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# Branes, Duality and Quantization

## Observation (S.G. + D. Zagier)

Consider Chern-Simons theory with complex gauge group $G_{\mathbb{C}}$ at "level" $k$ on a 3-manifold $W$ (possibly w/ Wilson lines)

For compact $G$: $k \in \mathbb{Z}$

For complex $G_{\mathbb{C}}$: $k \in \mathbb{C}$

$$Z^{CS}(G_{\mathbb{C}}; q) = \text{quantum } G_{\mathbb{C}}\text{-invariant of } W$$

$$q = \exp\left(\frac{2\pi i}{k}\right)$$

There is a hidden $SL(2,\mathbb{Z})$ symmetry:

$$Z^{CS}(G_{\mathbb{C}}; q) \sim Z^{CS}({}^L G; {}^L q)$$

where ${}^L q = \exp\left(\frac{2\pi i}{{}^L k}\right)$ and ${}^L k = -\frac{1}{k}$.

["quantum modular forms"]

(2)

$$\left.\begin{array}{l}\text{complex gauge group } G_{\mathbb{C}} \\ S\text{-duality}\end{array}\right\} \rightarrow \begin{array}{l}\text{realize } \overset{3D}{\text{CS}} \text{ theory in } \mathcal{N}=4 \\ \text{(topological) } \underset{4D}{\text{Yang-Mills}} \text{ theory}\end{array}$$

dimensional reduction of 4D YM theory to 3D

$\swarrow$ $\searrow$

on a circle

$M = S^1 \times W$

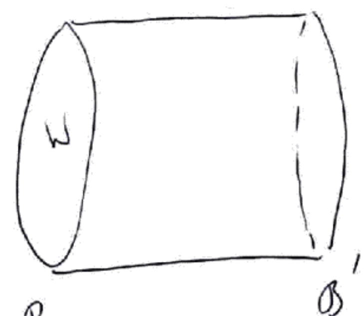
$\Downarrow$

effective 3D theory

invariant under $P: W \to \overline{W}$

on a "colored interval"

$M = I \times W, \quad I = [0,1]$




$\uparrow$ boundary conditions

$\Downarrow$

effective 3D theory

$T(B, B')$

not necessarily P-invariant

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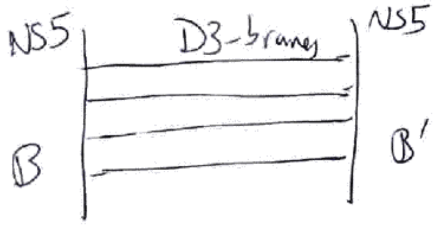
Boundary conditions from branes




$$\longrightarrow \; y = x^3 (= x^6)$$

For $\mathcal{B} = \mathcal{B}' =$ NS5-like b.c
= "zero section" b.c.,

$T(\mathcal{B},\mathcal{B}')$ = three-dim'l $\mathcal{N}=4$ Super Yang Mills
(8 supercharges)

* global symmetry group (in Euclidean space)

$$SO(3)_E \times SO(3)_N \times SU(3)_R$$

* topological twists
a) Rozansky-Witten twist
b) Donaldson-Witten twist
S-duality

* in general, to obtain the Hilbert space $\mathcal{H}$ of the effective 3d theory $T(\mathcal{B},\mathcal{B}')$ need to study

$$W = \mathbb{R} \times C$$

(Euclidean) time

Riemann surface

(4)

In 4d gauge theory,

$$M = \Sigma \times C$$

where $\Sigma = \mathbb{R} \times I$

time


$\simeq$ 2d topological $\sigma$-model $\Sigma \longrightarrow \mathcal{M}_H(G;C)$

$$\mathcal{H} = \mathrm{Hom}(\mathcal{B}, \mathcal{B}')$$

= space of open string states between branes $\mathcal{B}$ and $\mathcal{B}'$ on $\mathcal{M}_H(G;C)$

<u>Example (Chern-Simons theory)</u>:

$\mathcal{B}'$ = NS5-like b.c.

= mid-dimensional brane supported on

$\mathcal{M}(G;C) \subset \mathcal{M}_H(G;C)$ $\quad$ $(\mathrm{Bun}_G)$

$\mathcal{B}$ = canonical coisotropic brane supported on the entire $\mathcal{M}_H(G;C)$

$$F = \omega_I$$

This brane exists only in the gauge theory with quantized $\tau = ik$, $k \in \mathbb{Z}$.

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[S.G. + E. Witten]:

$\mathcal{H} = \mathrm{Hom}(\mathcal{B}_{c.c.}, \mathcal{B}') =$ Hilbert space of Chern-Simons theory w/ gauge group $G$, level $k$.

$=$ quantization of $\mathcal{M}_{flat}(G; \mathbb{C})$

* in this example, both branes $\mathcal{B}$ and $\mathcal{B}'$ are of type $(B, A, A)$.

<u>S-duality</u> $(B, A, A) \to (B, B, B)$

$$\mathcal{B}_{c.c.} \to \widetilde{\mathcal{B}}_{c.c.}$$

$$\mathcal{B}' \to \widetilde{\mathcal{B}}'$$

$\widetilde{\mathcal{B}}_{c.c.} =$ space-filling brane, with a nontrivial Chan-Paton bundle of rank $> 1$.

exists only for $\tau = \frac{i}{k}$ ! $(k \in \mathbb{Z})$

$\widetilde{\mathcal{B}}' =$ zero-brane, supported at the "most singular point" of $\mathcal{M}_H({}^L G; \mathbb{C})$

$$A_c\big|_{bdy} = \phi_c\big|_{bdy} = 0$$

with a Nahm pole for the fields $\vec{X} = (\sigma, \bar{\sigma}, \phi_y)$ that corresponds to the principal $SU(2)$ embedding

$$\rho_{princ} : SU(2) \to {}^L\mathfrak{g}$$