Quantum Geometric Langlands

vs.

Non-perturbative dualities in Sigma Models

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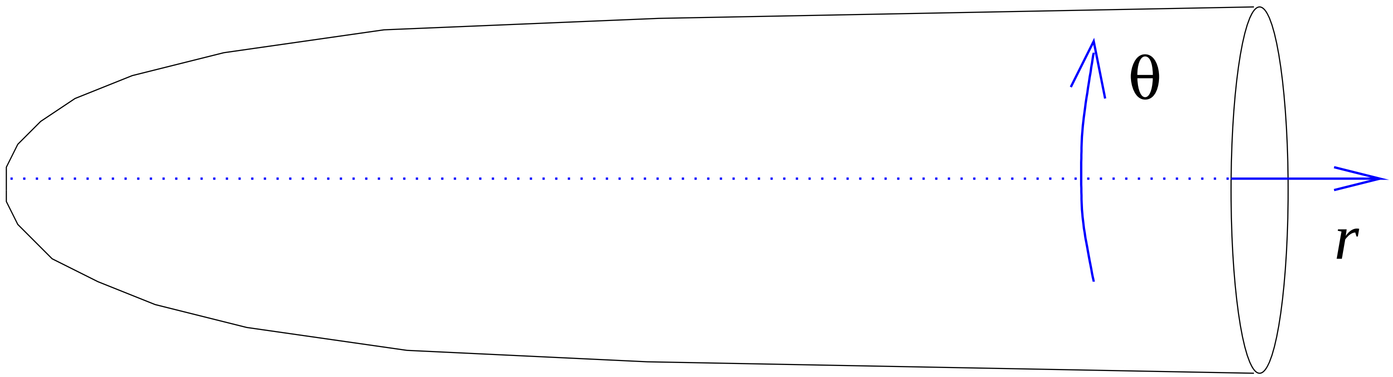
Motivation: FZZ duality - (Fateev, A.B. and Al.B. Zamolodchikov)

Quantized motion of strings on the "cigar", $ds^2 = dr^2 + \tanh^2(r)\, d\theta^2$,

$$S = \int_\Sigma \frac{d^2z}{4\pi} \left(\partial_z\rho\partial_{\bar{z}}\rho + \tanh^2(b\rho)\partial_z\theta\partial_{\bar{z}}\theta \right) = S_{\text{free}} + \int_\Sigma \frac{d^2z}{2\pi}\, e^{2b\rho}\partial_z\theta\partial_{\bar{z}}\theta + \dots$$



$\Leftrightarrow$ (for large curvature) motion of strings under exponentially growing force field ("tachyon condensate").

$$S = \int_\Sigma \frac{d^2z}{4\pi} \left(\partial_z\rho\partial_{\bar{z}}\rho + \partial_z\theta\partial_{\bar{z}}\theta + \lambda\, e^{\frac{1}{b}\rho} \cos(k\tilde{\theta}) \right) \qquad b^2 = \frac{1}{k-2}$$

Note: Exponential interactions $\propto e^{2b\rho}$ vs. $e^{\frac{1}{b}\rho}$!!!

Example: The FZZ duality II

- This duality means that string quantum fluctuations modify **geodesic motion** into **non-geometric propagation** — mirror symmetry.

- **New** type of non-perturbative effect: $\mathcal{O}\big(\exp\big(-e^{1/b^2}\big)\big)$.

Origin of FZZ duality

(Hikida, Schomerus): Combination of two dualities:

- Duality between the Cigar CFT and Liouville theory (Ribault, J.T.) — **Quantum geometric Langlands correspondence**

- Self-duality of Liouville theory (J.T.) — **Modular duality of quantum Teichmüller theory**

Origin of FZZ duality Ia: Duality Cigar - Liouville theory

What is **Liouville theory?**

$$S = \int_\Sigma \frac{d^2 z}{4\pi} \left(\partial_z \phi \partial_{\bar{z}} \phi + 4\pi\mu e^{2b\phi} \right).$$

Basic observables: $V_\alpha(z, \bar{z}) = e^{2\alpha\phi(z,\bar{z})}$. Theory fully characterized by correlation functions

$$\langle V_{\alpha_n}(z_n, \bar{z}_n) \dots V_{\alpha_1}(z_1, \bar{z}_1) \rangle$$

Related to **uniformization** of Riemann surfaces:

$ds^2 = e^{2b\phi} dz d\bar{z}$ has constant negative curvature

iff

ϕ satisfies Liouville equation of motion.

$\Leftrightarrow$

(classical/quantum) Liouville theory $\Leftrightarrow$ (classical/quantum) **Teichmüller theory**.

Origin of FZZ duality Ib: Duality Cigar - Liouville theory

Explicit relation between correlation functions (`Ribault, J.T.`)

$$\left\langle\, \Phi^{j_1}_{m_1\bar m_1}(z_1)\cdots\Phi^{j_n}_{m_n\bar m_n}(z_n)\,\right\rangle_{\rm Cigar} = \frac{2\pi^3 b}{\pi^{2n}(n-2)!}\;\delta\left(\textstyle\sum_{r=1}^n p_r\right)\delta_{\sum_{r=1}^n (m-\bar m)}\prod_{r=1}^n N^{j_r}_{m_r\bar m_r}$$

$$\times\int_C d^2y_1\cdots d^2y_{n-2}\;K(z_1,\ldots,z_n|y_1,\ldots,y_{n-2})$$

$$\times\left\langle\, V_{\alpha_n}(z_n)\cdots V_{\alpha_1}(z_1)V_{-\frac{1}{2b}}(y_{n-2})\cdots V_{-\frac{1}{2b}}(y_1)\,\right\rangle_{\rm Liou}\,,$$

where

$$K(z_1,\ldots,z_n|y_1,\ldots,y_{n-2}) = \prod_{r<s\le n}(z_r-z_s)^{m_r+m_s+\frac{k}{2}}(\bar z_r-\bar z_s)^{\bar m_r+\bar m_s+\frac{k}{2}}$$

$$\prod_{a<b\le n-2}|y_a-y_b|^k\prod_{r=1}^n\prod_{a=1}^{n-2}(z_r-y_a)^{-m_r-\frac{k}{2}}(\bar z_r-\bar y_a)^{-\bar m_r-\frac{k}{2}}\,.$$

Mathematics behind: Geometric Langlands correspondence

Origin of FZZ duality IIa: Self-duality of Liouville theory

Non-perturbative construction (J.T.) shows: **Liouville theory is self-dual**:

$$\left\langle V_{\alpha_n}(z_n,\bar{z}_n)\dots V_{\alpha_1}(z_1,\bar{z}_1)\right\rangle_{b,\mu} = \left\langle V_{\alpha_n}(z_n,\bar{z}_n)\dots V_{\alpha_1}(z_1,\bar{z}_1)\right\rangle_{\frac{1}{b},\tilde{\mu}},$$

where

$$\pi\,\frac{\Gamma(b^{-2})}{\Gamma(1-b^{-2})}\,\tilde{\mu} \,=\, \left(\pi\,\frac{\Gamma(b^2)}{\Gamma(1-b^2)}\,\mu\right)^{\frac{1}{b^2}}.$$

Origin of self-duality of Liouville theory — Necessity

Renormalization of exponential interactions

$$S = \int_\Sigma \frac{d^2 z}{4\pi} \left(\partial_z \phi \partial_{\bar{z}} \phi + 4\pi \mu e^{2b\phi} \right).$$

Consider term of n-th order in the perturbative expansion of Liouville theory,

$$\mu^n \int d^2 u_1 \ldots \int d^2 u_n \, e^{2b\phi(u_1,\bar{u}_1)} \ldots e^{2b\phi(u_n,\bar{u}_n)},$$

use OPE $e^{2b\phi(z,\bar{z})} e^{2b\phi(w,\bar{w})} \sim |z-w|^{-4b^2} e^{4b\phi(w,\bar{w})}$. Singular behavior comes from "clustering" of integration variables, similar to Dyson's integral:

$$\int_{S_1} dt_1 \ldots dt_n \prod_{r<s} |t_r - t_s|^{-2b^2} = \left(\frac{2\pi}{\Gamma(1-b^2)} \right)^n \Gamma(1-nb^2),$$

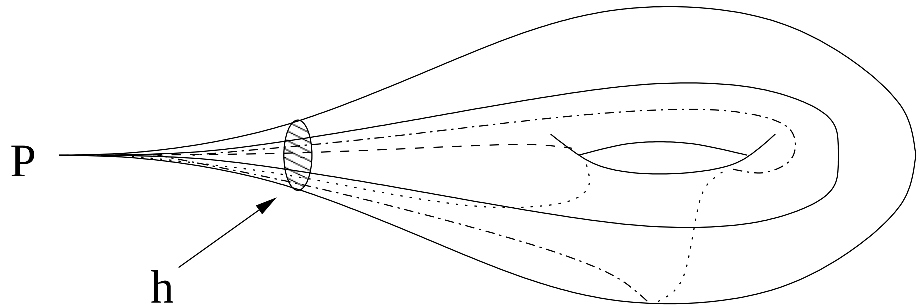
i.e. Poles for rational b^2, **small denominator problem** for irrational b^2.

Note: Dual interaction $e^{2\frac{1}{b}\phi}$ produces similar singularities which might solve the problem — need **nonperturbative** construction of Liouville theory (J.T.).

Non-perturbative construction of Liouville theory –

– quantum Teichmüller theory

What is quantum Teichmüller theory? (`Fock; Checkov/Fock; Kashaev`)
Use uniformization to introduce coordinates on Teichmüller spaces,



One length variable l_e for each edge of triangulation Δ. Form cross-ratio $z_e = l_a + l_c - l_b - l_d$. Natural Poisson bracket from Teichmüller theory (Weil-Petersson):

$$\{ z_e \, , \, z_f \} \, = \, n_{ef} \, , \quad \text{where } n_{ef} \in \{\pm 2, \pm 1, 0\} \text{ determined by triangulation.}$$

Quantization: Straightforward, operators z_e, $e \in \{\text{edges of } \Delta\}$, relations

$$[\, z_e \, , \, z_f \,] \, = \, n_{ef} \, ,$$

realized on $\mathcal{H}(\Sigma) = L^2(\mathbb{R}^{3g-3+n})$.

Modular duality of quantum Liouville – Teichmüller theory

Key issue: Change of coordinate system $\Leftrightarrow$ Change of triangulation.
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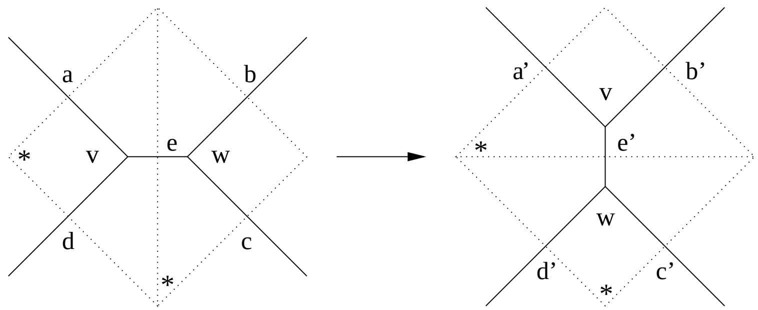

$$\mathsf{u}_{a'} = (1+q\mathsf{u}_e)\mathsf{u}_a \qquad \mathsf{u}_{e'} = \mathsf{u}_e^{-1} \qquad \mathsf{u}_{b'} = (1+q\mathsf{u}_e^{-1})^{-1}\mathsf{u}_b$$
$$\mathsf{u}_{d'} = (1+q\mathsf{u}_e^{-1})^{-1}\mathsf{u}_d \qquad \mathsf{u}_{c'} = (1+q\mathsf{u}_e)\mathsf{u}_c$$

Need operator W_e on $\mathcal{H}(\Sigma)$ such that

$$\mathsf{W}_e^{-1}\cdot\mathsf{u}_a\cdot\mathsf{W}_e \;=\; (1+q\mathsf{u}_e)\mathsf{u}_a \quad \text{etc..}$$

Such an operator can be constructed as $\mathsf{W} = W(\mathsf{u}_e)$ if $W(x)$ solves the functional equation

$$W(q^2x) \;=\; \big(1+qx\big)W(x)\,.$$

Mathematical root of self-duality

We need to solve a simple quantum mechanics exercise. Given operators p, q, $[\mathsf{p},\mathsf{q}] = (2\pi i)^{-1}$, construct representation of Weyl-algebra $\mathsf{u}\mathsf{v} = q^2\mathsf{v}\mathsf{u}$, $q = e^{i\pi b^2}$, as $\mathsf{u} = e^{2\pi b\mathsf{q}}$, $\mathsf{v} = e^{2\pi b\mathsf{p}}$. Does there exist a unitary operator W on $L^2(\mathbb{R})$ such that

$$\mathsf{W}^{-1}\cdot\mathsf{u}\cdot\mathsf{W} \;=\; \mathsf{u} + \mathsf{v}^{\frac{1}{2}}\,\mathsf{u}\,\mathsf{v}^{\frac{1}{2}} \;=\; (1+q\mathsf{v})\mathsf{u} \quad ?$$

Try ansatz $\mathsf{W} = W(\mathsf{v})$. Rewrite $(W(\mathsf{v}))^{-1}\cdot\mathsf{u}\cdot W(\mathsf{v}) \;=\; \frac{W(q^2\mathsf{v})}{W(\mathsf{v})}\mathsf{u}$. The ansatz would solve the problem if function W would satisfy the functional equation

$$W(q^2x) \;=\; \big(1+qx\big)W(x)\,.$$

This is solved formally by the power series

$$W_q(x) \;=\; \sum_{n=0}^{\infty}\frac{1}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}\big(-qx\big)^n\,.$$

This series does not converge, **small denominator problem !!!**

Fortunately there is a better solution to the functional equation

$$W(q^2 x) \;=\; \big(1 + qx\big) W(x)\,,$$

namely $W(e^{2\pi b y}) = e_b(y)$, $e_b(y)$: **self-dual quantum dilogarithm**, defined by

$$\begin{aligned} e_b(y) \;&=\; \exp\left(- \int_{\mathbb{R}+i0} \frac{dt}{4t} \, \frac{e^{-2ity}}{\sinh bt \sinh b^{-1} t} \right) \,. \\ &= \frac{W_q(x)}{W_{\tilde{q}}\big(x^{1/b^2}\big)} \quad \text{for} \;\; |q| < 1, \;\; \text{where} \;\; \tilde{q} = e^{\pi i b^{-2}}\,, \end{aligned}$$

Observation: **Self-duality** solves **small denominator problem** !

Main point:

Self-duality solves **small denominator problem** of Liouville theory !

Equivalence between Liouville theory and Teichmüller theory

Consider $\mathcal{T}_{g,n}$: Teichmüller space of Riemann surfaces with genus g and n conical singularities, deficit angles $\eta_k = b\alpha_k$. Teichmüller space has a Kähler structure. Let $\mathbf{m} = (m_1, \ldots, m_{3g-3+n})$ be complex analytic coordinates for $\mathcal{T}_{g,n}$.

Claim (J.T.):

There exists a canonical Kähler quantization of $\mathcal{T}_{g,n}$, characterized by the deformed **Bergmann kernel** $B_\Sigma(\bar{\mathbf{n}}, \mathbf{m})$ (Karabegov $\Rightarrow$ star product etc.). We then have

$$\boxed{B_\Sigma(\bar{\mathbf{m}}, \mathbf{m}) \; = \; \big\langle \, V_{\alpha_n}(z_n, \bar{z}_n) \ldots V_{\alpha_1}(z_1, \bar{z}_1) \, \big\rangle_{\Sigma_{\mathbf{m}}}}$$

Bear in mind:

- Quantum Liouville theory build from quantum dilogarithm: **self-dual**
- Duality: Solution of **small denominator problems** coming from exponential interactions.

The duality Liouville-Cigar: q-Geometric Langlands I

Cigar comes from $\widehat{\mathfrak{sl}}_2$-**WZNW model** via coset construction.

$\widehat{\mathfrak{sl}}_2$-**WZNW model:** Conformal field theory with current algebra symmetry $\widehat{\mathfrak{sl}}_{2,k}\times\widehat{\mathfrak{sl}}_{2,k}$,

- **Observables:** $\Phi^j(\mu|z)$,
- **Correlation functions:** $\left\langle\, \Phi^{j_n}(\mu_n|z_n)\dots\Phi^{j_1}(\mu_1|z_1)\,\right\rangle$

$$\left\langle\, \Phi^{j_n}(\mu_n|z_n)\cdots\Phi^{j_1}(\mu_1|z_1)\,\right\rangle_{\widehat{\mathfrak{sl}}_2} =$$
$$\frac{\pi b}{2(-\pi)^n}\,\delta^{(2)}\big(\textstyle\sum_{i=1}^n\mu_i\big)\,|\Theta_n|^2\left\langle\, V_{\alpha_n}(z_n)\cdots V_{\alpha_1}(z_1)V_{-\frac{1}{2b}}(y_{n-2})\cdots V_{-\frac{1}{2b}}(y_1)\,\right\rangle_{\rm Liou}$$

- Function Θ_n: explicitly known, simple.
- Variables μ_r related to $y_1,\dots,y_{n-2},u$ via $\displaystyle\sum_{i=1}^n\frac{\mu_i}{t-z_i}=u\frac{\prod_{j=1}^{n-2}(t-y_j)}{\prod_{i=1}^n(t-z_i)}$.
- Identification of parameters: $b^2=(k-2)^{-1}$, $\alpha_i=b(j_i+1)+1/b$.

The duality Liouville - $\widehat{\mathfrak{sl}}_2$-WZNW: q-Geometric Langlands II

Similar relation holds for the holomorphic/antiholomorphic building blocks of correlation functions on a Riemann surface X — the **conformal blocks.**

Liouville conformal blocks parameterized by monodromy representation of additional **degenerate** field $V_{-1/2b}(z)$ inserted into the conformal blocks: **Local system** λ.

The Liouville-$\widehat{\mathfrak{sl}}_2$-WZNW correspondence assigns to each Liouville conformal block a solution Ψ_λ to the KZ-equations:

$$(k-2)\partial_{m_r}\Psi_\lambda = \mathcal{D}_r\Psi_\lambda\,, \quad \mathcal{D}_r: \text{ Differential operator on } \mathrm{Bun}_G.$$

For $k \to 2$: Degeneration into eigenvalue equation $\mathcal{D}_r\Psi_\lambda = E_r(\lambda)\Psi_\lambda$ for quantized **Hitchin Hamiltonians** $\mathcal{D}_r$. System of these eigenvalue equations $\Leftrightarrow$ $\mathcal{D}$**-module** $\mathcal{E}_\lambda$.

So the Liouville-$\widehat{\mathfrak{sl}}_2$-correspondence gives $G = SL(2)$-case of correspondence:

$$\boxed{\text{Local systems } \lambda \text{ in } {}^LG} \leftrightarrow \boxed{\mathcal{D}\text{-modules } \mathcal{E}_\lambda \text{ on } \mathrm{Bun}_G}$$

That's the **geometric Langlands correspondence** (`Beilinson, Drinfeld...`) !

The duality Liouville - $\widehat{\mathfrak{sl}}_2$-WZNW: q-Geometric Langlands III

The Liouville-$\widehat{\mathfrak{sl}}_2$-WZNW correspondence is rich enough to reproduce geometric Langlands duality in the limit $k \to 2$:

Consider left hand side – **Liouville conformal block**

$$\left\langle \Psi_{\alpha_n}(z_n) \cdots \Psi_{\alpha_1}(z_1) \Psi_{-\frac{1}{2b}}(y_{n-2}) \cdots \Psi_{-\frac{1}{2b}}(y_1) \right\rangle_{\rm Liou}$$

Insert extra degenerate field to probe intermediate representations:

$$\psi(t) \equiv \left\langle \Psi_{-\frac{1}{2b}}(t) \Psi_{\alpha_n}(z_n) \cdots \Psi_{\alpha_1}(z_1) \Psi_{-\frac{1}{2b}}(y_{n-2}) \cdots \Psi_{-\frac{1}{2b}}(y_1) \right\rangle_{\rm Liou}$$

It satisfies

$$\big(\partial_t^2 - \langle\!\langle\, T(t) \,\rangle\!\rangle\big)\psi(t) = 0\,, \qquad \langle\!\langle\, T(t) \,\rangle\!\rangle \equiv \big\langle T(t) \Psi_{-\frac{1}{2b}}(t) \ldots \big\rangle \big/ \big\langle \Psi_{-\frac{1}{2b}}(t) \ldots \big\rangle$$

Limit $k \to 2 \Rightarrow$ **oper**, differential operator of the form $\partial_t^2 + \sum_{m=1}^{n} \big(\frac{\delta_m}{(t-z_m)^2} + \frac{c_m}{t-z_m}\big)$.

For $g \geq 0$: Conf. blocks parameterized by $3g-3+n$ parameters:
Same as for space of **opers** !

The duality Liouville - $\widehat{\mathfrak{sl}}_2$-WZNW: q-Geometric Langlands IV

Generalize left hand side to

$$\Big\langle\, \Psi_{\alpha_n}(z_n)\cdots\Psi_{\alpha_1}(z_1)\,\Psi_{-\frac{b}{2}}(q_n)\cdots\Psi_{-\frac{b}{2}}(q_1)\,\Psi_{-\frac{1}{2b}}(y_{n-2})\cdots\Psi_{-\frac{1}{2b}}(y_1)\,\Big\rangle_{\rm Liou}$$

This is mapped to

$$\Big\langle\, \Phi^{j_n}(\mu_n|z_n)\cdots\Phi^{j_1}(\mu_1|z_1)\Phi^{\frac{1}{2}}(\nu_n|x_n)\cdots\;\Phi^{\frac{1}{2}}(\nu_1|x_1)\,\Big\rangle_{\widehat{\mathfrak{sl}}_2}$$

For $k\longrightarrow 2$ we now get family of differential operators, for $g=0$:

$$\partial_t^2+\sum_{m=1}^{n}\left(\frac{\delta_m}{(t-z_m)^2}+\frac{H_m}{t-z_m}\right)+\sum_{m=1}^{n}\left(\frac{3}{4(t-q_m)^2}+\frac{p_m}{t-q_m}\right),$$

where $H_m=H_m(\mathbf{z},\mathbf{q},\mathbf{p})$ (`Garnier system`).

In general $(g\geq 0)$ we get $6g-6+2n$ complex parameters, as many parameters as **space of local systems** has!

The duality Liouville - $\widehat{\mathfrak{sl}}_2$-WZNW: q-Geometric Langlands V

Electric Hecke operators:

Insertion of $\Psi_{-\frac{1}{2b}}(t)$ in $\langle \dots \rangle_{\rm Liou}$	$\Leftrightarrow$	Insertions of $\int d\mu\, \Phi^{-\frac{k}{2}}(\mu\vert t)$ in $\langle \dots \rangle_{\widehat{\mathfrak{sl}}_2}$

- Compatible with Hecke-action in terms of bundle modifications.
- Relates conformal blocks with different amount of winding number violation.
- Reproduces Hecke eigenvalue property for $k \to 2$ since $\Psi_{-\frac{1}{2b}}(t)$ then factors out.

Magnetic Hecke operators:

Insertion of $\Psi_{-\frac{b}{2}}(t)$ in $\langle \dots \rangle_{\rm Liou}$	$\Leftrightarrow$	Insertions of $\Phi^{\frac{1}{2}}(x\vert t)$ in $\langle \dots \rangle_{\widehat{\mathfrak{sl}}_2}$

How about Hecke eigenvalue property for $k \neq 2$?

The duality Liouville - $\widehat{\mathfrak{sl}}_2$-WZNW: q-Geometric Langlands VI

Recall: quantum Liouville theory $\Leftrightarrow$ quantum **Teichmüller theory**.

On the level of conformal blocks this means

$$\big[\,\mathcal{H}_{\rm Liou}(\Sigma)\,,\,\pi_{\rm Liou}(\Sigma)\,\big]\;\simeq\big[\,\mathcal{H}_{\rm Teich}(\Sigma)\,,\,\pi_{\rm Teich}(\Sigma)\,\big]$$

- $\big[\,\mathcal{H}_{\rm Liou}(\Sigma)\,,\,\pi_{\rm Liou}(\Sigma)\,\big]$ Hilbert space of conformal blocks with natural mapping class group representation,
- $\big[\,\mathcal{H}_{\rm Teich}(\Sigma)\,,\,\pi_{\rm Teich}(\Sigma)\,\big]$ space of states of quantum Teichmüller theory with natural mapping class group representation,

How is **Hecke** action represented in quantum Teichmüller theory?

The duality Liouville - $\widehat{\mathfrak{sl}}_2$-WZNW: q-Geometric Langlands VII

- Insertion of $\Psi_{-\frac{1}{2b}}(t)$ translates into operator: $\mathbf{H}_t : \mathcal{H}_{\rm Teich}(\Sigma) \to \mathbb{C}^2 \otimes \mathcal{H}_{\rm Teich}(\Sigma)$.

- Variation of t generates **quantum local system**, collection of operators M_γ : $\mathbb{C}^2 \otimes \mathcal{H}_{\rm Teich}(\Sigma) \to \mathbb{C}^2 \otimes \mathcal{H}_{\rm Teich}(\Sigma)$ for all generators γ of the fundamental group.

- The operators M_γ, $\mathsf{M}_{\gamma'}$ do not commute unless $\gamma \circ \gamma' = 0$.

- Natural bases for $\mathcal{H}_{\rm Teich}(\Sigma)$ are defined by choosing max. set $\mathcal{C}$ of nonintersecting closed curves (pants decomposition) and requiring that

$$\mathsf{M}_\gamma \cdot \Psi_a \;=\; (M_a \otimes \mathrm{id})\, \Psi_a \,, \qquad \forall \gamma \in \mathcal{C} \,,$$

  where M_a is a $2{\times}2$-matrix.

This is mapped to the corresponding property of the $\widehat{\mathfrak{sl}}_2$-conformal blocks —

quantum Hecke eigenvalue property !

The duality Liouville - $\widehat{\mathfrak{sl}}_2$-WZNW: q-Geometric Langlands VIII

We see that:

quantum Teichmüller theory: proper home for **quantum local systems**

(operators on $\mathcal{H}(\Sigma)$ corresponding to representations of the fundamental group).

So quantum geometric Langlands $\leftrightarrow$

$$\boxed{\text{quantum local systems } \lambda \text{ in } {}^L G} \leftrightarrow \boxed{\text{Hecke eigen-}\mathcal{D}\text{-modules on } \mathrm{Bun}_G}$$

Concluding remarks

Note that **self-duality** of Liouville theory allows us to continue the q-geometric Langlands correspondence to the left:

$\mathfrak{sl}_{2,k_m}$ – WZNW model	$\leftrightarrow$	q-Teichmüller theory	$\leftrightarrow$	$\widehat{\mathfrak{sl}}_{2,k_e}$ – WZNW model

$$(k_m - 2) \;=\; b^2 \;=\; (k_e - 2)^{-1}\,.$$

On the left we see local systems emerging in the limit $k_e \to 2 \Leftrightarrow k_m \to \infty$ very naturally:

$$\partial_A \;\equiv\; \partial_t - \langle\!\langle\, J(t) \,\rangle\!\rangle\,, \qquad \langle\!\langle\, J(t) \,\rangle\!\rangle \;\equiv\; \frac{\langle J(t) \ldots \rangle_{\mathfrak{sl}_{2,k_m}}}{\langle\, \mathrm{id}\, \ldots \rangle_{\mathfrak{sl}_{2,k_m}}}\,,$$

The local system defined by this connection is an operator on the space of conformal blocks in general, but becomes classical for $k_m \to \infty$.

$\Rightarrow$ **Explanation** of geometric Langlands from

- $\widehat{\mathfrak{sl}}_2$-WZNW-Liouville duality
- Modular duality of quantum Teichmüller theory

Higher rank

There exists generalization of (quantum) Teichmüller theory where the role of $PSL(2,\mathbb{R})$ is taken by a real reductive group $G_{\mathbb{R}}$ (`Fock, Goncharov`).

The main result is the construction of an assignment

$$\mathcal{F}_{G,b} \;:\; \Sigma \;\mapsto\; \big[\,\mathcal{H}_{G,b}(\Sigma)\,,\,\pi_{G,b}(\Sigma)\,\big]\,,$$

where b is a (deformation) parameter and $\pi_{G,b}(\Sigma)$ is a representation of the mapping class group of Σ.

Fock and Goncharov show that

$$\boxed{\mathcal{F}_{G,b} \;=\; \mathcal{F}_{{}^LG,b^{-1}}}$$

Modular duality and **Langlands duality** are deeply related !!!