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Chiral de Rham complex (CDR) and generalized geometry

w/ Heluani 0812.4855

1) CDR ← sheaf of vertex algebras
2) generalized geometry, generalized complex str.
3) N=2 SUSY on CDR ⇒ generalized Calabi-Yau
4) speculations on the relation with non-linear sigma model

vertex algebra $(V, |0\rangle, \partial, Y)$

$$Y: V \to V[[z, z^{-1}]]$$
$$a \to a(z)$$

1) $\partial|0\rangle = 0$

id $\longleftrightarrow |0\rangle$, $\quad a(z)|0\rangle\Big|_{z=0} = a$

2) $[\partial, a(z)] = \partial_z a(z)$

3) $(z-w)^N [a(z), b(w)] = 0$, $N \gg 0$.

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$$a(z)\,b(w) = \sum_{j=1}^{N} \frac{c_j(w)}{(z-w)^j} + :a(z)\,b(w):$$

$$c_j(w) \to a_{(n)}b$$

$$_{(n)} : V \times V \to V$$

<u>conformal vertex algebra</u>

$L \to L(z)$, Virasoro

$L_0 \quad V = \bigoplus_i V_i$

$\Rightarrow V_0, V_1, |0\rangle, \partial, (-1), (0), (1)$

(Bressler) 1-truncated vertex algebra

$v, w \in V_1$

((Courant-Dorfman algebra))

$v_{(0)}w = v * w$ ← Leibniz on V_1

$v_{(1)}w = \langle v, w\rangle : V_1 \times V_1 \to V_0$

$v*(w*\lambda) = (v*w)*\lambda + w*(v*\lambda)$

$f, g \in V_0 \longrightarrow f_{(-1)}g = fg$ – commutative algebra

<u>CDR</u> M

$\mathbb{R}^n \quad \gamma^\mu(z), \beta_\mu(z), c^\mu(z), b_\mu(z)$

$$\gamma^\mu(z)\beta_\nu(w) = \frac{\delta^\mu_\nu}{z-w} + \cdots$$

$$c^\mu(z)\,b_\nu(w) = \frac{\delta^\mu_\nu}{z-w} + \cdots$$

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$$[\gamma^\mu{}_n, \beta_{\nu m}] = \delta_{n+m} \delta^\mu_\nu$$

$$\{c^\mu{}_n, b_{\nu m}\} = \delta_{n+m} \delta^\mu_\nu$$

$$\mathbb{C}[\gamma_0^1, \ldots, \gamma_0^n] \otimes \Lambda[c_0^1, \ldots, c_0^n] = \Omega^\cdot(\mathbb{R}^n)$$

$$z, \theta \qquad \theta^2 = 0$$

$$\Phi^\mu(z,\theta) = \gamma^\mu(z) + c^\mu(z)\theta$$

$$S_\mu(z,\theta) = b_\mu(z) + \theta \beta_\mu(z)$$

$$\tilde{\Phi}^\mu = f(\Phi^\mu), \quad S_\mu \text{ is 1-form}$$

$\Rightarrow$ sheaf of vertex algebras (CDR) $\uparrow_{N=1}$ $(V_0 V_1, \partial, (-1), (0), (1))$

$$V_0 \leftarrow C^\infty(M)$$

$$V_1 \leftarrow \Gamma(T+T^*)$$

$$[v_1+w_1, v_2+w_2] = (v_1+w_1) * (v_2+w_2) + D\delta(\ )\langle v_1+w_1, v_2+w_2\rangle$$

$$v + w = v^\mu S_\mu + w_\mu D\Phi^\mu$$

$$D = \partial_\theta - \theta\partial_z$$

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$$(v_1+w_1)*(v_2+w_2) = \{v_1, v_2\} + \mathcal{L}_{v_1} w_2 - i_{v_2} dw_1$$ Dorfman bracket

$$\langle v_1+w_1, v_2+w_2\rangle = i_{v_1} w_2 + i_{v_2} w_1$$

Leibniz bracket

$$\Gamma(T+T^*) \times \Gamma(T+T^*) \longrightarrow C^\infty(M)$$

$A, B \in$

$$A*(fB) = f(A*B) + \langle A, df\rangle B$$

$$\langle A, d\langle B, C\rangle\rangle = \langle A*B, C\rangle + \langle B, A*C\rangle$$

Courant algebroid

$$\begin{array}{ccc} E & \xrightarrow{\pi} & TM \\ \downarrow & & \downarrow \\ M & & M \end{array}$$

π - anchor

Leibniz

$\Gamma(E)$, $*$, $\langle\,,\,\rangle$

$$\langle\,,\,\rangle : \Gamma(E) \times \Gamma(E) \to C^\infty(M)$$

$$d : C^\infty(M) \to \Gamma(E)$$

$$A*B + B*A = d\langle A, B\rangle$$

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Courant algebroid $\longrightarrow$ sheaf of SUSY vertex algebras $\mathcal{U}^{ch}(E)$

$$C^\infty(M) \hookrightarrow \mathcal{U}^{ch}(E)$$

$$\Gamma(\Pi E) \hookrightarrow \mathcal{U}^{ch}(E)$$

Comment $C^\infty(T^* \mathcal{L}M), \{\ ,\ \}$

$$\omega = \int d\sigma\, d\theta\ d\Phi^\mu \wedge dS_\mu + \dots$$

$$J = v^\mu S_\mu + \omega_\mu D\Phi^\mu$$

$T+T^*, *\ \langle\ ,\ \rangle$

$$[A,B]_c = \tfrac{1}{2} A * B - \tfrac{1}{2} B * A$$

2002 (Hitchin)

$$(T+T^*)\otimes\mathbb{C} \qquad \mathcal{J}^2 = -1, \ \mathcal{J} \text{ respects } \langle\ ,\ \rangle$$

$$= L + \bar{L}$$

almost generalized complex structure

generalized complex structure

max. isotropic wrt $\langle\ ,\ \rangle$

L is involutive w.r.t. $*$:

$$A, B \in \Gamma(L) \Rightarrow A * B \in \Gamma(L).$$

$$\mathcal{J} = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix}$$

$\uparrow$

gen. comp. str. $\Leftrightarrow$ $J : TM \to TM$

J is complex structure

$$\mathcal{J} = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

$\omega \in \Omega^2(M)$ $\Leftrightarrow$ ω - symplectic

$E \;(T+T^*)$ $\quad N=2$ algebra on global section of CDR

$\Rightarrow$ a) M is generalized complex manifold

b) generalized Calabi-Yau condition.

$$(T+T^*)_{\mathbb{C}} = L + \bar{L}$$

$(v+w)\cdot\rho = i_v\rho + w\wedge\rho$, Clifford action

ρ - pure spinor

$\forall A \in \Gamma(L)$, $A\cdot\rho = 0$.

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generalized CY condition

$$\exists\ \rho \in \Omega^{\cdot}(M),\quad d\rho = 0 \quad (\text{or } (d+H\wedge)\rho = 0)$$

ρ - pure spinor. $(\rho, \bar\rho) \neq 0$.

1) complex example for GCY

$$\rho = \Omega^{n,0}, \text{ closed } \quad d\Omega^{n,0} = 0$$

2) symplectic example

$$\rho = e^{i\omega}, \quad d\rho = 0.$$

topological twist

Lie algebroid

$$(\mathcal{U}^{ch}(E), Q_{BRST}) \rightarrow (\Lambda^{\cdot} \bar{L}, d_L)$$

sheaf of Poisson vertex algebras:
local functionals in $C^\infty(T^* \mathcal{L}M)$
(Hamiltonian)

✓ Dorfmann bracket, N=2 GCM

$H \leftarrow$ hamiltonian

$$\frac{dO}{dt} = \{O, H\}$$

CDR (sheaf of susy vertex alg.)
(quantum Hamiltonian)

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CY $\rightarrow$ two copies 1 N=2

H $\frac{3}{2}$ dim M.

$$\frac{dO}{dt} = [O, H]$$

full Hamiltonian of nonlinear sigma model

$$L_0 + \bar{L}_0 = h$$