Kavli Institute for Theoretical Physics

Magnetic Field Generation in Experiments, Geophysics and Astrophysics

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Rotating Hydromagnetic Instabilities with Anisotropic Diffusivities at Large Prandtl Number

Main task

To study
how anisotropic diffusive coefficients
(viscosity and thermal diffusivity)
can influence various models of rotating magnetoconvection
in horizontal planar fluid layer.

Stability analysis

We choose basic state,
which suitably represents Earth's core conditions,

to study the stability of the system and the conditions for onset of various hydromagnetic instabilities.

Diffusive instabilities

We focused our attention on special case of instabilities which are affected by diffusive processes.

For study of the dynamo in the Earth's core following physics is important

(1) dynamics of the system (Navier Stokes equation)
(2) Faraday electromagnetic induction (induction equation)
(3) Thermodynamics (equation of heat induction)

All of these equations contain terms which represent diffusion.

Active role of diffusive processes

We must realize ourselves

that naive understanding of the role of diffusive processes
- only in weak damping of arising instabilities -
is not sufficient.

Basic equilibrium

Diffusive processes are very often neglected.

Dynamics of the Earth's core is given by three basic forces:
Magnetic, Archimedean and Coriolis ($\boldsymbol{M}$, $\boldsymbol{A}$ and $\boldsymbol{C}$) forces.

However,

diffusive processes may weaken basic forces in the sense:

viscosity weakens only Coriolis force,
magnetic diffusivity weakens only magnetic force,
and thermal diffusivity weakens only Archimedean force.

Weak diffusive processes lead to

$$\boldsymbol{M} \rightarrow \boldsymbol{M} - \delta \boldsymbol{M}, \quad \boldsymbol{C} \rightarrow \boldsymbol{C} - \delta \boldsymbol{C}, \quad \boldsymbol{A} \rightarrow \boldsymbol{A} - \delta \boldsymbol{A}.$$

Turbulent diffusive coefficients

If we suppose turbulent state in the core then transport phenomena are connected with turbulent processes.

Unlike molecular diffusion (it is property of material), turbulent diffusion is highly dependent on characteristic flows. These flows are influenced by dominant forces M, A, and C

and

the shapes of turbulent eddies are affected by these forces.

Shape of turbulent eddies

By Braginsky and Meytlis (1990)

the eddies have a form of pancakes elongated in the direction of magnetic field (azimuthal direction) and in the direction of angular velocity **Ω**.

In geophysical fluid flows
the eddies are also influenced by density stratification determined by direction of gravity ***g***.

Deformation of the eddies

The eddies, influenced by Coriolis and magnetic forces, are deformed into the shapes in which the effect of these forces is minimal.

Elongating of the eddy in certain direction means that the velocity has the dominant component in this direction.

This means, that the Coriolis force $(\sim \Omega \times \boldsymbol{v})$ and magnetic force magnitude $(\sim |\boldsymbol{v} \times \boldsymbol{B}|)$ are very small and the forces can be neglected in this direction.

Anisotropic diffusive coefficients studies

Since the study by Braginsky and Meytlis (1990) there have been some approaches to investigate the role and consequences of introduced anisotropic diffusive coefficients into Dynamo and Magnetoconvection processes,

e.g. by St. Pierre (1996), Matsushima et al (1999), Donald and Roberts (2004), Phillips and Ivers (2000, 2005), …

Strategy of investigation

Our strategy is to modify old models
or to set up new models of rotating magnetoconvection
in the sense of introducing anisotropic diffusive coefficients.

The results are the conditions for the onset of instabilities
which are determined by new parameters -
- parameters of anisotropy of diffusive coefficients.

Our former results related to anisotropic studies

There are many common results for various basic magnetic fields

Our studies of hydromagnetic instabilities arising **in unstably stratified horizontal planar layer rotating around vertical axis** give some similar results related to the influence of anisotropic diffusive coefficients (in SA anisotropy case) for various basic magnetic fields, i. e. (a) Azimuthal magnetic field linearly growing with distance from rotation axis

(b) Vertical homogeneous magnetic field

(c) Horizontal homogeneous magnetic field.

SA anisotropy related to the greater diffusivities in vertical direction than in horizontal directions (a type anisotropy) facilitates convection and shortens sizes of rolls in horizontal directions.

o type anisotropy with the smaller diffusivity in vertical direction than in horizontal directions inhibites convection and elongates sizes of rolls in horizontal directions.

Anisotropy

In the case of turbulent transport phenomena
the shape of transporters – eddies
distinguish transport efficiency in various directions.

For that reason it is convenient to change the isotropic
transport phenomena, usual in molecular case,
into anisotropic ones.

It means:
for transport coefficients we have the transition
scalar quantities → tensor quantities

Various types of anisotropy

In our models of this contribution we introduced two types of anisotropy

(1) $\nu_{zz} > \nu_{xx} = \nu_{yy}$, SA - anisotropy,

determined by vertical direction of $\boldsymbol{g} \downarrow\uparrow \hat{\boldsymbol{z}}$

(2) $\nu_{zz} = \nu_{yy} > \nu_{xx}$, BM - anisotropy,

determined by vertical axis of rotation $\Omega \uparrow\uparrow \hat{\boldsymbol{z}}$

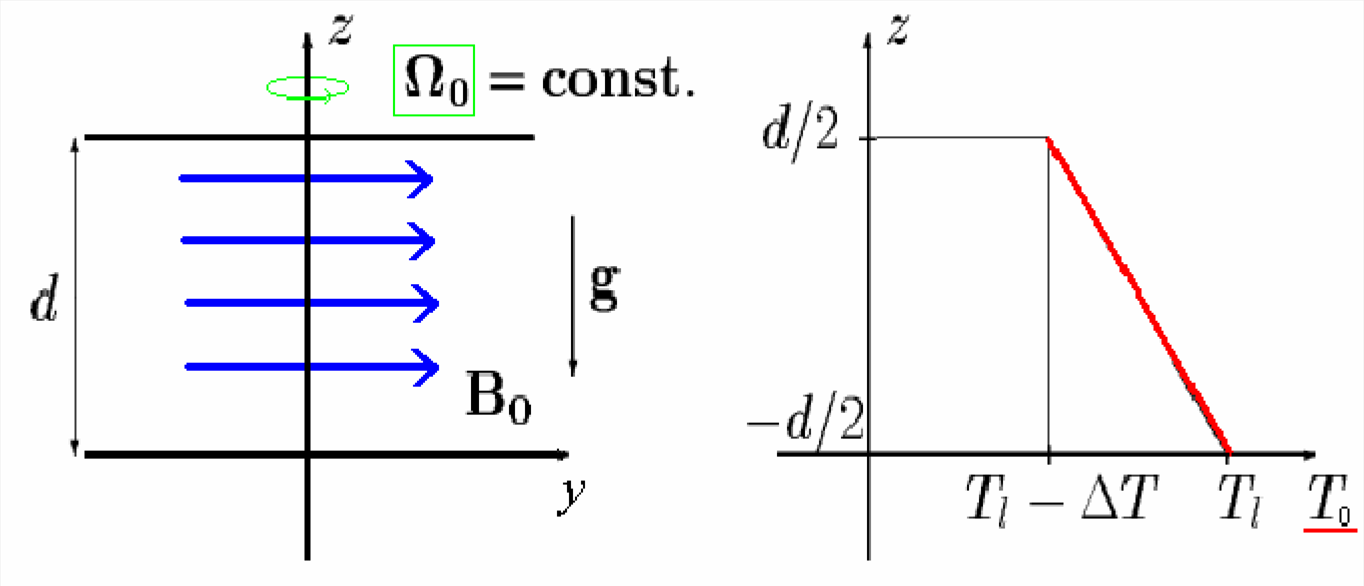
(or horizontal $\Omega \uparrow\uparrow \hat{\boldsymbol{x}}$ when BM $\equiv$ SA)

and horizontal direction of magnetic field $\boldsymbol{B} \uparrow\uparrow \hat{\boldsymbol{y}}$

SA - stratification anisotropy (analogy to the lower atmosphere)

BM - anisotropy corresponds to Braginsky and Meytlis (1990) model of the Earth's core turbulence ($\nu_{zz} \sim \nu_{\varphi\varphi} >> \nu_{ss}$).

Model of rotating magnetoconvection



Basic state

$$\boldsymbol{U}_0 = \boldsymbol{0}, \quad \boldsymbol{B}_0 = B_M \hat{\boldsymbol{y}}, \quad T_0 = T_l - \Delta T \frac{z + d/2}{d}.$$

Basic equations and dimensionless parameters

$$R_o \partial_t \boldsymbol{u} + \hat{\boldsymbol{z}} \times \boldsymbol{u} = -\nabla p + \Lambda (\nabla \times \boldsymbol{b}) \times \hat{\boldsymbol{y}} + R \tilde{\vartheta} \hat{\boldsymbol{z}} + E_z \nabla_\alpha^2 \boldsymbol{u},$$

($\hat{\boldsymbol{x}} \times \boldsymbol{u}$ represents Coriolis force for horizontal rotation axis)

$$\partial_t \boldsymbol{b} = \nabla \times (\boldsymbol{u} \times \hat{\boldsymbol{y}}) + \nabla^2 \boldsymbol{b},$$

$$(1/q_z)\partial_t \tilde{\vartheta} = \hat{\boldsymbol{z}} \cdot \boldsymbol{u} + \nabla_\alpha^2 \tilde{\vartheta},$$

$$\nabla \cdot \boldsymbol{u} = 0, \quad \nabla \cdot \boldsymbol{b} = 0.$$

$$E_z \nabla_\alpha^2 \boldsymbol{u} = \begin{cases} E_z \left[(\alpha_\nu - 1) \partial_{xx} + \nabla^2 \right] \boldsymbol{u} \\ E_z \left[(1 - \alpha_\nu) \partial_{zz} + \alpha_\nu \nabla^2 \right] \boldsymbol{u} \end{cases} \qquad \nabla_\alpha^2 \tilde{\vartheta} = \begin{cases} \left[(\alpha_\vartheta - 1) \partial_{xx} + \nabla^2 \right] \tilde{\vartheta} & \text{(BM type)} \\ \left[(1 - \alpha_\vartheta) \partial_{zz} + \alpha_\vartheta \nabla^2 \right] \tilde{\vartheta} & \text{(SA type)} \end{cases}$$

modified Rayleigh number $R = \dfrac{g \alpha_T \Delta T d}{2 \Omega_0 \kappa_{zz}}$, Ekman numbers $E_z = \dfrac{\nu_{zz}}{2\Omega_0 d^2}$, $E_x = \dfrac{\nu_{xx}}{2\Omega_0 d^2}$

Elsasser n. $\Lambda = \dfrac{\mathrm{B}_\mathrm{M}^2}{2\Omega_0 \mu \rho_0 \eta}$, Roberts ns $q_z = \dfrac{\kappa_{zz}}{\eta}$, $q_x = \dfrac{\kappa_{\mathrm{xx}}}{\eta}$, modified Rossby n. $R_\mathrm{o} = \dfrac{\eta}{2\Omega_0 d^2}$

anisotropic parameters $\alpha_\vartheta = \dfrac{\kappa_{\mathrm{xx}}}{\kappa_{\mathrm{zz}}}$, $\alpha_\nu = \dfrac{\nu_{\mathrm{xx}}}{\nu_{\mathrm{zz}}}$ (we supposed $\alpha_\vartheta = \alpha_\nu$).

Method of solution

Linear stability analysis is used.

We split velocity and magnetic field perturbations, $\boldsymbol{u}$ and $\boldsymbol{b}$, into poloidal and toroidal parts

$$\boldsymbol{u} = a^{-2}[\nabla\times(\nabla\times\tilde{w}\hat{\boldsymbol{z}}) + \underline{\nabla\times\tilde{\omega}\hat{\boldsymbol{z}}}] \text{ and } \boldsymbol{b} = a^{-2}[\nabla\times(\nabla\times\tilde{b}\hat{\boldsymbol{z}}) + \underline{\nabla\times\tilde{j}\hat{\boldsymbol{z}}}].$$

All perturbations $(\tilde{w},\ \tilde{\omega},\ \tilde{b},\ \tilde{j}$ and $\tilde{\vartheta})$ have the form

$$\tilde{f}(z,x,y,t) = \Re\mathrm{e}[F(z)\exp(ilx + imy)\exp(\lambda t)]$$

(where horizontal components of wave number, l and m, determine $a^2 = l^2 + m^2$).

Boundaries are stress free and perfectly thermally and electrically conducting.

Method of solution

We derived the basic dispersion relationship □ l^2 for horizontal rotation axis

$$Ra^2 \frac{q_z\left(K^2+\lambda\right)}{\left(q_z K_\alpha^2+\lambda\right)} = \frac{K^2\left[E_z\left(K^2+\lambda\right)K_\alpha^2+\Lambda m^2\right]^2+\pi^2\left(K^2+\lambda\right)^2}{E_z\left(K^2+\lambda\right)K_\alpha^2+\Lambda m^2}$$

where $K^2 = a^2+\pi^2, \quad K_\alpha^2 = \alpha a^2+\pi^2 \quad (SA), \quad K_\alpha^2 = \alpha a^2+\pi^2+(1-\alpha)m^2 \quad (BM)$

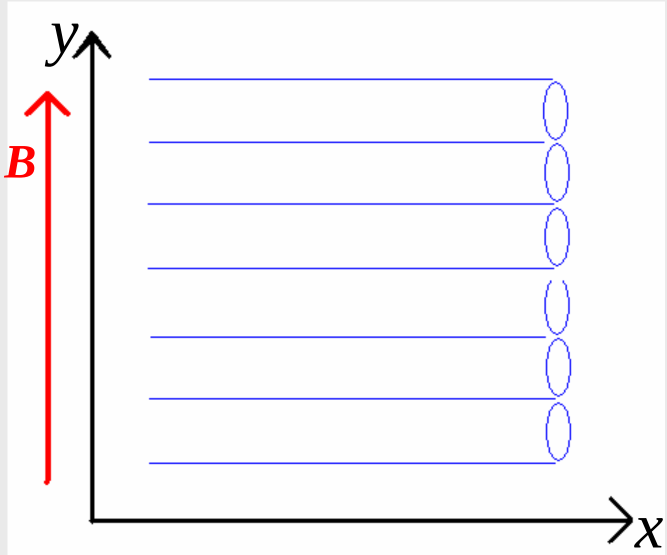
The aim of investigation is to determine, for given $E,\ \Lambda,\ q$ and $\alpha,$ the preferred mode.

1st step $\rightarrow$ to find, for every l and m, the marginal modes for which $\mathrm{Re}(\lambda)=0$.

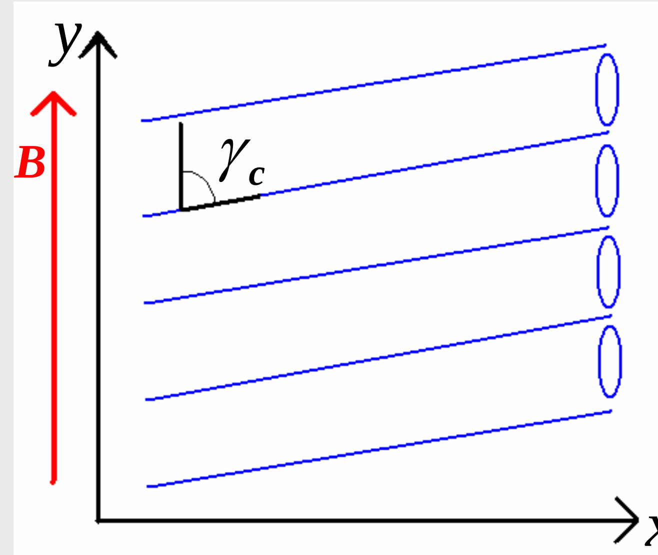
2nd step $\rightarrow$ having found the marginal modes, to find the critical modes, for both the steady and overstable modes – the modes for which $R^s(l,m)$ and $R^o(l,m)$ are least.

3rd final step $\rightarrow$ to determine which of R_c^s and R_c^o is smaller.

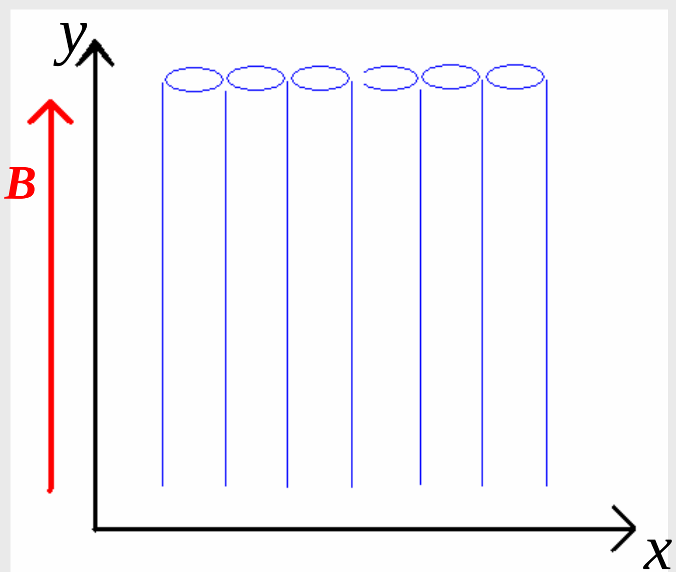
The orientation of the rolls of convection



$\exp\,(imy)$



$\exp\,(ilx + imy)$



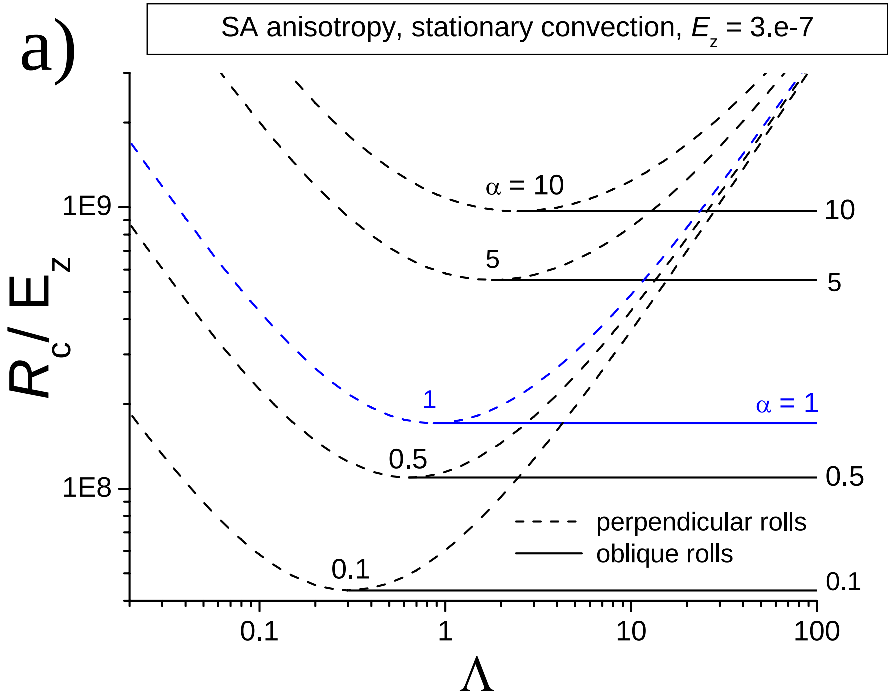
$\exp\,(ilx)$

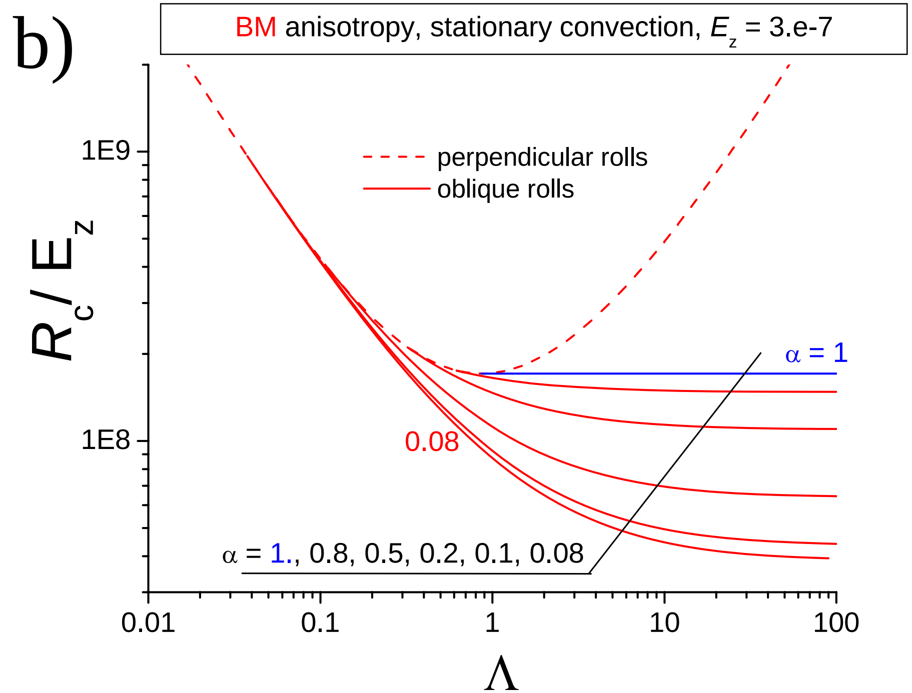
γ_c – the angle between the rolls' axis and magnetic field ***B***. If rolls´ patterns in horizontal direction are determined by $\exp\,(ilx + imy)$

then the angle is given by $\gamma_c = \operatorname{atan}\left(\dfrac{m}{l}\right)$

$l = 0 \;\Rightarrow\; \gamma_c = \pi/2$ for the rolls $\perp$ to ***B***

$m = 0 \;\Rightarrow\; \gamma_c = 0$ for the rolls $\parallel$ to ***B***





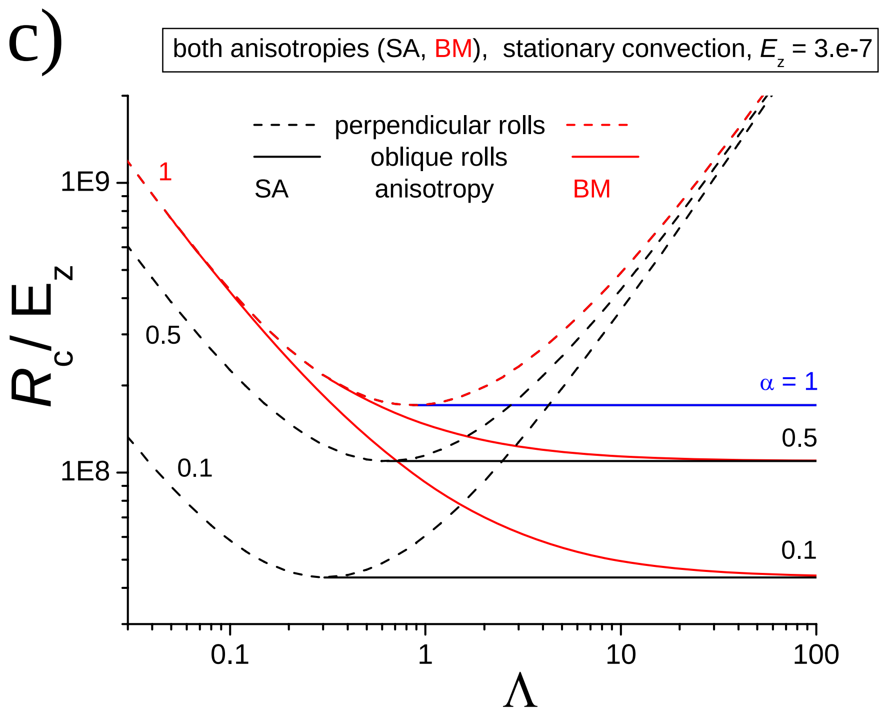
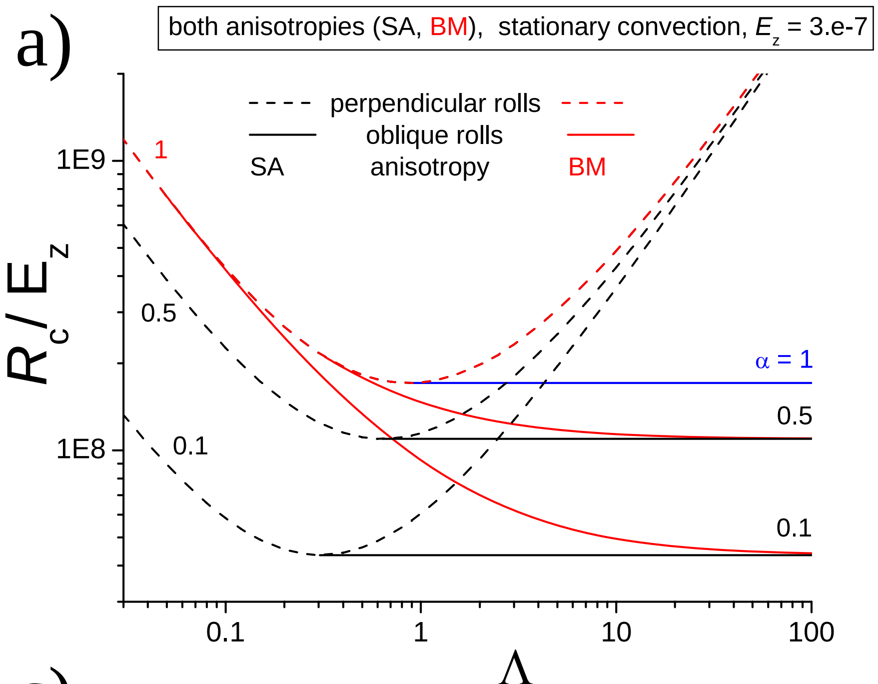
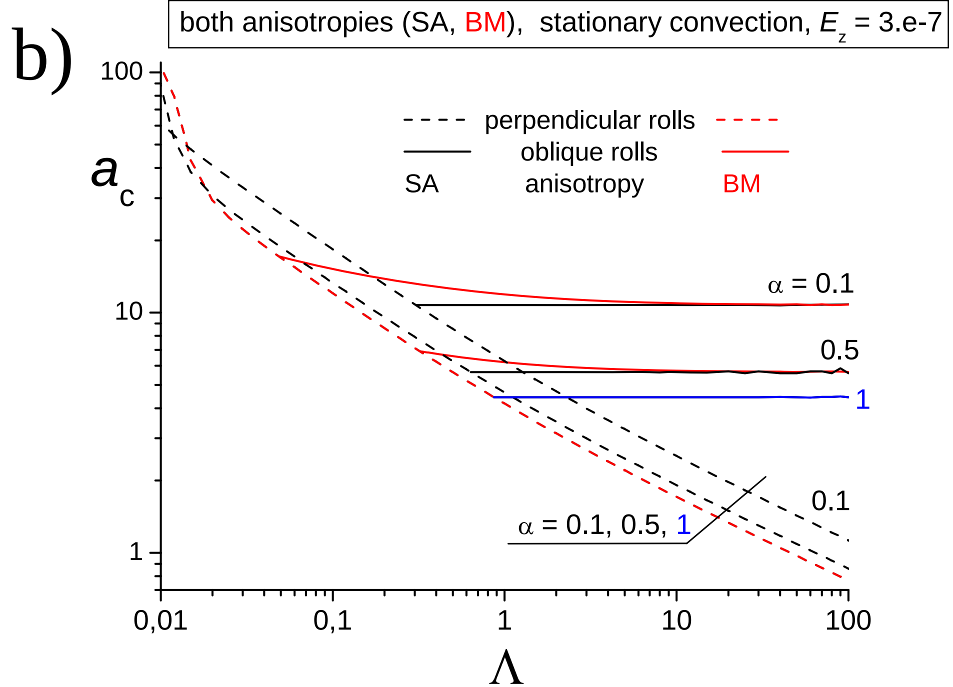


Fig. 1 Comparison of cases of anisotropic diffusive coefficients of SA and BM type with isotropic case. Dependence of critical Rayleigh number, R_c, (a) in case of SA anisotropy on Elsasser number, Λ, (b) in case of BM anisotropy, and (c) in both cases of anisotropy (SA, BM) for modes with rolls perpendicular (dashed lines) and oblique (solid lines) to the direction of basic magnetic field.





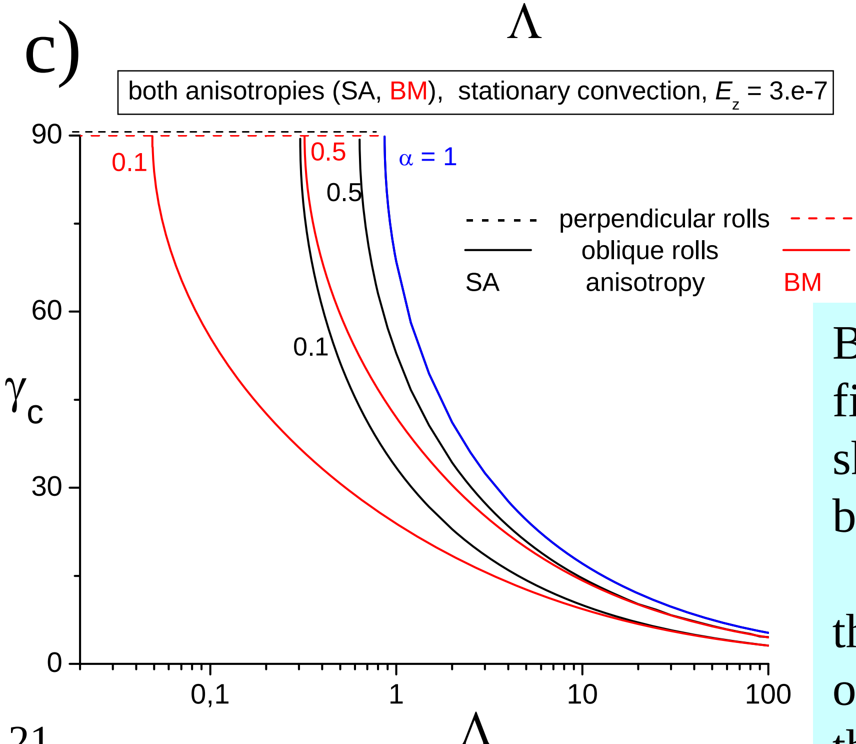


Fig. 2 Comparison of anisotropic and isotropic cases. Dependence of (a) critical Rayleigh number, R_c, (b) horizontal wave number, a_c, and (c) angle between rolls and magnetic field, γ_c, on Elsasser number, Λ, in case of both types of anisotropy (BM and SA).

Both types of anisotropy (SA, BM) of diffusive coefficients facilitates the onset of convection firstly by the shortening of rolls in horizontal direction and secondly by inclination of rolls to the magnetic field direction.

SA anisotropy facilitates convection more effectively than BM anisotropy, but mainly due to the inclination of rolls, because SA rolls are rather perpendicular to the magnetic field lines than BM rolls.

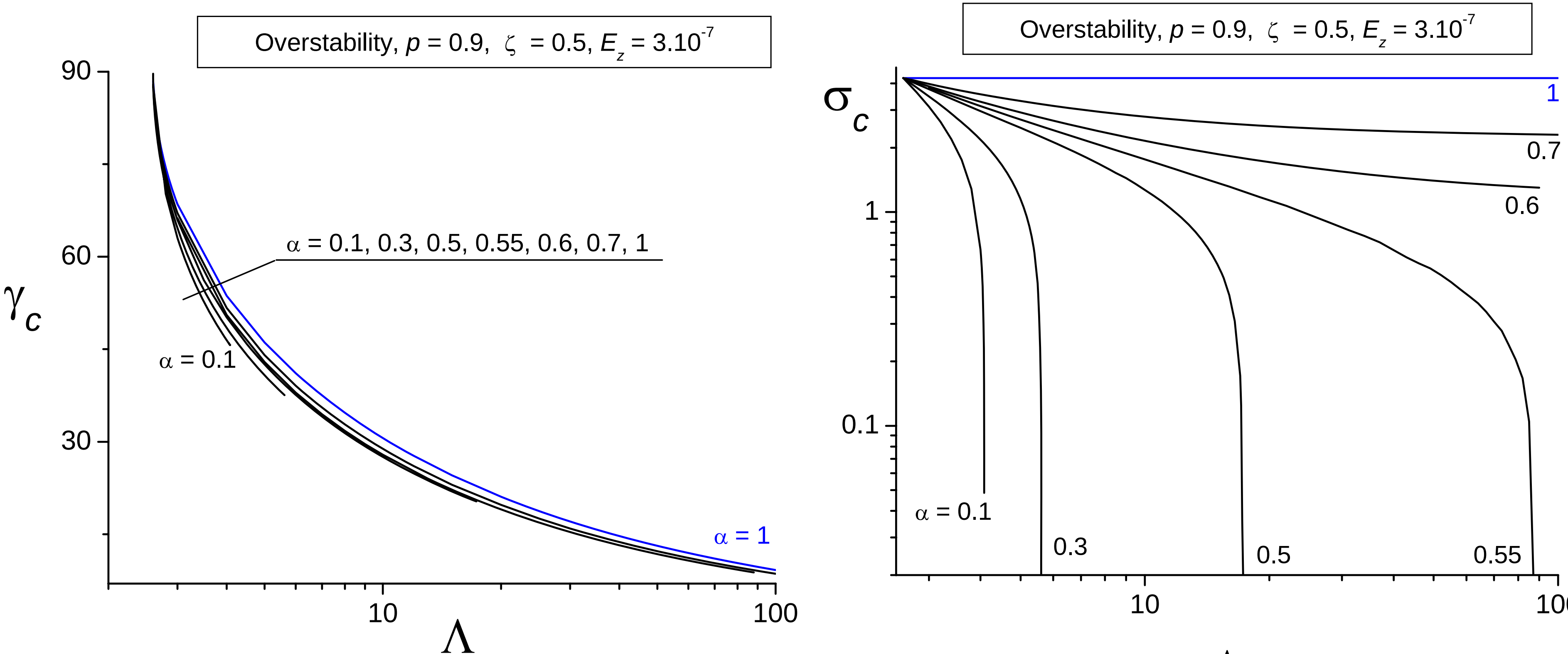
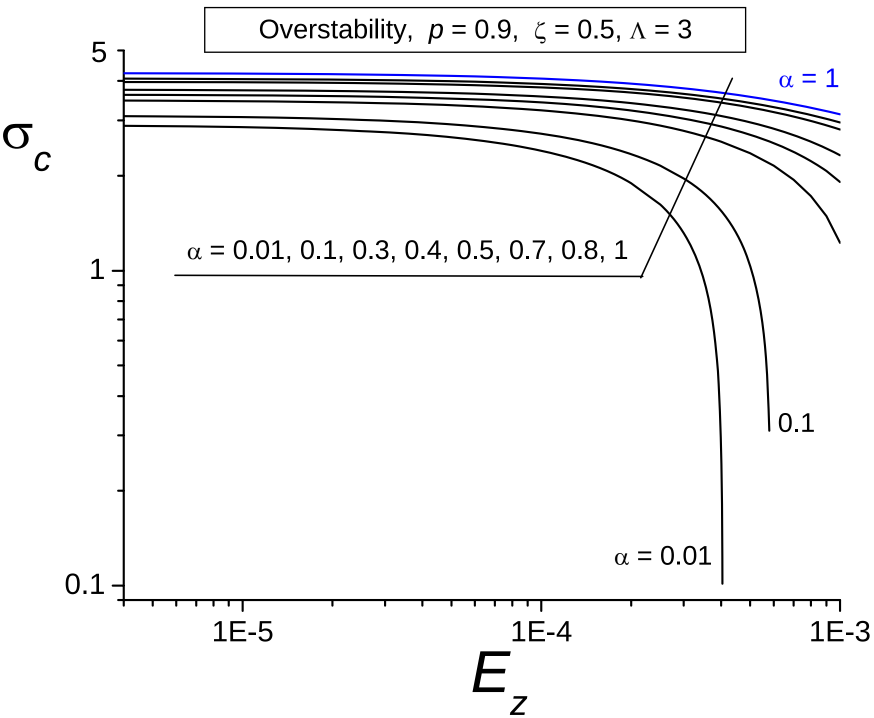
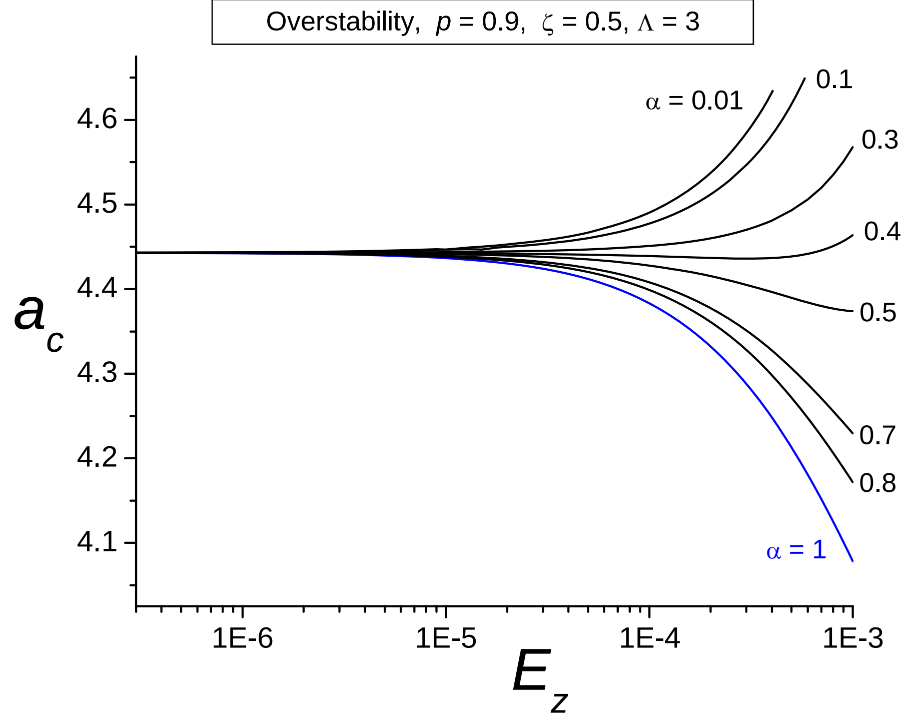


Fig. 3 The influence of BM type anisotropy on nonstationary convection.

Dependence of a) critical angle between rolls and magnetic field, γ_c, and b) critical frequency, σ_c, on Elsasser number in the case of constant Ekman number $E_z = 3 . 10^{-7}$.

This picture does not include graphs of dependence R_c *vs* Λ and a_c *vs* Λ, because there is no influence of anisotropy on these dependences for OO modes, overstable oblique rolls. R_c is not dependent on Λ (and nor α for OO modes). This constant for OO modes is $6\zeta\sqrt{3}\pi^2$ (= 51.28 for ζ=0.5). The critical horizontal wave numbers has the same feature, constant a_c value is $\pi\sqrt{2} = 4.44$ for OO modes (as well as for SO modes).





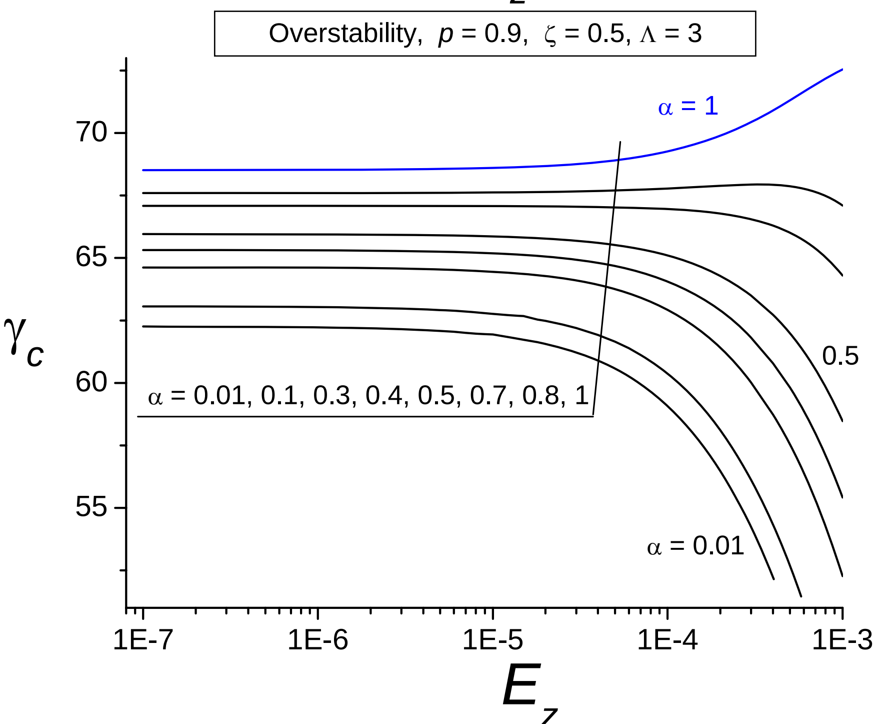


Fig. 4 Influence of BM type anisotropy on nonstationary convection.

Dependence of a) critical frequency, σ_c,
critical horizontal wave number, a_c,
and c) critical angle between rolls and magnetic field, γ_c, on Ekman number in the case of constant Elsasser number $\Lambda = 3$.

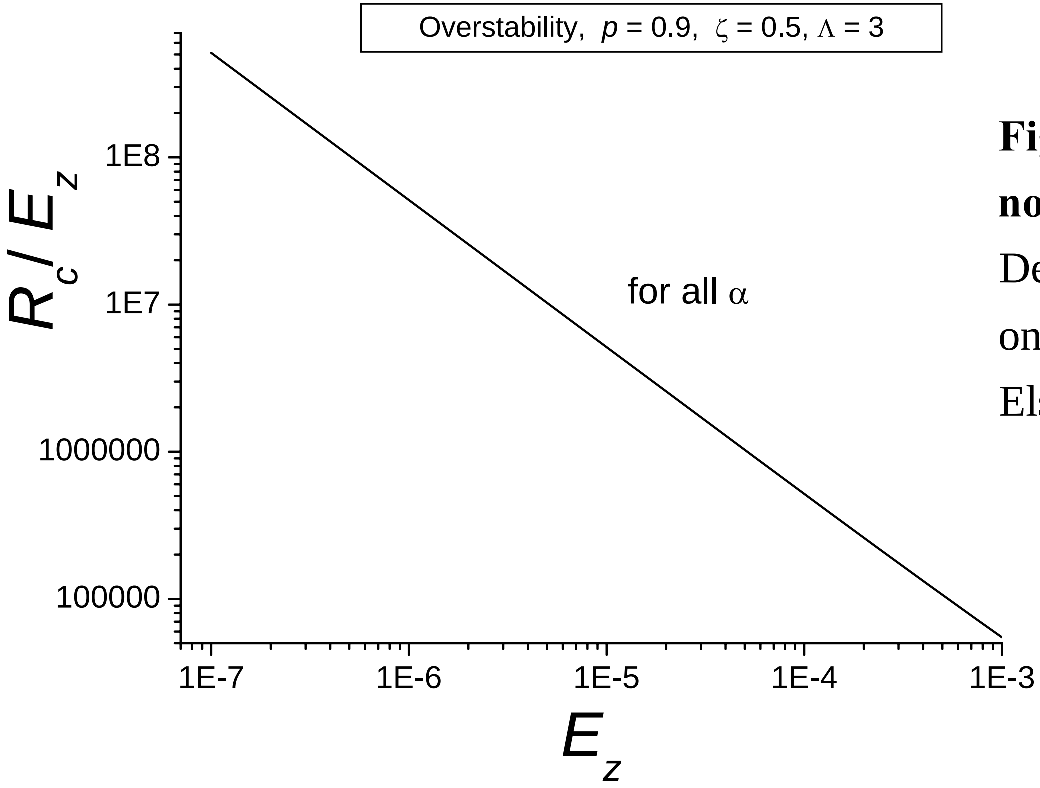
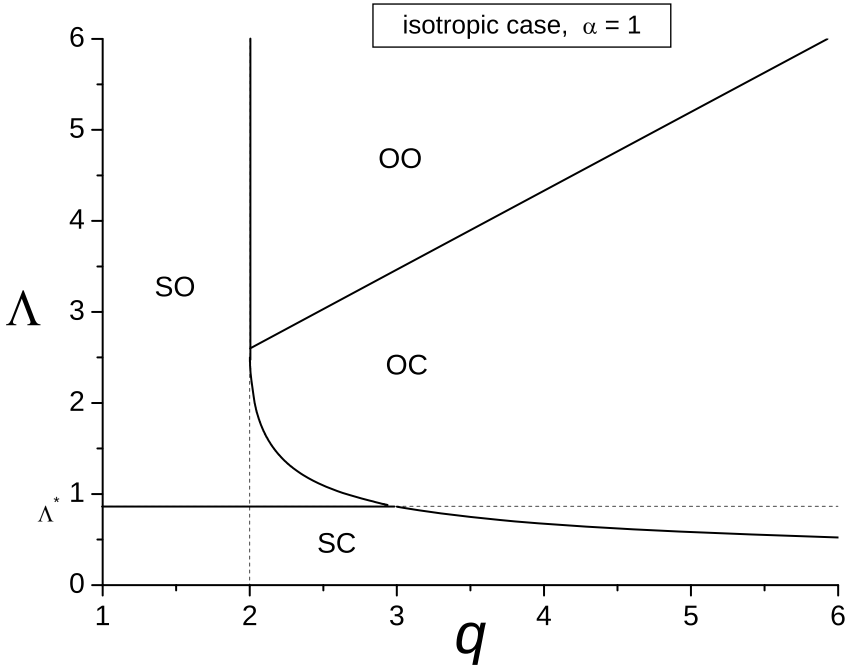
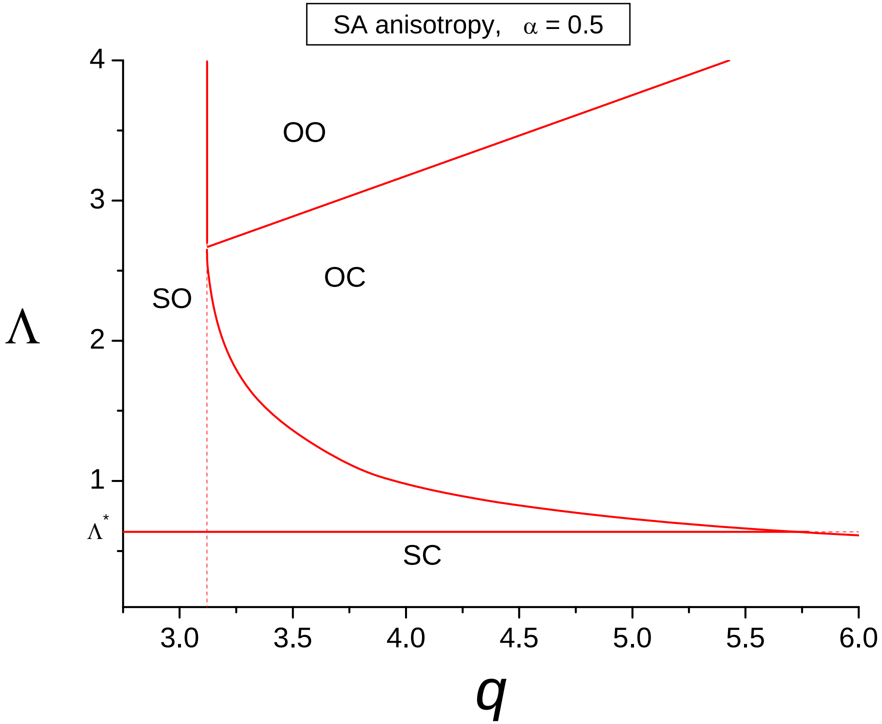


Fig. 5 Influence of BM type anisotropy on nonstationary convection.

Dependence of critical Rayleigh number, R_c / E_z, on Ekman number in the case of constatnt Elsasser number $\Lambda = 3$.

The example for BM anisotropy (which is typical also for SA anisotropy as well as for isotropic case) when R_c for OO modes is constant for great range of parameters, i. e. R_c is independent on E_z, α, Λ... for overstable oblique modes of convection.





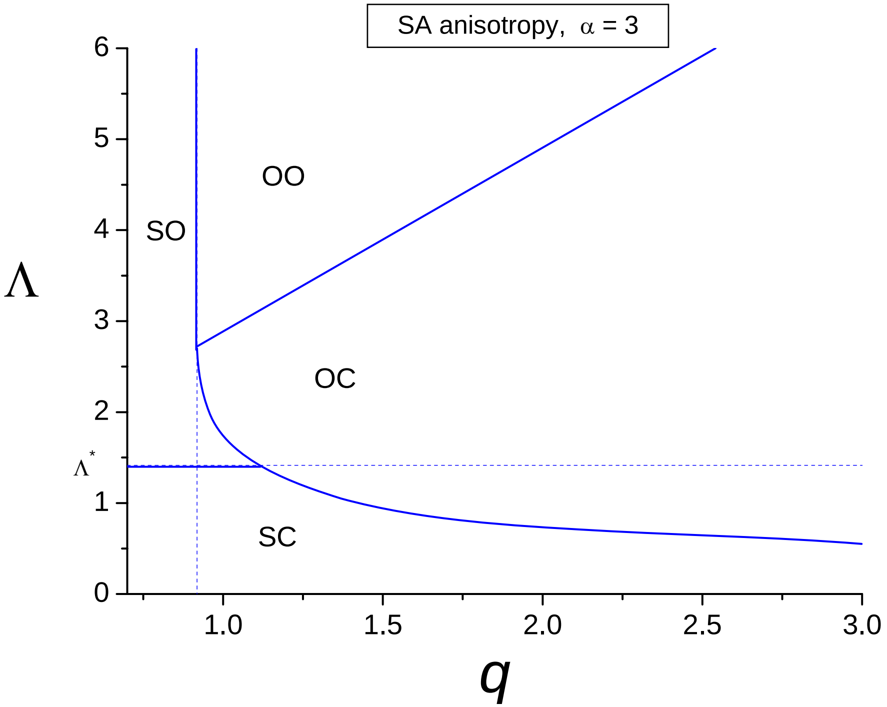
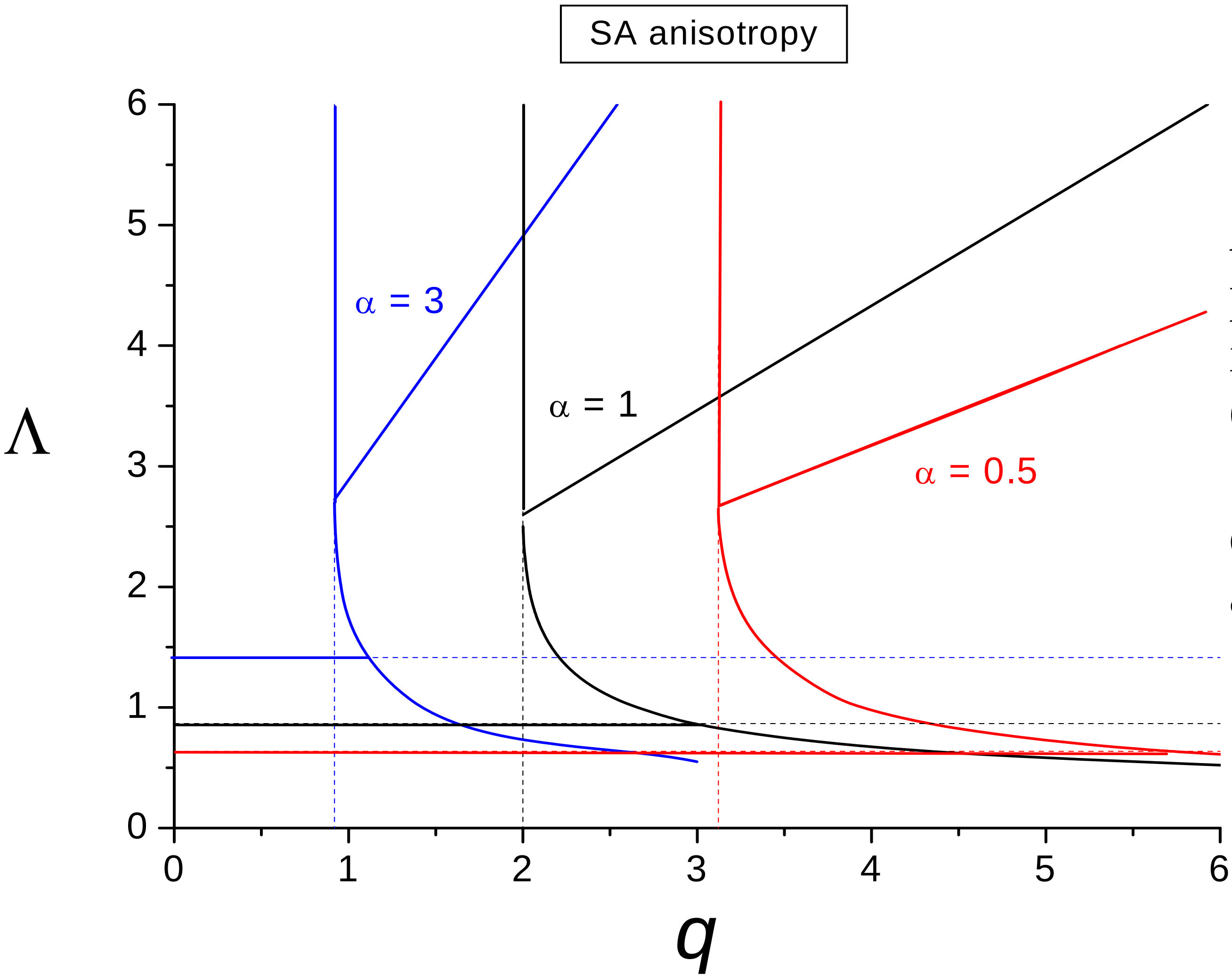


Fig. 6 Λq regime diagram for stationary and overstable modes in $E = 0$ case (inviscid or too rapidly rotating case).

a) isotropic case $\alpha = 1$. (black),

b) SA anisotropy $\alpha = 0.5$ (red)

c) SA anisotropy $\alpha = 3$ (blue)

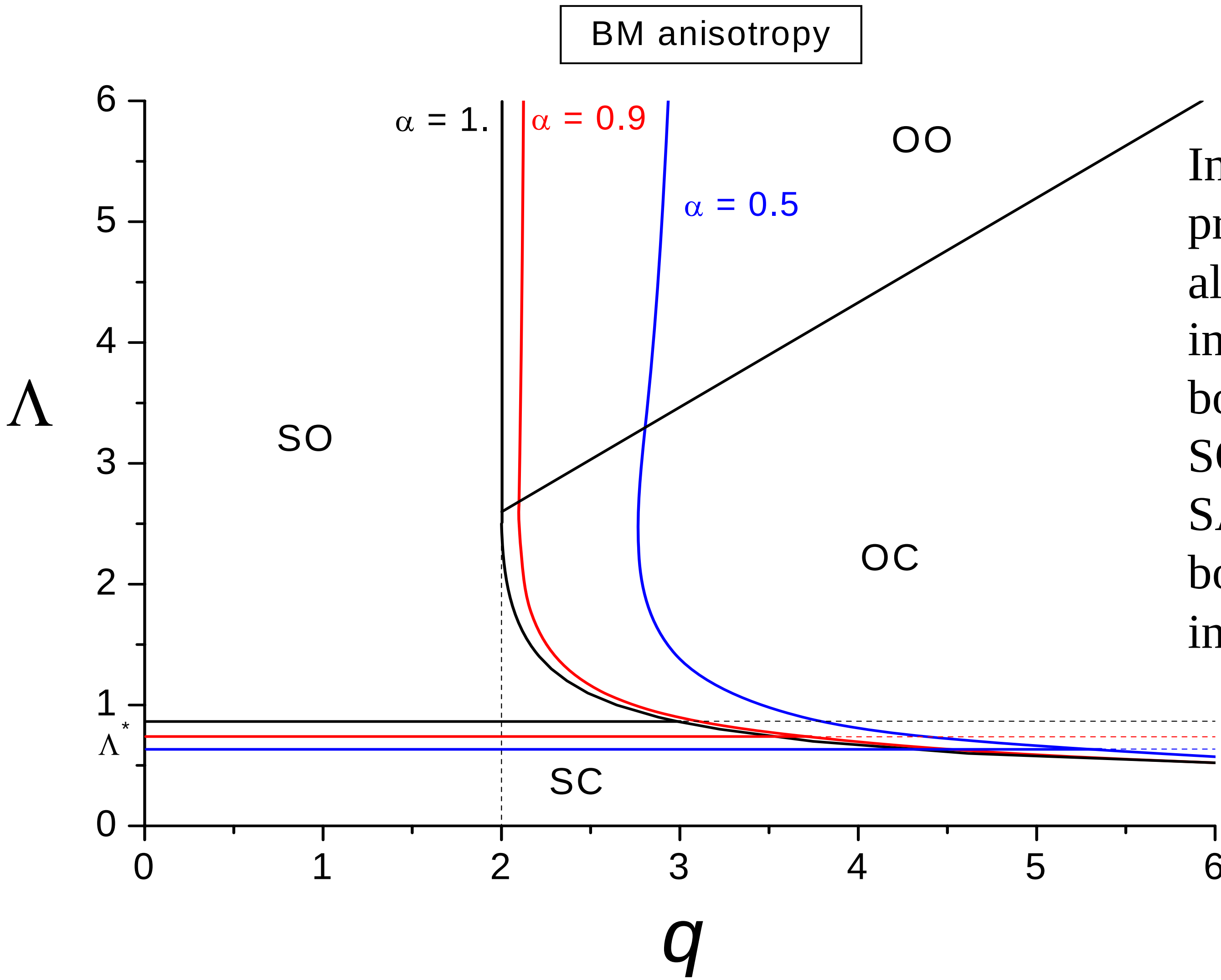


All boundaries of preference between modes, i.e. SC/SO, SC/OC, OC/OO, SO/OC and SO/OO are strongly dependent on α in SA anisotropy

Fig. 7 The influence of SA anisotropy on Λq regime diagram of stationary and overstable modes in $E = 0$ case. Different domains of preferred type of convection are labelled by SC, SO, OC, OO:

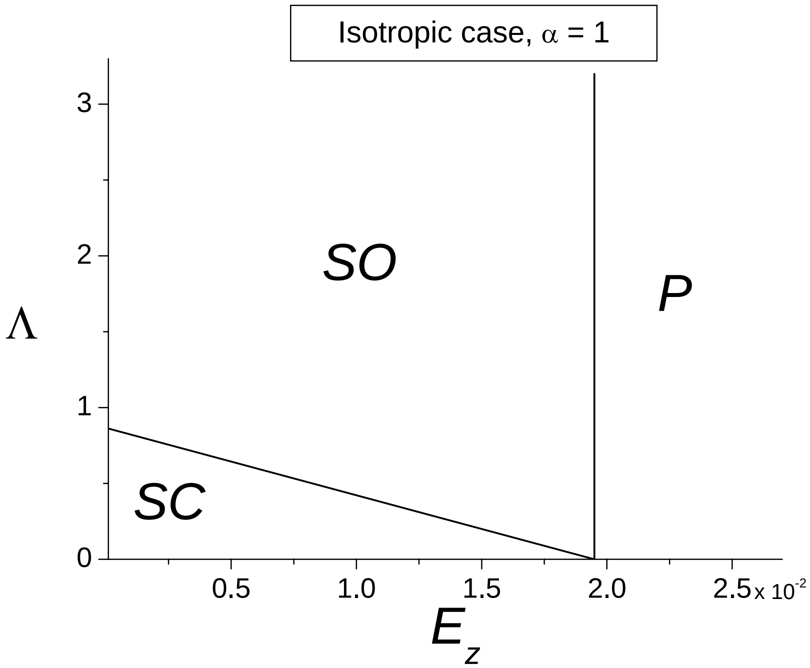
SC - **S**teady **C**ross roll is preferred; **SO** - **S**teady **O**blique roll is preferred;

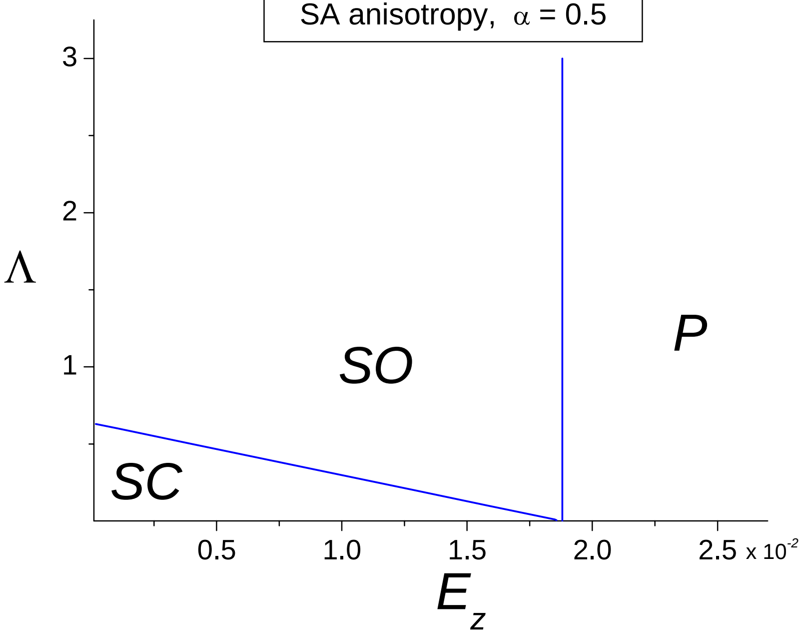
OC - **O**verstable **C**ross roll is preferred, **OO** - **O**verstable **O**blique roll is preferred.



In BM anisotropy the preference boundaries are also dependent on α, but in more complex way for boundaries SO/OO and SO/OC. Contrary to SA anisotropy, the boundary OC/OO is independent on α

Fig. 8 The influence of BM anisotropy on Λq regime diagram of stationary and overstable modes in $E = 0$ case. Different domains of preferred type of convection are labelled by SC, SO, OC, OO:
SC - **S**teady **C**ross roll is preferred; **SO** - **S**teady **O**blique roll is preferred;
OC - **O**verstable **C**ross roll is preferred, **OO** - **O**verstable **O**blique roll is preferred.





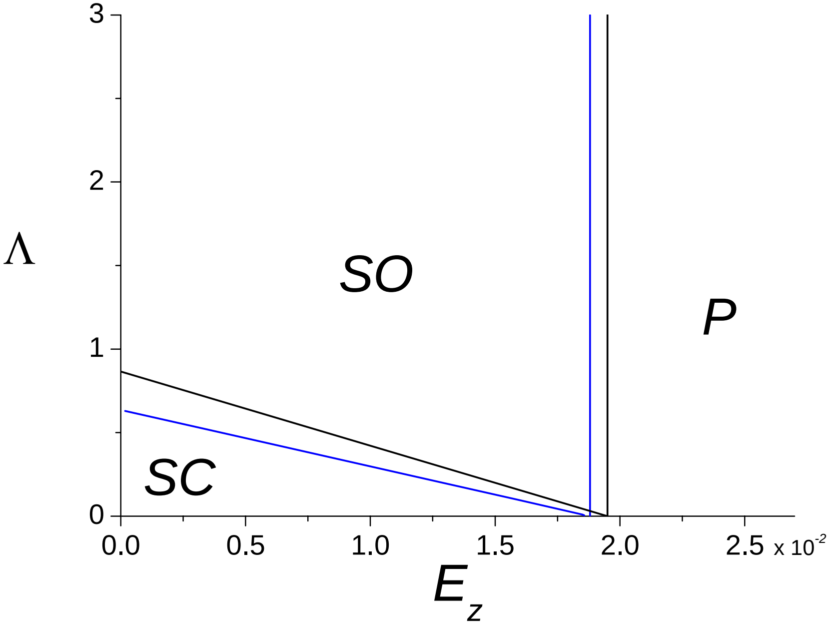


Fig. 9 The influence of SA anisotropy on ΛE regime diagram for steady modes in the case of vertical axis of rotation. Different domains of preferred type of convection are labelled by SC, SO, P:

SC - **S**teady **C**ross roll is preferred; **SO** - **S**teady **O**blique roll is preferred; **P** Steady **P**arallel roll is preferred.

The greatest differences (with physical sense) for $\alpha = \mathrm{O}(1)$ between isotropic case and SA anisotropy cases are at $E = 0$. In limiting cases of α the values of Λ at $E = 0$ are $\left[\Lambda^*\right]_{\alpha\to 0} = \mathrm{O}(\alpha^{1/2})$, $\left[\Lambda^*\right]_{\alpha\to\infty} = \mathrm{O}(\alpha^{1/4})$ and the values of E at $\Lambda = 0$ are $\left[E^*\right]_{\alpha\to 0} = \mathrm{O}(\alpha^{1/2})$, $\left[E^*\right]_{\alpha\to\infty} = \mathrm{O}(\alpha^{-1/2})$.

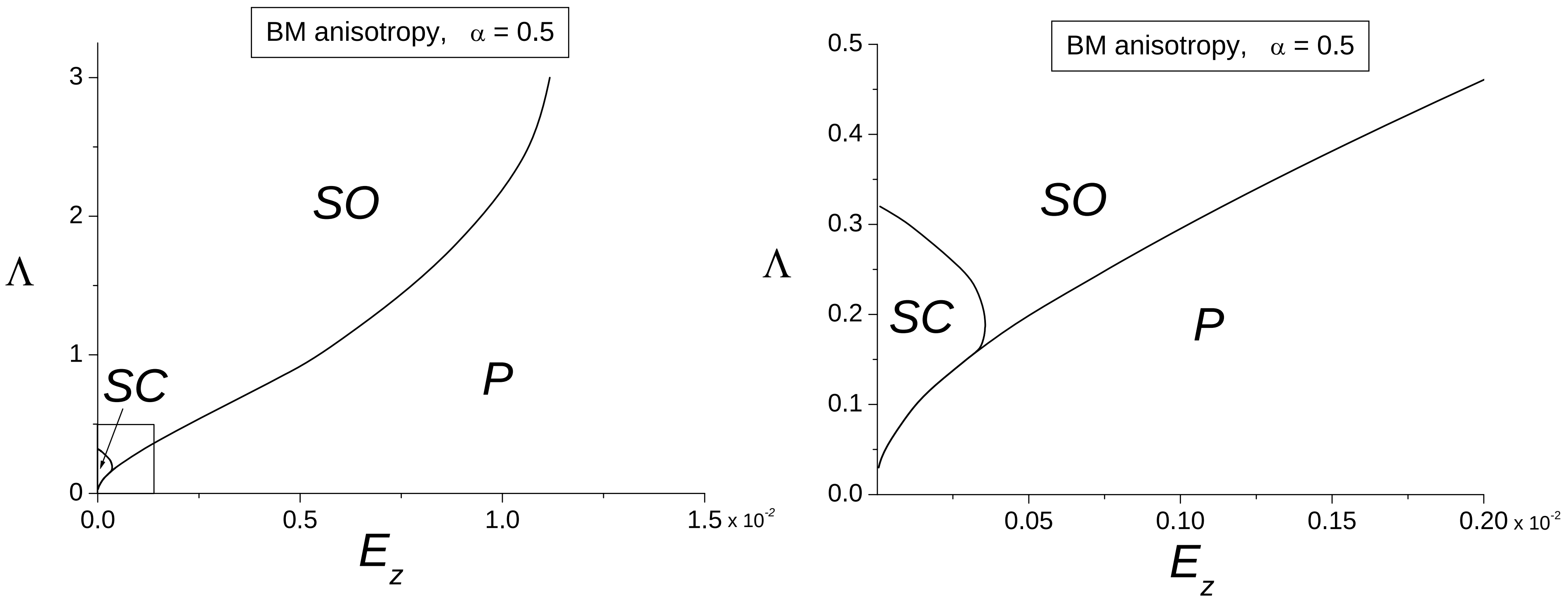
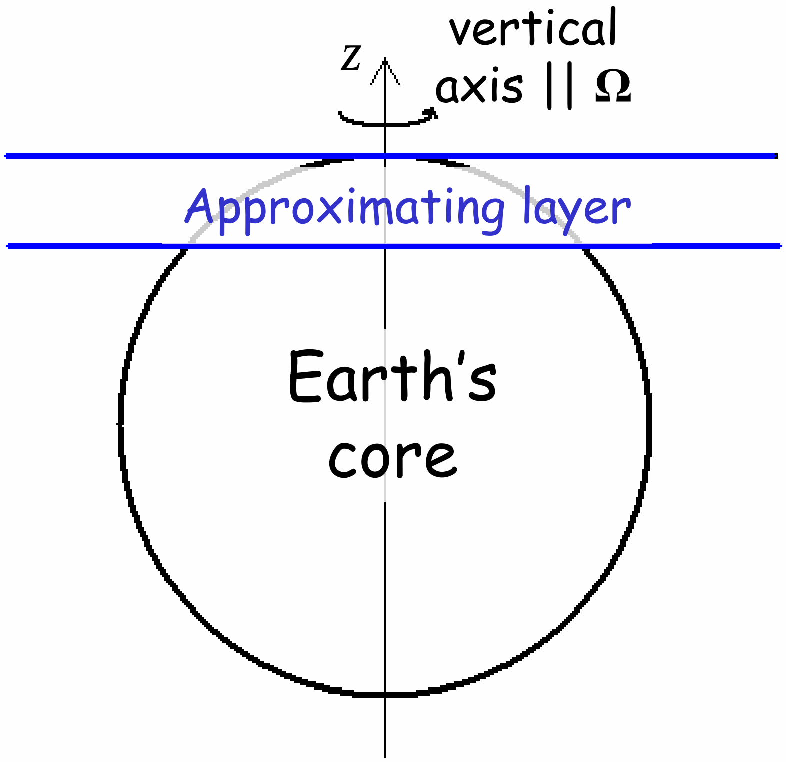


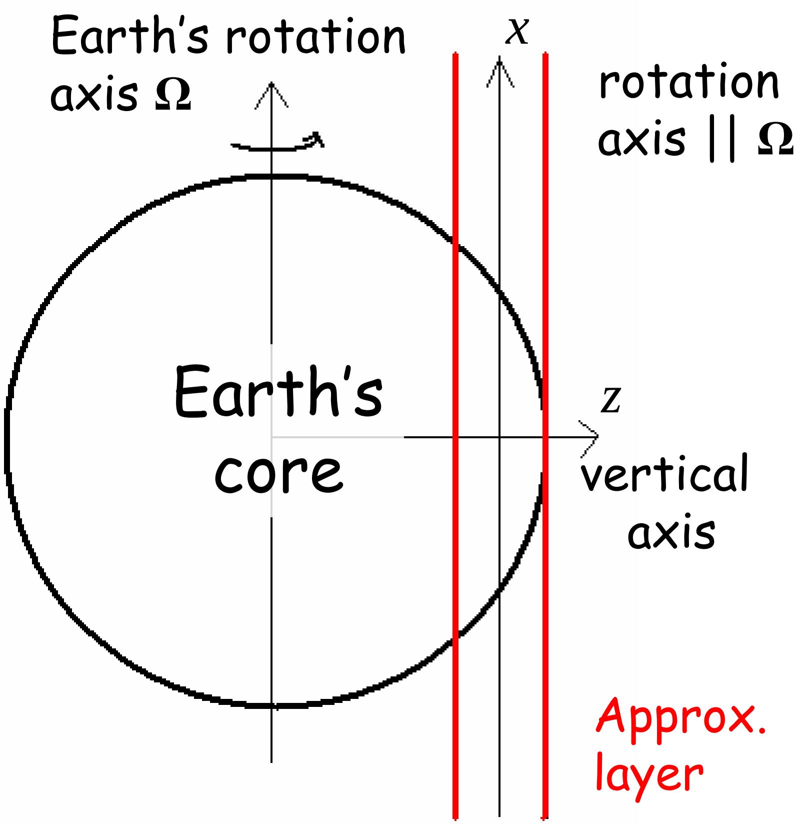
Fig. 10 The influence of BM anisotropy on ΛE regime diagram for steady modes in the case of vertical axis of rotation. Different domains of preferred type of convection are labelled by SC, SO, P: **SC** - **S**teady **C**ross roll is preferred; **SO** - **S**teady **O**blique roll is preferred; **P** Steady **P**arallel roll is preferred.

BM anisotropy cases are dramatically different from isotropic case. (SC modes and not P modes are preferred at $E = 0$. BM anisotropy has no sense for $\Lambda = 0$.)

Polar regions

Equatorial regions

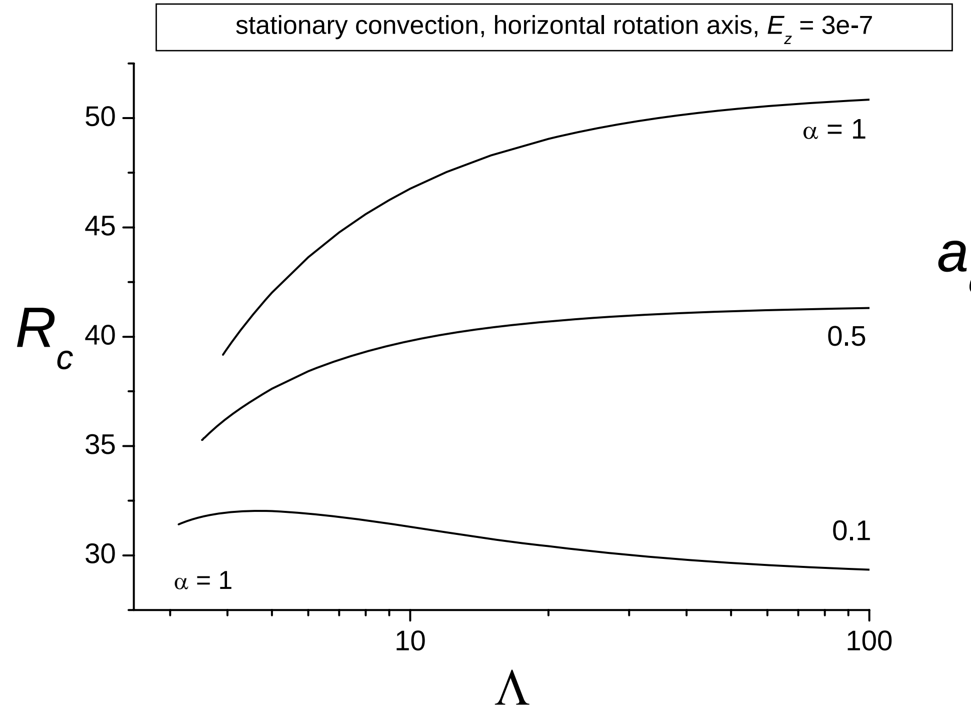


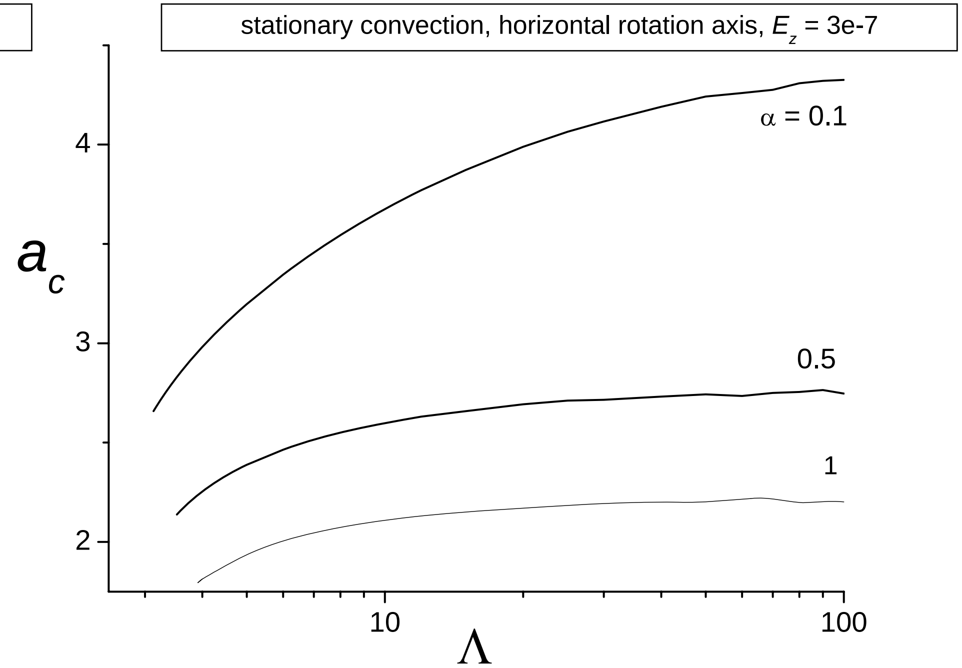


$\boldsymbol{\Omega}_0 = \Omega\hat{\boldsymbol{z}}$

$\boldsymbol{\Omega}_0 = \Omega\hat{\boldsymbol{x}}$

Basic state : $\boldsymbol{U}_0 = \boldsymbol{0}, \quad \boldsymbol{B}_0 = B_0\hat{\boldsymbol{y}}, \quad T_0 = T_l - \Delta T\,\frac{z + d/2}{d}.$





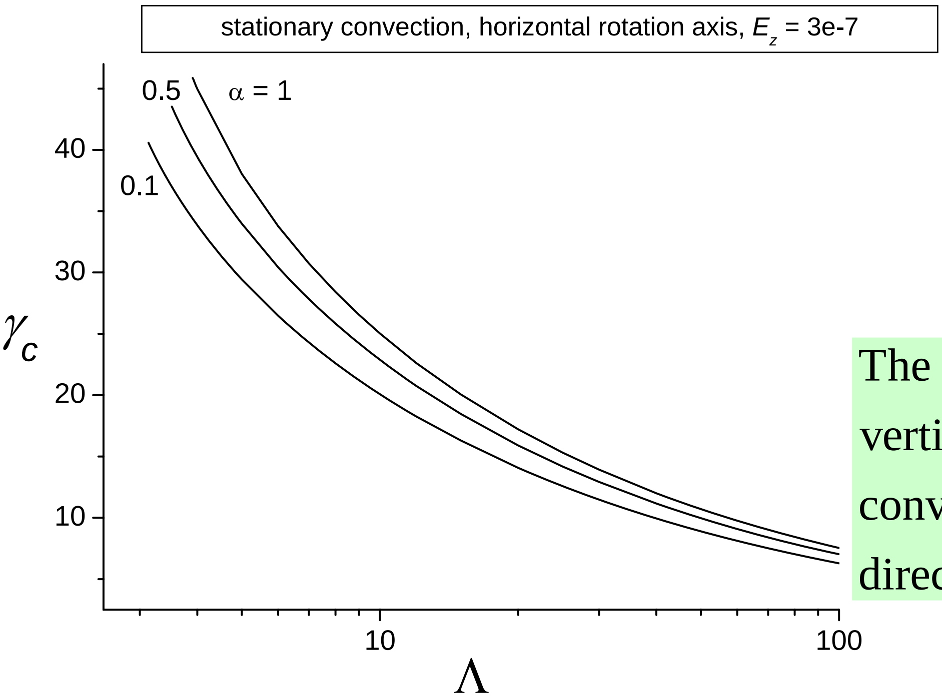
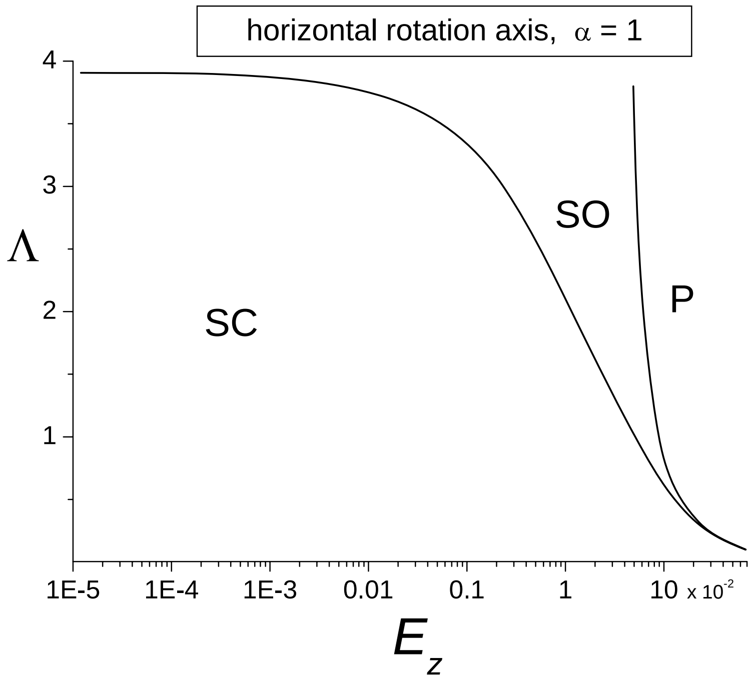


Fig. 11 Comparison of anisotropic and isotropic cases for rotation axis lying in horizontal plane. Dependence of (a) critical Rayleigh number, R_c, (b) horizontal wave number, a_c, and (c) angle between rolls and magnetic field, γ_c, on Elsasser number, Λ, in case of anisotropy (BM and SA).

The anisotropy related to the greater diffusivities in vertical direction than in horizontal directions facilitates convection and shortens sizes of rolls in horizontal directions like in the cases of vertical rotation axis



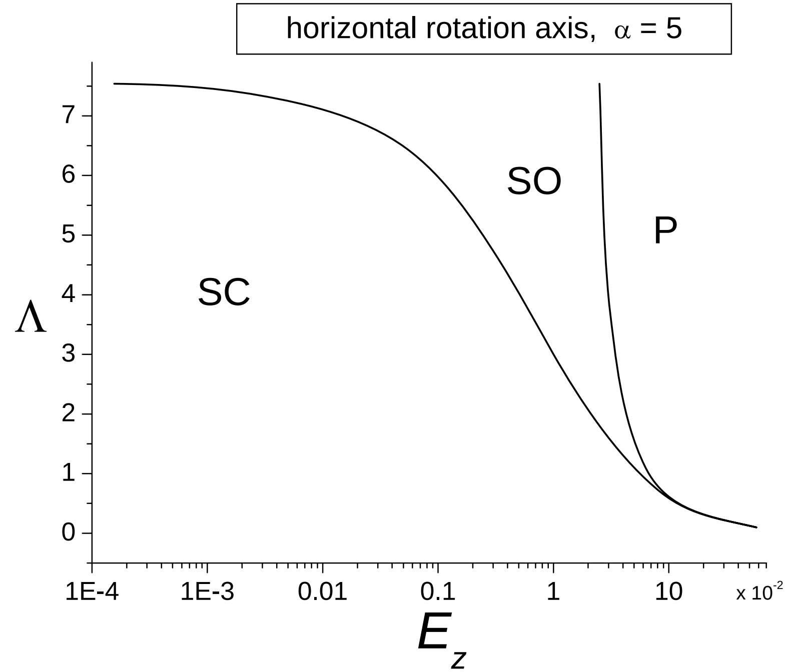
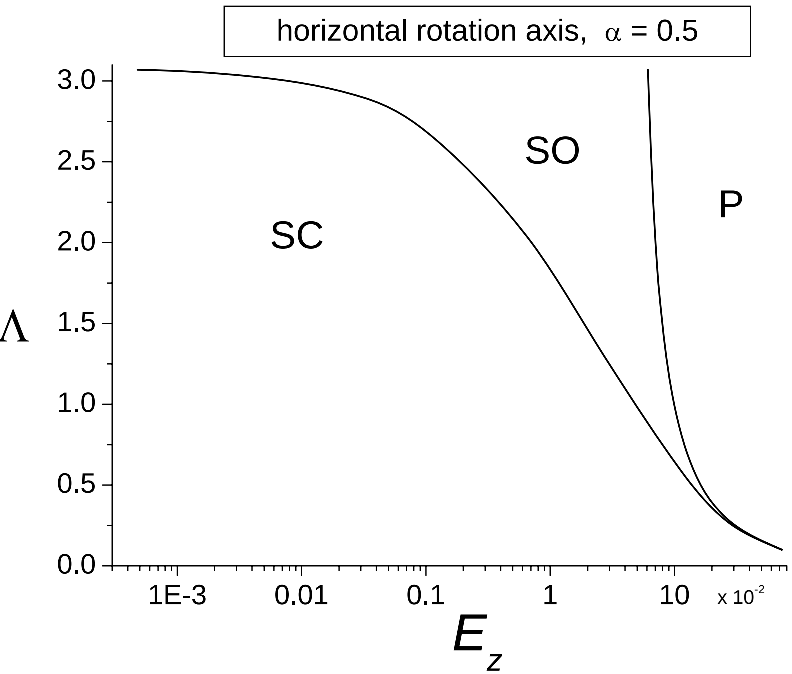


Fig. 12 ΛE regime diagram for steady modes in the case of horizontal axis of rotation. Different domains of preferred type of convection are labelled by SC, SO, P:

SC - **S**teady **C**ross roll is preferred;

SO - **S**teady **O**blique roll is preferred;

P Steady **P**arallel roll is preferred.



The preference boundaries between various modes of convection are more complex than in the case of vertical rotation axis.

Conclusions

Comparisons between isotropic and various anisotropic cases of diffusive coefficients for horizontal magnetic field and for vertical and horizontal axis of rotation

- Both SA anisotropy (of atmospheric type) and BM anisotropy facilitate the convection and shorten the horizontal width of convective rolls in either vertical or horizontal rotation axis case

- SA anisotropy facilitates convection more effectively than BM anisotropy, but mainly due to the inclination of rolls, because SA rolls are rather perpendicular to the magnetic field lines than BM rolls

- Arising instabilities, SC, SO, OC and OO modes, are preferred in various ranges of parameters, e.g. E, Λ or q, Λ. Boundaries between their preference ranges are strongly dependent on types of anisotropy of diffusive coefficients as well as on the rotation axis orientation.