Validating Open Quantum Simulators by Lindblad Resummation Techniques

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1 Motivation

2 Lindblad perturbation theory

3 Application: Jaynes-Cummings lattice
Circuit QED lattices

Houck Lab (Princeton): realizing JC lattice models
Quantum Simulation

physical model
(no exact solution, classical sim. not possible)

approximate descriptions

quantum simulator
(experiment)

experimental data

known

validation

“exact” results

see also: Cirac & Zoller, Nature Phys. 8, 264-266 (2012)
Open-system quantum simulator
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<td>$H = H^\dagger$</td>
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<td>$L \neq L^\dagger$</td>
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after: E.M. Kessler et al., PRA 86, 012116 (2012)
Lindblad perturbation theory and Resummation

\[ f = \sum_2 + \sum_2^2 + \sum_3 + \ldots \]
Open quantum systems

\[ \frac{d}{dt} \rho(t) = -i[H, \rho] + \sum_n \gamma_n \left( A_n \rho A_n^\dagger - \frac{1}{2} \{A_n^\dagger A_n, \rho\} \right) \]

\[ \mathbb{D}[A_n] \rho \]

Lindblad master equation

\text{(Lindblad, Kossakowski, Gorini, and Sudarshan)}

Observables: \[ \langle M \rangle = \text{tr} \left( M \rho_s \right) \]
Steady state, stationary Lindblad eq.

\[
\frac{d}{dt} \rho(t) = -i[H, \rho] + \sum_n \gamma_n D[A] \rho = \mathbb{L} \rho
\]

Liouvillian super-operator

Steady state:

\[
0 = \frac{d}{dt} \rho(t) = \mathbb{L} \rho \quad \rightarrow \quad \mathbb{L} \rho_s = 0
\]

stationary sol. to Lindblad master eq.

\[\mathbb{L} |u_\nu\rangle = \lambda_\nu |u_\nu\rangle\]

Stationary Lindblad eq.

\[\text{Re} \lambda \leq 0\]

steady state
Lindblad Perturbation Theory (non-deg.)

Stat. Lindblad master eq. \( \mathcal{L} |u_\nu\rangle = \lambda_\nu |u_\nu\rangle \)

decompose Liouvillian: \( \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \)

- controlled, analytical approximation

\[ \lambda_\nu^1 = (w_\nu^0 | \mathcal{L}_1 | u_\nu^0) \]

- directly study spectrum of Liouville super-operator

\[ |u_\nu^1\rangle = \sum_{\mu \neq \nu} \frac{(w_\mu^0 | \mathcal{L}_1 | u_\nu^0)}{\lambda_\mu^0 - \lambda_\nu^0} |u_\mu^0\rangle \]

- resummation scheme

Li, Petruccione, JK, Scientific Reports 4, 4887 (2014)
Recursion relations

Eigenvalues:

\[ \lambda_{\nu}^n = (w_{\nu}^0 | L_1 | u_{\nu}^{n-1}) - \sum_{m=1}^{n-1} \lambda_{\nu}^m (w_{\nu}^0 | u_{\nu}^{n-m}) \]

Eigenstates:

\[ |u_{\nu}^n \rangle = -\frac{1}{L_0 - \lambda_{\nu}^0} \left[ L_1 |u_{\nu}^{n-1}\rangle + \sum_{m=1}^{n} \lambda_{\nu}^m |u_{\nu}^{n-m}\rangle \right] \]

**Steady state:** \( \lambda_0^j = 0 \) (all orders)

\[ |\rho_s \rangle = \sum_j |\rho_j \rangle \quad \Rightarrow \quad |\rho_j \rangle = -L_0^{-1} L_1 |\rho_{j-1} \rangle \quad \Rightarrow \quad |\rho_{j} \rangle = L_{j} |\rho_0 \rangle \]

\[ |\rho_{j} \rangle = \underbrace{\cdots \cup L_{j} \cup}_{j \text{ times}} |\rho_0 \rangle \quad \mathbb{I} = \sum_{\mu} |u_{\mu} \rangle (w_{\mu}) \quad \text{(assuming completeness)} \]
Resummation scheme

\[ |\rho_s\rangle = \sum_{j=0}^{\infty} U^j |\rho_0\rangle \quad \leadsto \quad ? \]

**Idea:** extract diagonal part

- **start:** \( U^1 \) off-diag. \( \Rightarrow \Sigma_1 = 0 \) and \( T_1 = U \)
- **recursion:** \( T_{j-1} U = \Sigma_j + T_j \)
  - diag. part: \( (\Sigma_j)_{\mu\nu} = \delta_{\mu\nu} (T_{j-1} U)_{\mu\nu} \)
  - \( \Sigma_j |u_\mu^0\rangle = \Sigma_{j;\mu} |u_\mu^0\rangle \)

**solution:** \( T_j = [[[\cdots [[[U]U]\cdots U]U]]} \) (\( j \) times) \[ [A] \]: off-diagonal part of \( A \)

**Ambiguity:** \( U^3 = U(U^2) \) or \( U^3 = (U^2)U \)

\[ U^j = U^{j-1} U \quad \text{systematic replacement rule} \]
Resummation scheme (cont’d)

\[ |\rho_s) = \sum_{j=0}^{\infty} \mathbb{U}^j |\rho_0) = \mathfrak{f} \sum_{j=0}^{\infty} \mathbb{T}_j |\rho_0) \]

\[ \mathfrak{f} = 1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_2^2 + \Sigma_5 + \Sigma_2 \Sigma_3 + \Sigma_3 \Sigma_2 \cdots \]

\[ = \sum_{n=0}^{\infty} (\Sigma_2 + \Sigma_3 + \cdots)^n = (1 - \Sigma)^{-1} \]

\[ \sum \text{ (irred. diagrams only)} \]

\[ |\rho_s) = \sum_{j=0}^{\infty} \mathbb{U}^j |\rho_0) = \sum_{j=0}^{\infty} \frac{1}{1 - \Sigma} \mathbb{T}_j |\rho_0) \]

\[ |\rho_s^{(j)} ) = \sum_{\mu_j} \sum_{\nu_1, \ldots, \nu_{j-1} \neq \mu_j} |u_{\mu_j}^0 \rangle \left( \frac{1}{1 + \Sigma} \right)_{\mu_j \nu_{j-1}} (\mathbb{U})_{\nu_{j-1} \nu_{j-2}} \cdots (\mathbb{U})_{\nu_2 \nu_1} (\mathbb{U})_{\nu_1 0} \]

\[ |u_0^0 \rangle \quad |u_{\mu_1 \neq \mu_j}^0 \rangle \quad |u_{\mu_2 \neq \mu_j}^0 \rangle \quad |u_{\nu_{j-1} \neq \mu_j}^0 \rangle \quad |u_{\mu_j}^0 \rangle \]
3 Application: Jaynes-Cummings lattice
open Jaynes-Cummings system

coplanar waveguide resonator + sc qubit
JC lattices

1d chain

global coupling

2d square lattice

REVIEWS

A. Tomadin and R. Fazio

Houck, Tureci, JK
Nature Phys. 15, 115002 (2012)

Schmidt, JK
Lindblad master eq. \[ \frac{d}{dt} \rho = \mathbb{L} \rho = 0 \]

\[ \mathbb{H} : N \times N \quad \Rightarrow \quad \mathbb{L} : N^2 \times N^2 \]

"worse" than exact diagonalization for closed system

e.g., 4 resonators
(up to 3 photons each)
4 qubits

\[ \mathbb{L} : 16 \text{ millions} \times 16 \text{ millions} \]

Promising numerical schemes:
Cluster-MFT, DMRG, TEBD, Variational Methods, …
JC lattice model: pert. treatment

\[ H = \sum_r h_r^{JC} + t \sum_{\langle r, r' \rangle} (a_r^\dagger a_{r'} + \text{h.c.}) \]

\[ h_r^{JC} = \delta \omega a_r^\dagger a_r + \epsilon (a_r^\dagger + a_r) + \delta \Omega \sigma_r^+ \sigma_r^- + g (a_r \sigma_r^+ + a_r^\dagger \sigma_r^-) \]

- resonators
- drive
- qubits
- JC coupling

\[ \dot{\rho} = -i[H, \rho] + \gamma \sum_r \mathcal{D}[a_r] \rho + \Gamma \sum_r \mathcal{D}[\sigma_r^-] \rho \]

- photon loss
- qubit relax.

Perturbation: 
JC interaction \( g \) or photon hopping \( t \)

done here

problem: no exact solution for driven, damped JC site
Structure of PT corrections

\( |r_{mn}^k\rangle = (|b_k^\dagger\rangle)^m (|\beta_k^\dagger\rangle)^n |0\rangle \langle 0| / \sqrt{m!n!}, \)

\( \lambda_{mn}^k = -i \delta \omega_k (m - n) - \frac{\tau}{2} (m + n) \)

\( \otimes_r \) driven, damped spin \( |u_{\mu_r}^r\rangle \)

0th order:

\( |\rho_0\rangle = \otimes_k |r_{00}^k\rangle \otimes \otimes_r |u_{00}^r\rangle \)

1st order:

\( |\rho_1\rangle = \sum_{k,r} |\rho_{kr}^1\rangle \otimes |\rho_{kr}^0\rangle \)

\( \rightarrow \) cluster

2nd order:

\( |\rho_2\rangle = \sum_{k,k',r,r'} |\rho_{kk'rr'}^2\rangle \otimes |\rho_{kk'rr'}^0\rangle \)

\( \cdots \)
Comparison of results

\[
\text{Deviation } |\langle \delta \sigma^- \rangle| \quad \text{ vs } \quad \text{\(\frac{\delta \omega}{\Gamma} \)}}
\]

- \(N = 2\)
- \(N = 4\) (no resummation)
Summary

**Motivation:** validation of circuit QED quantum simulators (Houck Lab)

1. simple stationary Lindblad perturbation theory
   use resummation to go beyond finite order

2. application to JC lattices
   comparison w/ exact solutions, results

**Goal:** validation of circuit QED quantum simulators
FIG. 5. Comparison between the perturbative results and the exact results obtained by quantum trajectories methods. (a) The steady-state expectation value $|\langle \sigma^- \rangle|$ of the spin lowering operator is plotted as a function of the detuning $\delta \omega$ for $\delta \Omega = \delta \omega$, $g/\Gamma = 3$, $\epsilon/\Gamma = 20$, $\kappa_0/\Gamma = 10$, $\gamma/\Gamma = 4$. The exact results (points) for $N = 2$ and $N = 4$ are well approximated by the perturbative results with SE corrections (solid lines). (b) The deviation $|\langle \delta \sigma^- \rangle|$ of the steady-state expectation value $|\langle \sigma^- \rangle|$ from the exact results is plotted as a function of the detuning $\delta \omega$ using the same set of parameters. In general, the perturbative results without SE corrections (dashed lines) show a much larger deviation from the exact results than the one with SE corrections (solid lines).