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Introduction and Overview

S-duality of 4D $N=4$ Super-Yang-Mills

G_c - compact Lie group

G - complexification (not G, $G_{\mathbb{C}}$)

$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$

$$(G_c, \tau) \xleftrightarrow{\text{S-duality}} ({}^L G_c, -\tfrac{1}{\tau}) \quad \rightsquigarrow \text{geometric Langlands}$$

Langlands dual group.

If G_c is abelian, this is "Fourier transform".

Kapustin-Witten

Particular topological twist, "$GL^{\#}$-twist."

Supercharges

$$Q = u\, Q_e + v\, Q_r \qquad t = \frac{v}{u} \in \mathbb{P}^1$$

$$\Psi = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}\,\frac{t - t^{-1}}{t + t^{-1}}, \quad \text{S-duality}: \Psi \to \frac{-1}{n_g \Psi}$$

$$t \to \frac{|\tau|}{\tau}\, t$$

$n_g = 1$ ADE

$= 2$ BCF

$= 3$ G_2

②

$$({}^L G_{\mathbb{C}},\ t = i,\ \psi = \infty) \xleftrightarrow{\text{S-duality}} (G_{\mathbb{C}},\ t = 1,\ \psi = 0)$$
$$\theta = 0 \qquad\qquad\qquad \theta = 0$$

$M_4 = C \times \Sigma$, $\quad C, \Sigma$ - Riemann surfaces

Compactify it on C, get effective theory on Σ.

This is a supersymmetric sigma-model on Σ with target space $\mathcal{M}_H(G)$ or $\mathcal{M}_H({}^L G)$, the moduli space of semi-stable Higgs bundles on C.

(E, ϕ), $\quad \begin{matrix} E \\ \downarrow \\ C \end{matrix}$ (holo) principal G-bundle on C, $\phi \in H^0(C, \mathfrak{g}_E \otimes K_C)$

all $G_{\mathbb{C}}$. $\qquad \mathfrak{g}_E = E \times_G \mathfrak{g}$

(hyperKähler manifolds)

$$\begin{tikzcd} \mathcal{M}_H({}^L G) \arrow[dr] & & \mathcal{M}_H(G) \arrow[dl] \\ & B & \end{tikzcd}$$

B - vector space

SYZ mechanism.

generic fibers are dual tori

K-W: what happens to branes?

B-model on left; needs a complex structure.

Three complex structures I, J, K; I = complex structure describing the moduli space of Higgs bundles

But in the mirror symmetry, we want complex structure J...

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In complex structure J, $\mathcal{M}_H({}^LG) \underset{\text{diffeo}}{=} \mathcal{Y}({}^LG)$, the moduli space of semi-stable flat ${}^LG$-bundles on C:

$\mathcal{E} = (E, \nabla)$, $\begin{matrix} E \\ \downarrow \\ C \end{matrix}$ - holo. principal ${}^LG$-bundle, ∇ - holo connection on E.

$\boxed{\text{B-branes on } \mathcal{Y}({}^LG)}$ (= complex structure J on $\mathcal{M}_H({}^LG)$)

$\updownarrow$ homological mirror symmetry.

$\boxed{\text{A-branes on } \mathcal{M}_H(G) \text{ w.r.t. symplectic structure } \omega_K}$; in complex structure I, $\mathcal{M}_H(G)$ is birational to $T^*\mathcal{M}(G)$ where $\mathcal{M}(G)$ is the moduli space of semi-stable G-bundles. The cotangent space $T^*\mathcal{M}(G)$ has a canonical symplectic structure ω_K, which transports back to $\mathcal{M}_H(G)$.

Wilson operators act on $\boxed{\text{B-branes on } \mathcal{Y}({}^LG)}$

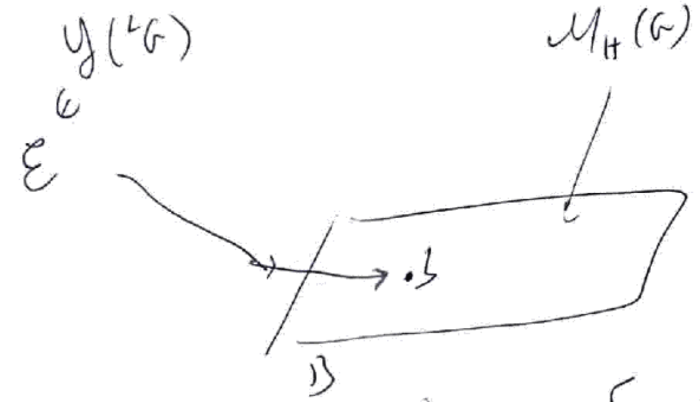
't Hooft operators act on $\boxed{\text{A-branes on } \mathcal{M}_H(G)}$

and these operators correspond under the duality

B-branes $\longleftrightarrow$ category of coherent sheaves on $\mathcal{Y}({}^LG)$.

Simplest one is skyscraper sheaf $\mathcal{O}_{\mathcal{E}}$ supported at a point $\mathcal{E} \in \mathcal{Y}({}^LG)$.

(4)



${}^LF_b$ = fiber ; dual torus to F_b (the fiber on right over b)

A point of ${}^LF_b$ (such as $\mathcal{E}$) is the same as a line bundle $\mathcal{L}$ on F_b and a flat $U(1)$-connection ∇ on $\mathcal{L}$.

$$\mathcal{E} \mapsto (\mathcal{L}_\mathcal{E}, \nabla_\mathcal{E}) \text{ on } F_b.$$

This is the data of an A-brane on $\mathcal{M}_H(G)$

(since F_b are Lagrangian tori on $\mathcal{M}_H(G)$).

$\mathcal{O}_\mathcal{E}$ is an "eigenobject" of Wilson operators

$$\begin{array}{c} \tilde{E} \\ \downarrow \\ C \times \{\mathcal{E} = (E,\nabla)\} \\ \| \\ \mathcal{Y}({}^LG) \end{array}$$

$p \in C \to \tilde{E}_p \rightsquigarrow \tilde{V}_p$

Z 

C 

$r \in \mathrm{Rep}({}^LG)$
$p \in C$

$$W_{r,p}(\mathcal{O}_\mathcal{E}) = \tilde{V}_{p,\mathcal{E}} \otimes \mathcal{O}_\mathcal{E}$$

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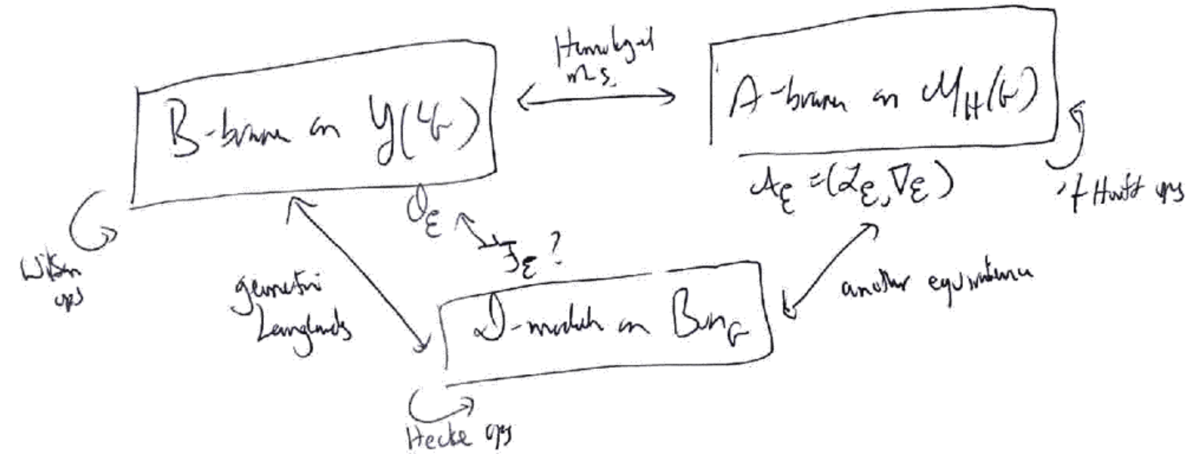
3rd category: $\mathcal{D}$-modules on Bun_G.



Bun_G - moduli "stack" of G-bundles on C.

$$\left.\begin{array}{l} F_A - \phi \wedge \phi = 0 \\ D_A \phi = D_A {*\phi} = 0 \end{array}\right\} \quad F_{\tilde{A}} = 0, \text{ where } \tilde{A} = A + i\phi$$

A-brane $\mathfrak{B}$ on $\mathcal{M}_H(G)$

$$\mathfrak{B} \longmapsto \mathrm{Hom}(\mathfrak{B}_{c.c.}, \mathfrak{B})$$

$\mathcal{D}$

$\mathfrak{B}_{c.c.}$ = canonical coisotropic brane

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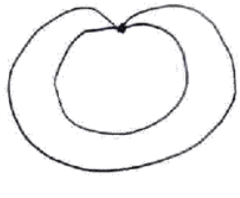
Orbifold singularities: Witten-Frenkel arXiv: 0710...

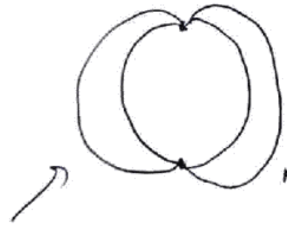
$G = SL_2$, ${}^L G = SO_3 \supset O_2 \supset \mathbb{Z}_2$

$\mathbb{C}^2/\mathbb{Z}_2$ singularity



B-branes: $\mathcal{O}_\varepsilon^+$, $\mathcal{O}_\varepsilon^-$ (via $\mathbb{Z}_2$-action)

dual fiber:



nonsingular in ambient space.

$\mathcal{O}_\varepsilon^+$ $\mathcal{O}_\varepsilon^-$

$\underbrace{\phi_0, \ldots, \phi_3}_{1\text{-form}}, \phi_4, \phi_5$

$\sigma = (\phi_4 - i\phi_5)/\sqrt{2}$

$\bar\sigma = (\phi_4 + i\phi_5)/\sqrt{2}$

$\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}$-valued scalars

$$D\sigma + t[\phi,\sigma] = 0$$
$$D\sigma - t^{-1}[\phi,\sigma] = 0$$
$$t D^*\phi + [\sigma,\bar\sigma] = 0$$
$$-t^{-1} D^*\phi + [\sigma,\bar\sigma] = 0$$

$t \neq \pm i \Rightarrow D\sigma = 0$, $[\phi,\sigma] = 0$, $[\sigma,\bar\sigma] = 0$

$\Rightarrow$ generically $\sigma \equiv 0$.

Our model has $t = i$:

$D_{A+i\phi}\,\sigma = 0 \Rightarrow \sigma \in$ Lie algebra of infinitesimal symmetries of $\mathcal{A} = A + i\phi$

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Gaiotto-Witten studied S-duality of general boundary conditions
in 4D SYM.

(1) $\sigma, \bar{\sigma}$ can have a pole at boundary
Nahm equation $\quad SL_2 \to {}^L G.$
("Arthur's SL_2" in geometric Langlands)

(2) Coupling to 3D SCFT's on boundary