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M. Zabzine

General 3D TFTs, BV and knot invariants

Chern-Simons theory (pert. theory)

$$Z = \int \mathcal{D}A \; e^{\frac{1}{\hbar}\int A dA + \frac{2}{3}A^3}$$

$$\underset{AS}{\rightsquigarrow} \; Z = \int \mathcal{D}A \; e^{\frac{1}{\hbar}\int A^A D A^B \eta_{AB} + \frac{2}{3} f_{ABC} A^A A^B A^C}$$

color inf.

$$\overset{\hbar^{\#}}{=} \sum_{\Gamma} \hbar^{\frac{v}{2}} V_\Gamma = \sum_{\Gamma} c_\Gamma b_\Gamma$$

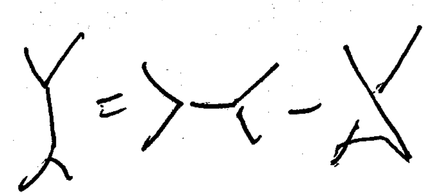
← integrals over propagators

↖ trivalent graphs

$\Gamma =$  $\quad c_\Gamma = f_{A_1A_2A_3} f_{B_1B_2B_3} \eta^{A_1B_1} \eta^{A_2B_2} \eta^{A_3B_3}$

$b_\Gamma = \int\int$

IHX relation



$c_I = c_H - c_X$

Jacobi ident.

<u>graph complex</u> — without ext legs

— orient graphs

— $(\Gamma, o) = -(\Gamma, -o)$

$$\partial\big(\,)\!-\!(\,\big) = \pm \times$$

$$\sum_\Gamma c_\Gamma \Gamma = \text{graph cycle.}$$

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c_Γ weight system.

$$\boxed{\sum_{\Gamma^*} b_\Gamma \Gamma^* \quad \text{graph cocycle}}$$

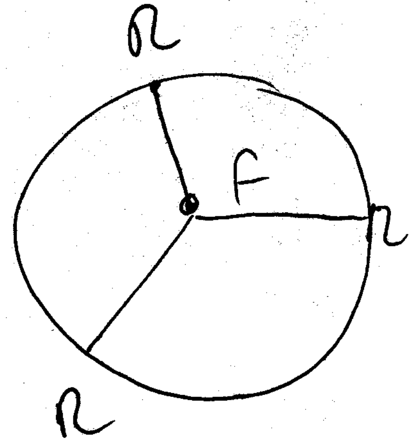
$$Z = \left\langle \sum_\Gamma c_\Gamma \Gamma, \sum_{\Gamma^*} b_\Gamma \Gamma^* \right\rangle$$

$$W_R(C) = \mathrm{Tr}_R\, P\exp\left(\oint A\right)$$

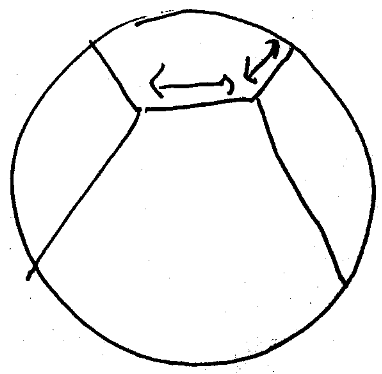
Kontsevich integral

$$W_R(C) = \sum_\Gamma c_\Gamma b_\Gamma$$

depend on $\mathfrak{g}$, R



$$c_\Gamma = R^\alpha_{A_1\beta} R^\beta_{A_2\gamma} R^\gamma_{A_3\alpha} f^{A_1A_2A_3}$$

∂ 

$\sum_{\Gamma} c_\Gamma \Gamma$ graph cycle

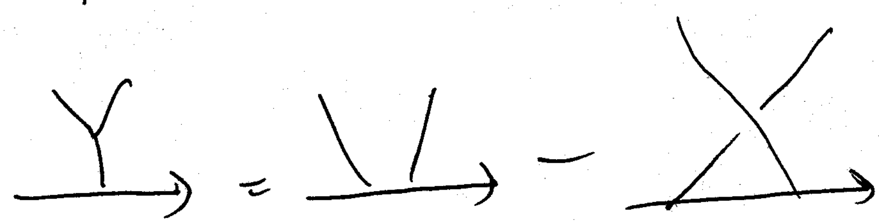
$\sum_{\Gamma^*} b_\Gamma \Gamma^*$ graph cocycle

$J_R(C)$

$\parallel$

$\langle W_R(C) \rangle = \sum \hbar^n V_n$ ⟵ knot invariant (Vassiliev invs.)

c_Γ = weight system (for knots)

STU 

Physics: many 3D TFTs $\rightsquigarrow L_\infty$

Math: all weight systems come from Lie alg.? (No)

gauge theories – BV formalism

$$\omega = \delta\Phi \wedge \delta\Phi^+ \leftsquigarrow -1$$

$$\{S, S\} = 0 \quad \text{classical master equation}$$

$$\int_{\mathcal{Z}} e^{-S}$$

AKSZ ('95)

Lie algebra $\quad \mathfrak{g}[1] \qquad \omega = \oint da^A \wedge da^B \eta_{AB}$

$\uparrow$ Symp structure of degree 2

$\Theta = f_{ABC}\, a^A a^B a^C \quad$ deg 3

$\{\Theta, \Theta\} = 0 \iff$ Jacobi identity

$$(\underset{1}{\mathfrak{g}[1]},\ \underset{2}{\omega},\ \underset{3}{\Theta})$$

$T[1]\Sigma_3 \to \mathfrak{g}[1].$

$\theta \quad \xi$

$d\xi$

$\mathbb{A}^A(\xi, \theta)$

$$\omega = \int d^3\theta\, d^3\xi\, \delta\mathbb{A}^A \wedge \delta\mathbb{A}^B \eta_{AB} \leftarrow -1$$

$$S = \int d^3\theta\, d^3\xi\, \Big(\mathbb{A}^A D\mathbb{A}^B \eta_{AB} + \tfrac{2}{3} \underbrace{f_{ABC}\, \mathbb{A}^A \mathbb{A}^B \mathbb{A}^C}_{\Theta}\Big)$$

$$D = \theta^\alpha \frac{\partial}{\partial \xi^\alpha}$$

$$(\mathcal{M}, \underset{2}{\omega}, \underset{3}{\Theta}) \qquad \{\Theta, \Theta\} = 0 \quad \Big\} \text{ Courant algebroid}$$

$$S = \int d^3\theta\, d^3\xi \left(\Phi^A D \Phi^B \omega_{AB} + \Theta(\Phi) \right)$$

$W_R(C)$

$$R_A R_B - R_A R_B = f_{AB} R_C \qquad \text{Lie alg.}$$

$$Q = f_{AB}^{\ C} a^A a^B \frac{\partial}{\partial a^C} \qquad Q^2 = 0$$

$$q = f_{AB}^{\ C} a^A a^B \frac{\partial}{\partial a^C} + 2 \underbrace{R_{A\ \beta}^{\ \alpha} a^A}_{R^\alpha_\beta(a)} \xi^\beta \frac{\partial}{\partial \xi^\alpha} \qquad q^2 = 0.$$

$(\mathcal{M}, \omega, \Theta)$

Q = Hamiltonian vector field for Θ

$$\boxed{Q \cdot R = -RR}$$

$T[1]M$

$Q = V^\mu \frac{\partial}{\partial x^\mu}$

$q = V^\mu \frac{\partial}{\partial x^\mu} + \Gamma^\alpha_{\mu\beta}(x) V^\mu \xi^\alpha \frac{\partial}{\partial \xi^\beta}$

connection on E

bundle $E \downarrow M$

$q^2 = 0 \iff d\Gamma + [\Gamma, \Gamma] = 0$

M complex manifold

$T^{(1,0)}[1]M$

$Q = V^{\bar{\imath}} \frac{\partial}{\partial x^{\bar{\imath}}} \Rightarrow$ representation : holomorphic vector bundles over M.

$(\mathcal{M}, \omega, \Theta)$

$R \quad E \to \mathcal{M}$

$QR = -RR$

$W_R(C) = \mathrm{Tr}\, P\exp \int_C dt\, d\theta\, R$

BV-invariant

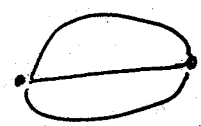
$$Z = \int_{\mathcal{L}} \mathcal{D}\Phi \; e^{\int \Phi^A D \Phi^B \omega_{AB} + H(\Phi)}$$

$x_0 \in \mathcal{M}$

$\partial_{A_1} \cdots \partial_{A_k} H(x_0)$

$\sum_\Gamma C_\Gamma(x_0)\, b_\Gamma \in H_Q(\mathcal{M})$

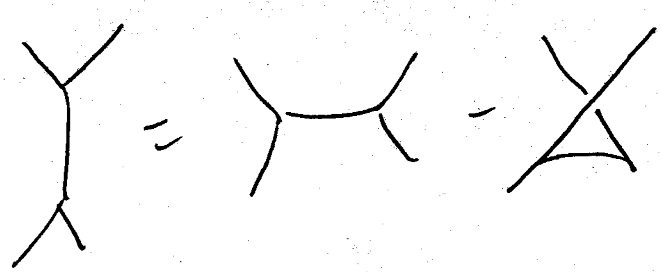
function

$C_\Gamma(x_0)$ 

$\partial_{A_1}\partial_{A_2}\partial_{A_3} H(x_0)\, \partial_{B_1}\partial_{B_2}\partial_{B_3} \cdot H(x_0)\, \omega^{A_1B_1}\omega^{A_2B_2}\omega^{A_3A_3}$

$b_\Gamma \leftarrow$ the same as in CS theory

IHX



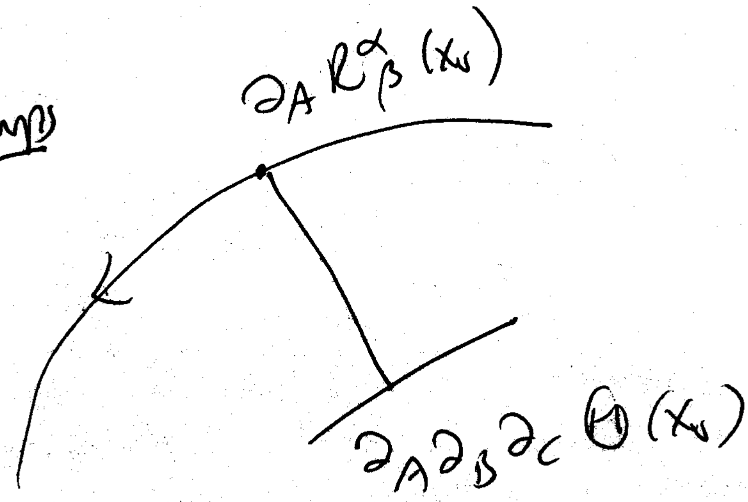
+ Q exact terms

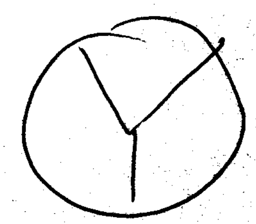
$Q^A \partial_A = \omega^{AB} \partial_B (H)(x_0)$

$\sum_\Gamma C_\Gamma(x_0)\Gamma$ - cycle in $HQ(\mathcal{M})$

$$\int_{\mathcal{L}} \mathcal{D}\Phi \; e^{\int \Phi^A D\Phi^B \omega_{AB} + \Theta(\Phi)} \; \mathrm{Tr}\, P\exp \int_{T[1]C} \Omega$$

<u>Wilson loops</u>





$$\sum_\Gamma c_\Gamma(x_0)\, b_\Gamma$$

$\uparrow$ ω in Chern-Simons

$c_\Gamma(x_0)$ = new weight system with values in $H_Q(\mathcal{M})$

Let M be hyperkähler (or holo sympl. manifold)

$\mathcal{M} = T^*[2]\, T^*[1] M.$

$$\omega = dx^\mu + dp_\mu + dq_\mu + dr^\mu + \Omega_{ij}\, dx^i \wedge dx^j$$

degree 2

0 2

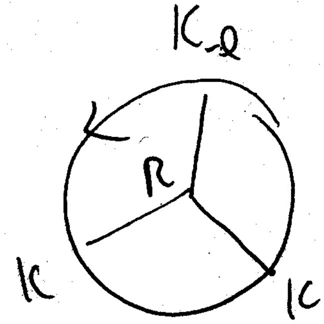
$\Theta = p_{\bar{i}} v^{\bar{i}} \leftrightsquigarrow \bar{\partial}$

$$S = \int d^3\theta\, d^3\xi \left(P_\mu D X^\mu + v^\mu D q_\mu + \Omega_{ij} X^i D X^j + P_{\bar{i}} v^{\bar{i}} \right)$$

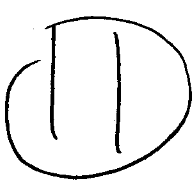
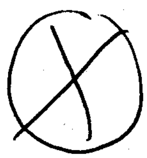
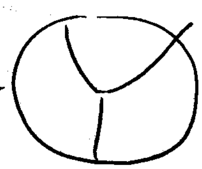
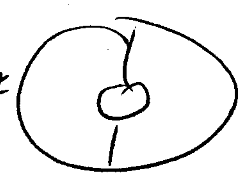
$$\begin{array}{l} E \quad \text{holo bundle} \\ \downarrow \\ M \end{array}$$

$R_{\bar{i} l_1 l_2 l_3} = R_{\bar{i} l_1 l_3}{}^{n}\, \Omega_{n l_2}$

$\nabla_l (K_{\bar{i}})^I{}_J \equiv K_{\bar{i} l}{}^I{}_J$



$\sim (\ )^{(0,3)}$

$\#$  $+ \#$  $+ \#$  $+ \#$  $CH_{\bar{\partial}}(M)$

$$\mathrm{Tr}\; P\exp\Big(\oint K_{\bar{\lambda}} \cancel{A}^{\bar{\lambda}}$$