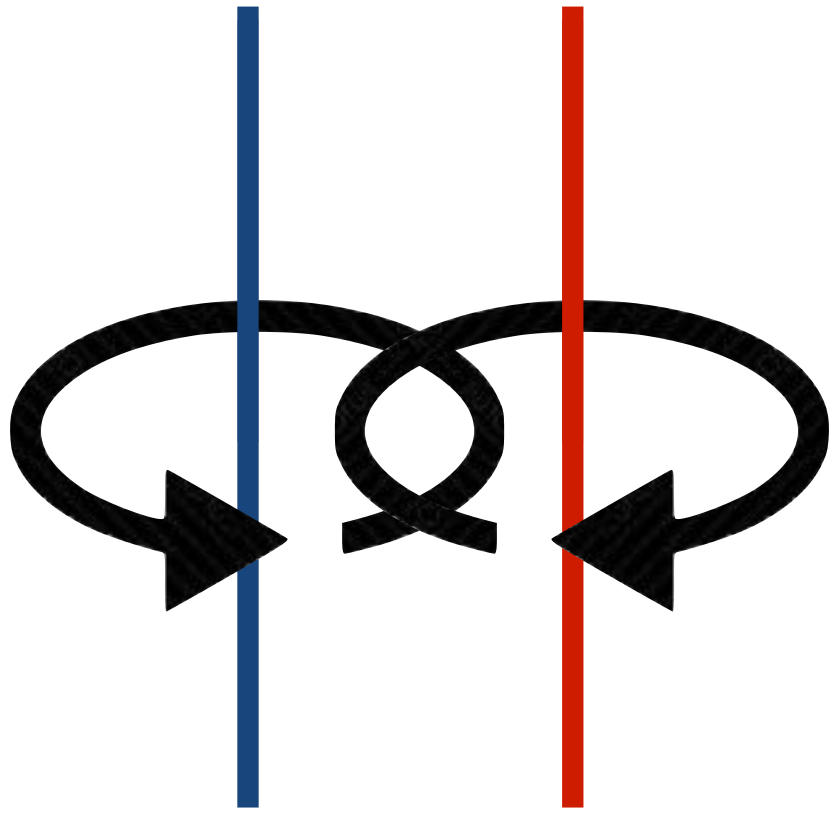
## Vorticity

phase jumps → vorticity

$$\psi = \sqrt{\rho}\exp[i\phi_v/\hbar] \qquad \boldsymbol{v} = \frac{i\hbar}{2}\frac{\psi\boldsymbol{\nabla}\bar{\psi} - \bar{\psi}\boldsymbol{\nabla}\psi}{|\psi|^2} = \nabla\phi_v$$

**topological defects: rotons**

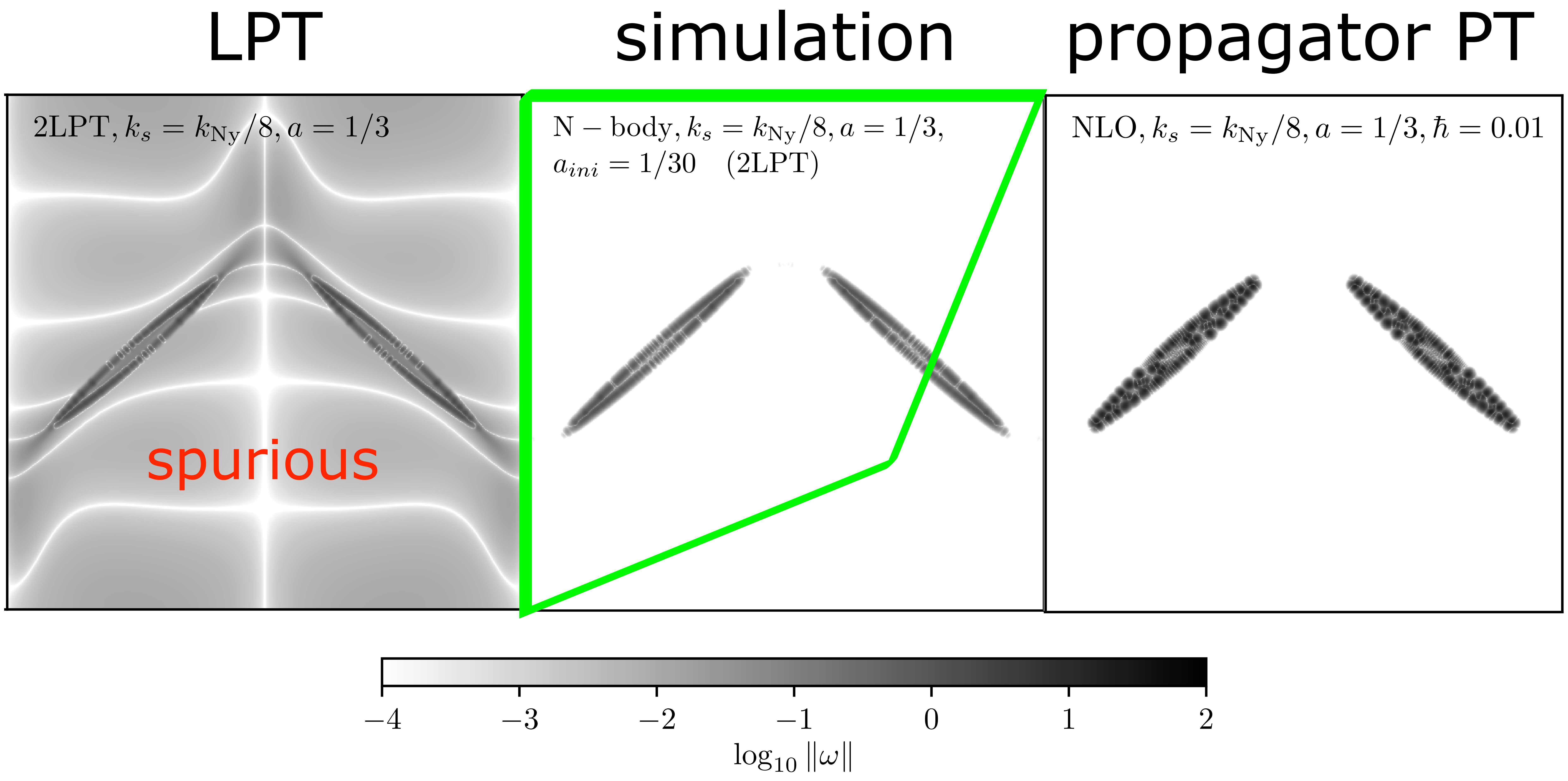
$$\frac{1}{2\pi\hbar}\oint_{C(a)} \boldsymbol{\nabla}\phi_{\mathrm{v}} \cdot \mathrm{d}\boldsymbol{x} = n_+ - n_- = 0$$



preserve Kelvin-Helmholtz invariant

# Phased Wave Example

## Vorticity

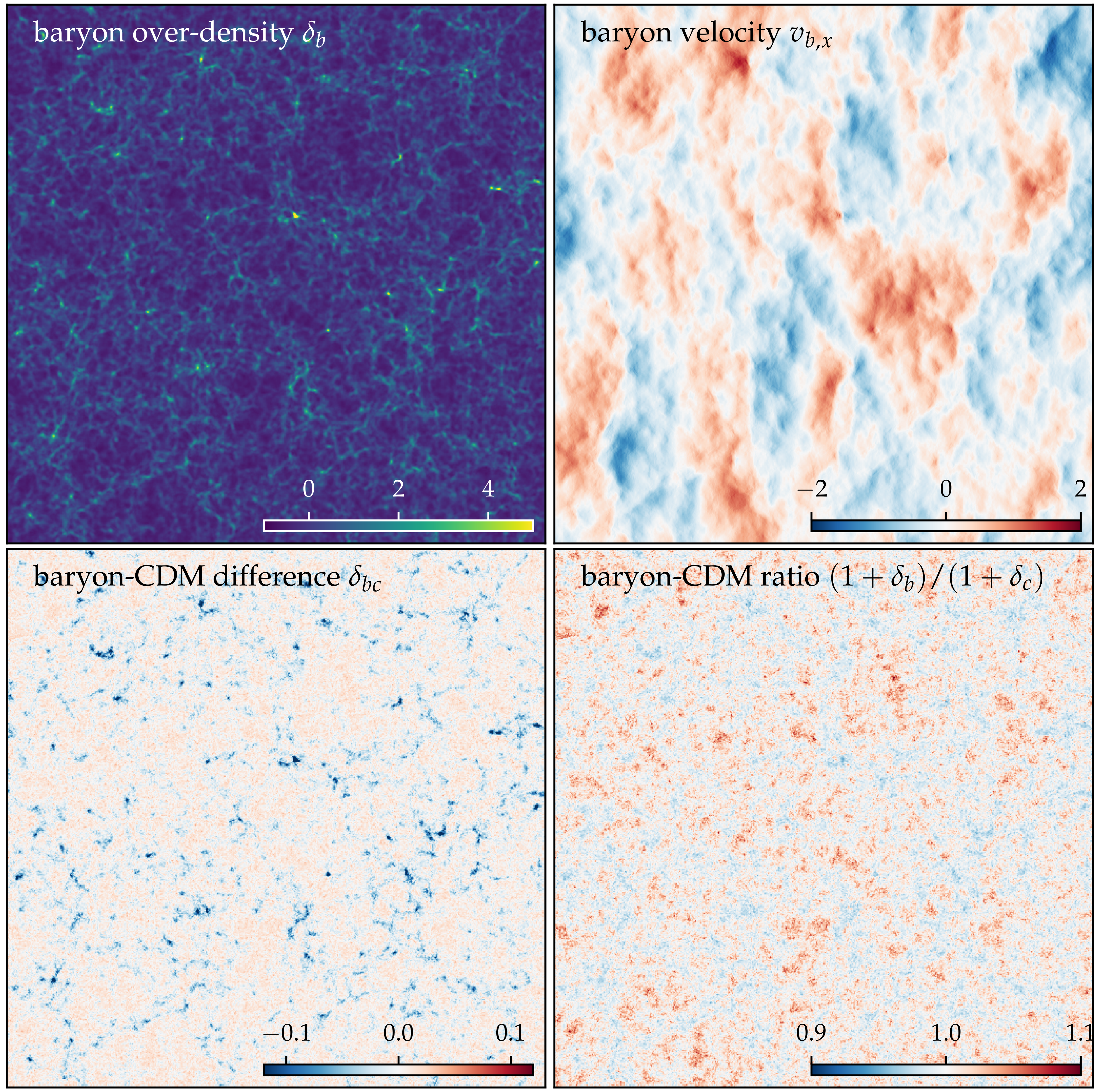


# Dark Matter + Baryons: ICs

PPT initial conditions for Eulerian codes

evolve one Ψ for each component (valid for non-decaying modes)

Rampf, **CU**, Hahn ’20
Hahn, Rampf, **CU** `20

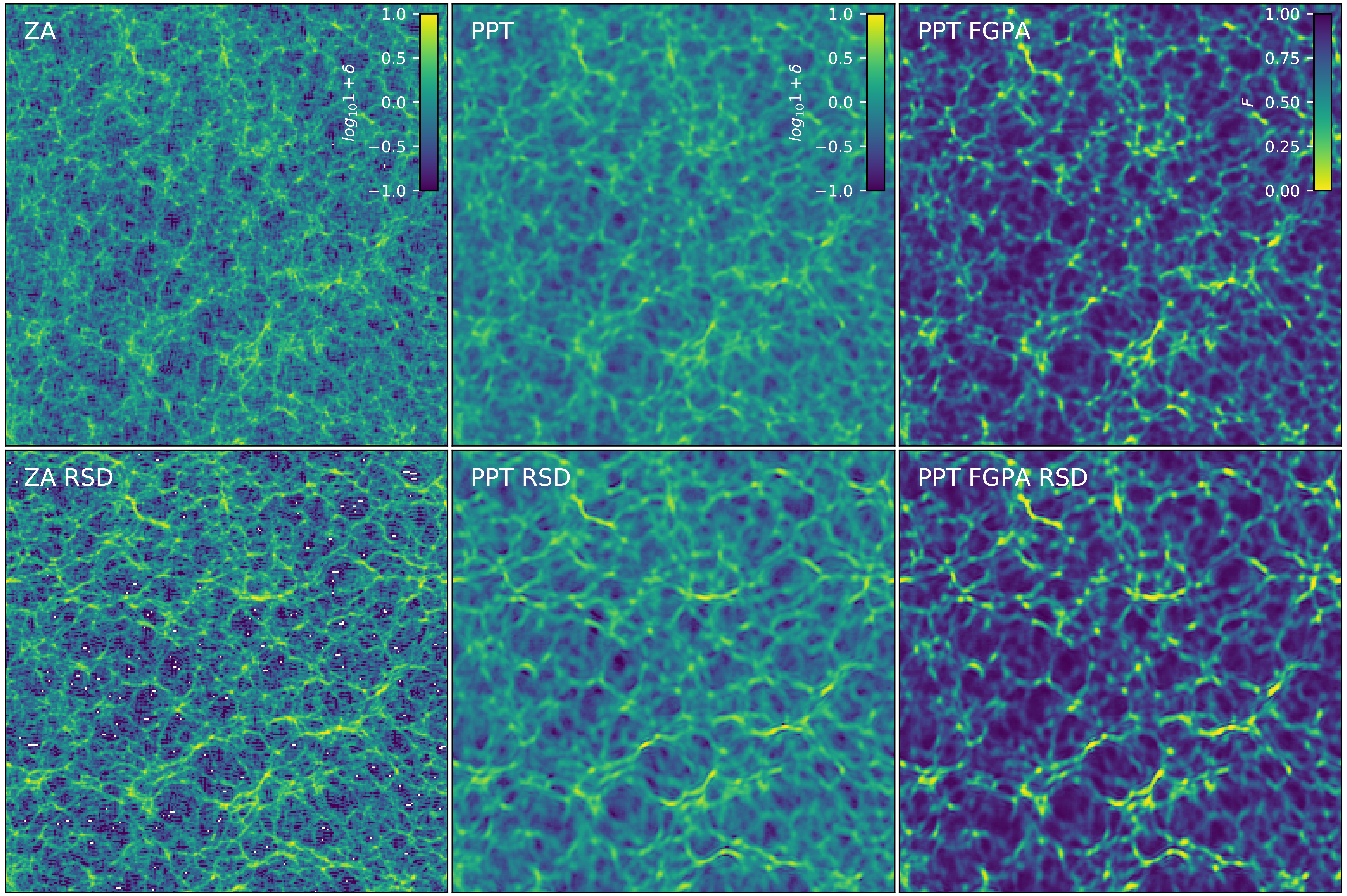


# The Sky from Ψ

density ρ    quasar flux F(ρ)

real space

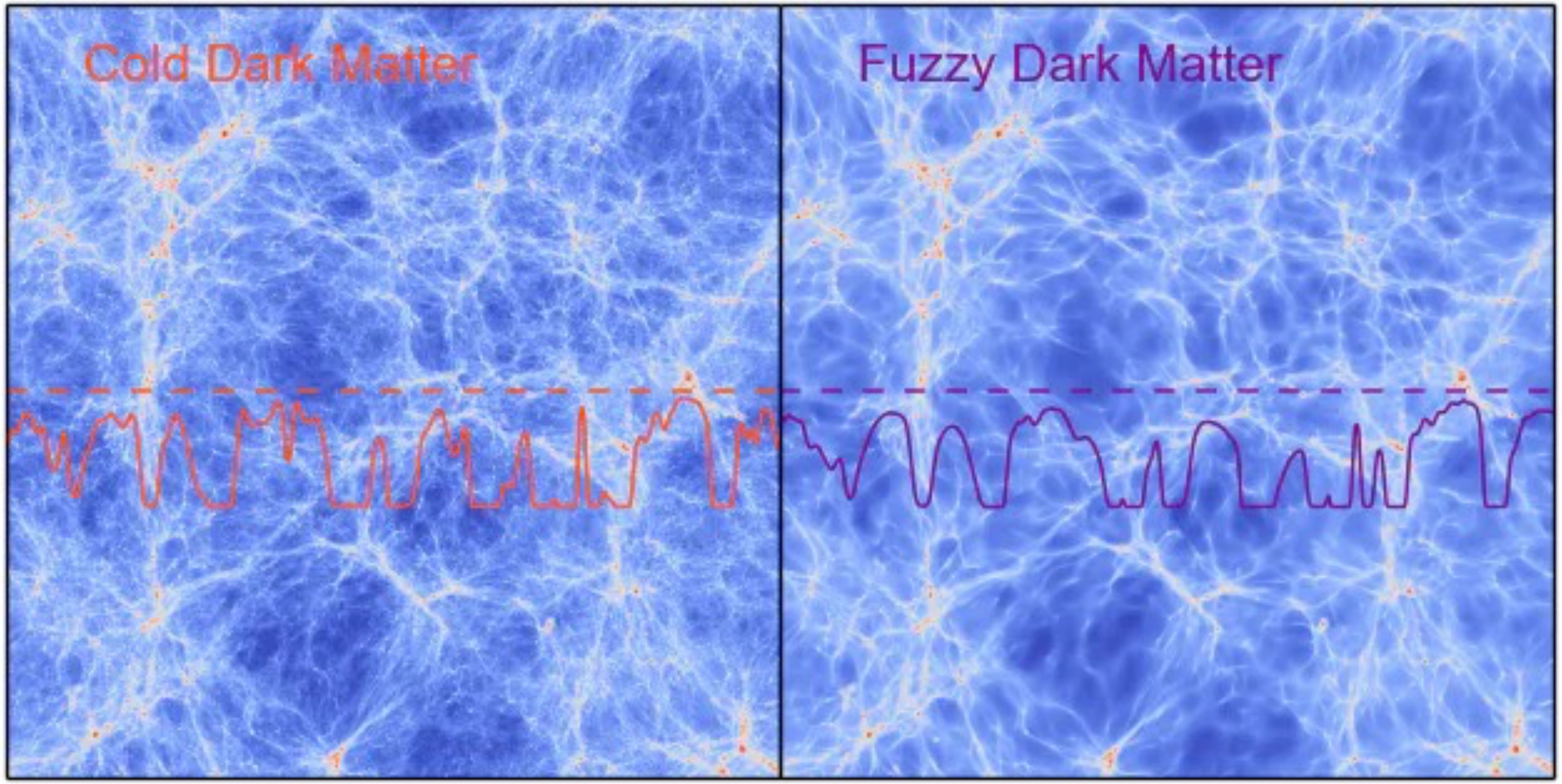
redshift space




Porqueres ++ ‘20

# Fuzzy Dark Matter Constraints

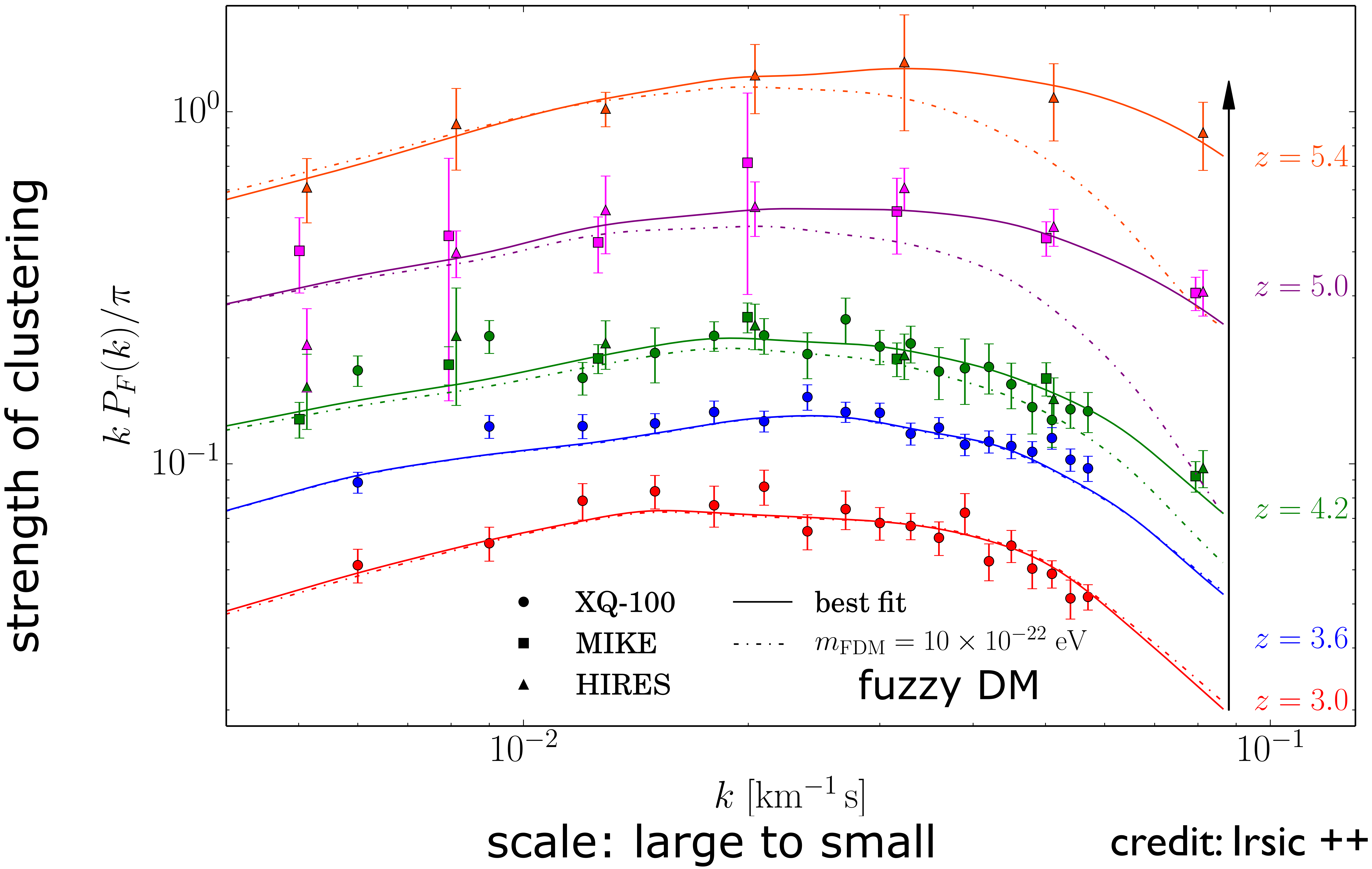
Lyman-alpha forest: light absorption by hydrogen gas within the intergalactic medium at high redshifts




simulation & plot from Vid Irsic

# Fuzzy Dark Matter Constraints

lower mass limit by Lyman-alpha forest




credit: Irsic ++ '17

## Perfect fluid perturbation theory

Continuity $a\partial_\tau \delta = -\boldsymbol{\nabla}[(1 \cancel{+ \delta})\boldsymbol{\nabla}\phi]$

Bernoulli $a\partial_\tau \Delta\phi = -\frac{\Delta}{2}(\cancel{\boldsymbol{\nabla}\phi})^2 - a^2 \Delta V \propto -\delta$

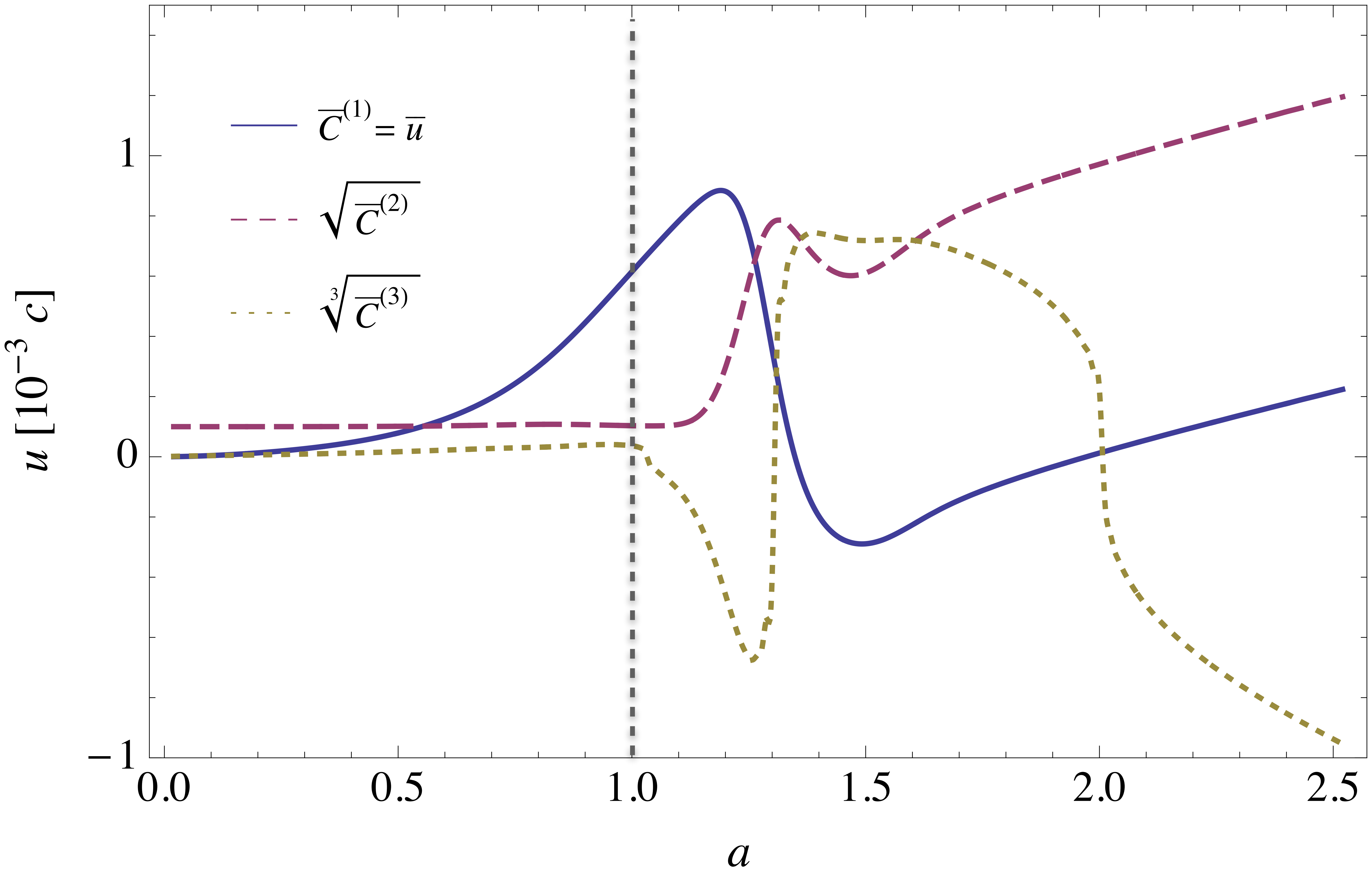
determine linear solution
& plug back in

$$\delta(\tau, \boldsymbol{k}) = \sum_n a^n(\tau)\delta_n(\boldsymbol{k})$$

$$\delta_n(\boldsymbol{k}) = \int \frac{\mathrm{d}^3\boldsymbol{p}_1 \ldots \mathrm{d}^3\boldsymbol{p}_n}{(2\pi)^{3(n-1)}} \, \delta_\mathrm{D}(\boldsymbol{k} - \boldsymbol{p}_{1\ldots n}) \, F_n(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n) \;\; \delta_1(\boldsymbol{p}_1) \ldots \delta_1(\boldsymbol{p}_n)$$

get recursion relations

## Cumulant hierarchy



## Schrödinger-Poisson equation

$$i\hbar\,\partial_t\psi(\boldsymbol{x},t) = \hat{H}\psi(\boldsymbol{x},t)$$

$$\Delta V(\boldsymbol{x},t) \propto \overbrace{|\psi(\boldsymbol{x},t)|^2}^{\rho(\boldsymbol{x},t)} - 1$$

self-gravitating field

$$\psi = \sqrt{\rho}\exp(i\phi/\hbar)$$

$$f_W(\boldsymbol{x},\boldsymbol{p}) = \int \frac{d^3\tilde{x}}{(2\pi\hbar)^3}\exp\left[2\frac{i}{\hbar}\boldsymbol{p}\cdot\tilde{\boldsymbol{x}}\right]\psi(\boldsymbol{x}-\tilde{\boldsymbol{x}})\bar{\psi}(\boldsymbol{x}+\tilde{\boldsymbol{x}})$$

$\hbar$ as free parameter

**Features**

- same # degrees of freedom as fluid
- fluid model as limit $\hbar \to 0$
- no singularity
- nonzero higher cumulants

**Problems**

- not manifestly positive
- time evolution not quite like Vlasov

**cure: add coarse-graining** $\sigma_x\sigma_p \gtrsim \hbar/2$

$$\bar{f}_W(\boldsymbol{x},\boldsymbol{p}) = \int \frac{d^3\tilde{x}\,d^3\tilde{p}}{(\pi\sigma_x\sigma_p)^3}\exp\left[-\frac{(\boldsymbol{x}-\tilde{\boldsymbol{x}})^2}{2\sigma_x^2} - \frac{(\boldsymbol{p}-\tilde{\boldsymbol{p}})^2}{2\sigma_p^2}\right] f_W(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{p}})$$

## Closing a cumulant hierarchy with finitely generated cumulants

- **Idea: finite # of fundamental functions rather than finite # of cumulants**

$$G[\boldsymbol{J}] = \int \mathrm{d}^3p \ \ \exp\left[i\boldsymbol{p}\cdot\boldsymbol{J}\right] f \,, \quad C^{(n)}_{i_1\cdots i_n} := (-i)^n \left.\frac{\partial^n \ln G[\boldsymbol{J}]}{\partial J_{i_1}\ldots\partial J_{i_n}}\right|_{\boldsymbol{J}=0}$$

- cumulant generator = (linear) operators on fundamental functions

$$\ln G[\boldsymbol{J}] = \mathcal{O}_n(\boldsymbol{J}) \ln n + i\mathcal{O}_\phi(\boldsymbol{J})\phi$$

- **Idea: make evolution for higher cumulants automatically fulfilled**

$$\partial_t \ln G[\boldsymbol{J},\boldsymbol{x}] = \frac{i}{a^2 m}\left(\boldsymbol{\nabla}_J\cdot\boldsymbol{\nabla}_x \ln G + \boldsymbol{\nabla}_J \ln G\cdot\boldsymbol{\nabla}_x \ln G\right) - im\boldsymbol{J}\cdot\boldsymbol{\nabla}_x V$$

- given **initial conditions** & evolution for lower cumulants

$$\partial_t \ln n = \frac{-1}{a^2 m}\left[\boldsymbol{\nabla}^2\phi + \boldsymbol{\nabla}\ln n\cdot\boldsymbol{\nabla}\phi\right] \qquad \partial_t\phi = -\frac{1}{a^2 m}\left\{\frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \tilde{C}^{(2)}\right\} - mV$$

- **Schrödinger: one wave function to rule them all**

$$\ln G[\boldsymbol{J}] = \cosh\left(\frac{\hbar}{2}\boldsymbol{J}\cdot\boldsymbol{\nabla}\right)\ln n(\boldsymbol{x}) + 2\frac{i}{\hbar}\sinh\left(\frac{\hbar}{2}\boldsymbol{J}\cdot\boldsymbol{\nabla}\right)\phi(\boldsymbol{x})$$

# Multi-Stream Regime

## Vorticity

small scales
quantised

large scales
classical

from Kopp++ PRD '17