The construction of exact Taylor States (for the Geodynamo)

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Geodynamo modelling

Nondimensional N-S equation e.g. (Fearn 98)

$$R_o\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{u}\right) + \hat{\mathbf{z}}\times\mathbf{u} = -\boldsymbol{\nabla}\Pi + R_a\,q\,T\,\mathbf{r} + E\nabla^2\mathbf{u} + [\boldsymbol{\nabla}\times\mathbf{B}]\times\mathbf{B}$$

- Parameters:

Ekman number E = O(10^{-15})
Rossby number Ro=O(10^{-8})
Rayleigh number Ra >> 1, q= O(10^{-5})

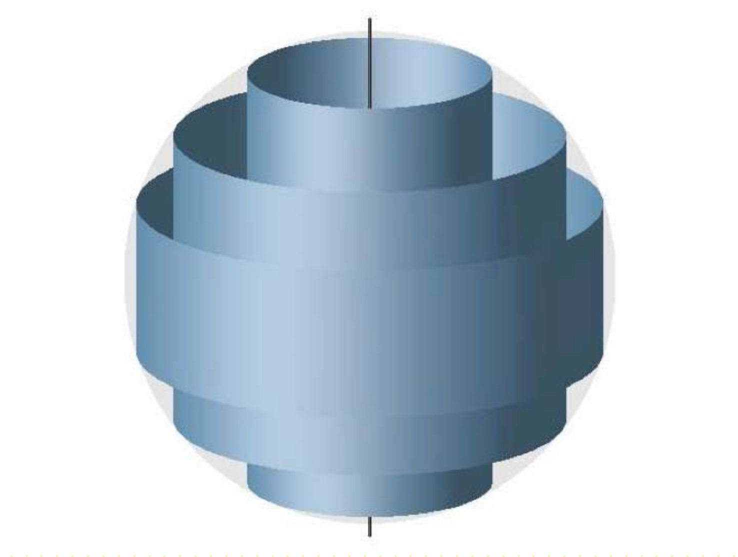
- Magnetostrophic balance:

$$\hat{\mathbf{z}}\times\mathbf{u} = -\boldsymbol{\nabla}\Pi + R_a\,q\,T\,\mathbf{r} + [\boldsymbol{\nabla}\times\mathbf{B}]\times\mathbf{B}$$

Immediate consequences

$$\hat{\mathbf{z}} \times \mathbf{u} = -\nabla \Pi + R_a q T \mathbf{r} + [\nabla \times \mathbf{B}] \times \mathbf{B}$$

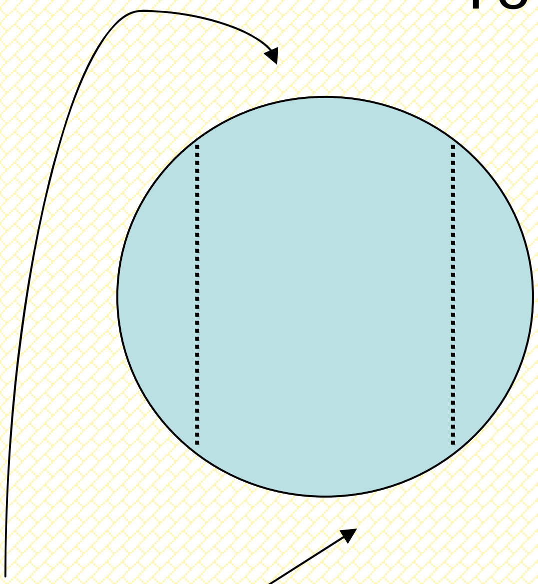
1. Take azimuthal component
2. Average over cylinders aligned with rotation axis



Vanishing of Coriolis term

To show that the cylindrical average of

$$(\hat{z} \times u)_\phi = u_s \quad \text{vanishes}$$



End caps

$$\int_{Cylinder} u_s \, dS = \int_{Cylinder} u_s \, dS + \int_{caps} u_n \, dS$$

$$= \int_{Cylinder+caps} u_n \, dS = \int_V \nabla \cdot u \, dV = 0$$

Assumptions:

- incompressible flow (anelastic works also, Smylie et. al. 1984);
- impenetrable boundary

Taylor's constraint

$$\mathcal{T}(s) \equiv \int_{C(s)} ([\nabla \times \mathbf{B}] \times \mathbf{B})_\phi \, s \, d\phi \, dz = 0$$

(J.B. Taylor, 1963)

1. Any field satisfying the constraint is termed a Taylor state (TS).

2. Infinitely many constraints, one for each cylindrical radius s.

Taylor States

- Geodynamo models produce fields that are not Taylor states….but look realistic

(a) Correct asymptotic regime or

(b) Chance ?

- Properties of TS that we'd like to know

(a) characteristic spectrum?

(b) common features?

(c) do geodynamo model fields look similar to TS?

Observational perspective

What can we learn about interior structure from surface observations?

B_r, B_θ, B_ϕ unknown *a priori*

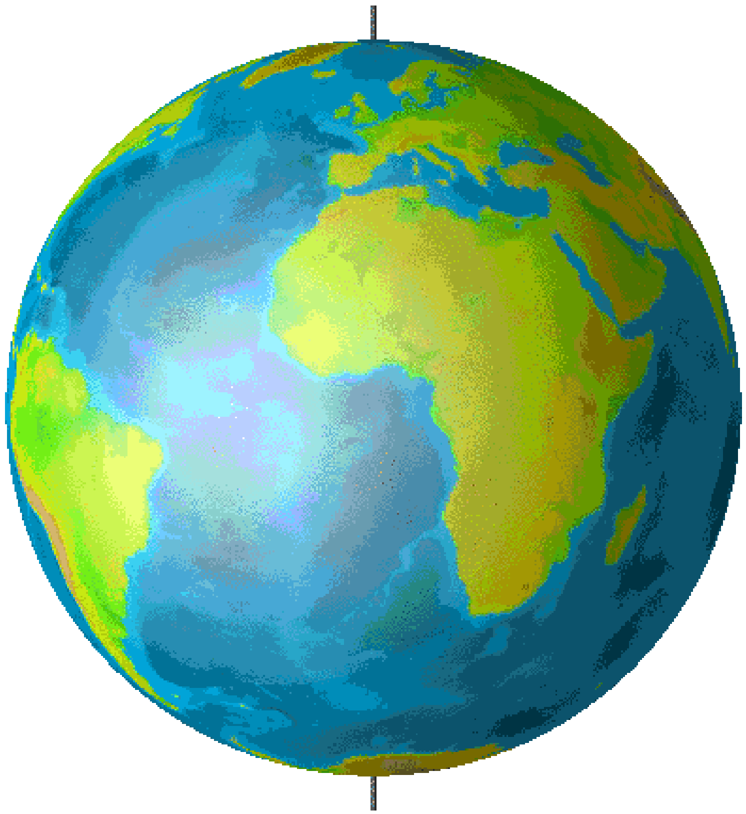
But $\int_V B \cdot dS = 0$ for any volume V,
so only S,T unknown

Now $\mathcal{T}(s) \equiv \int_{C(s)} ([\nabla \times \mathbf{B}] \times \mathbf{B})_\phi \, s \, d\phi \, dz = 0$

Reduction to only one unknown scalar?

Add in observations at r=1
How constrained is the interior field?

Geomagnetic observations



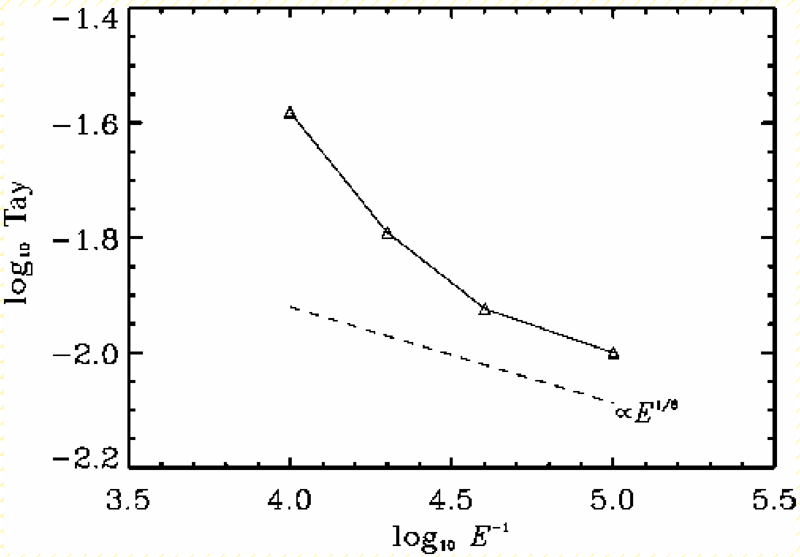
Question: “What is beneath the core-mantle boundary?”

Quest for the ~~holy grail~~ Taylor state

- Much effort from 1970's to find ***any*** Taylor state
- Limited progress with axisymmetric models, solve mean-field equations with small E.

(Soward & Jones, 1983; Hollerbach & Ierley, 1991; Fearn & Proctor, 1987)

- 3D geodynamo models with small E.



$$Tay = \frac{rms \int_{C(s)} (\nabla \times B \times B)_\phi \, d\phi dz}{rms \sqrt{\int_{C(s)} (\nabla \times B \times B)_\phi^2 \, d\phi dz}}$$

Rotvig & Jones, 2002

Constructing a Taylor State

Write B in finite modal expansion $\mathbf{B} = \sum_{1}^{N} c_i \mathbf{B}_i$

N^2 contributions to

$$\mathcal{T}(s) \equiv \int_{C(s)} ([\nabla \times \mathbf{B}] \times \mathbf{B})_\phi \, s \, d\phi \, dz$$

If each independent algebraic form, set each to zero to satisfy Taylor's constraint (worst case).

N^2 constraints (now finite), N degrees of freedom.

➔ No solution.

New results I

Since $\nabla \cdot B = 0$,expand in poloidal/toroidal

$$B = \nabla \times \nabla \times (S\hat{r}) + \nabla \times (T\hat{r})$$

and spherical harmonics with some appropriate polynomial radial basis..

$$S = \sum_{l,m,n} a_{l,m,n} Y_l^m(\theta,\phi) P_n^l(r)$$

$$T = \sum_{l,m,n} b_{l,m,n} Y_l^m(\theta,\phi) P_n^l(r)$$

with $P_n^l(r)$ chosen such that B is smooth (not trivial in spherical polar coordinates)

New Results II

Adopting this expansion then…

$$\mathcal{T}(s) \equiv \int_{C(s)} ([\boldsymbol{\nabla} \times \mathbf{B}] \times \mathbf{B})_{\phi}\, s\, d\phi\, dz = s^2\sqrt{1-s^2}\left(A_0 + A_1 s^2 + A_2 s^4 + \ldots\right)$$

…and if we use finite truncation for **B**, the series terminates with

Number terms << N (Livermore et. al. 2008)

➔ Taylor states ubiquitous

Of course, as truncation $\longrightarrow \infty$

then number of constraints also $\longrightarrow \infty$

.

Counting constraints

Spherical harmonics are polynomials

Radial basis functions are polynomials

Every contribution to

$$\mathcal{T}(s) \equiv \int_{C(s)} ([\nabla \times \mathbf{B}] \times \mathbf{B})_{\phi}\, s\, d\phi\, dz$$

is a polynomial in s (up to a factor of $\sqrt{(1-s^2)}$)

Key point: polynomials are closed under addition.

Toy problem: $\mathbf{B} = \sum_{1}^{4} c_i \mathbf{B}_i$

$$T(s) = c_1c_2s^2\sqrt{1-s^2}\left(4+5s^2\right) + c_1c_3s^2\sqrt{1-s^2}\left(1+7s^2\right) +$$
$$c_2c_4s^2\sqrt{1-s^2}\left(8+3s^2\right) + c_3c_2s^2\sqrt{1-s^2}\left(2+11s^2\right)$$

gives 2 homogeneous constraints in 4 unknowns

$$4c_1c_2 + c_1c_3 + 8c_2c_4 + 2c_3c_2 = 0$$
$$5c_1c_2 + 7c_1c_3 + 3c_2c_4 + 11c_3c_2 = 0$$

$$T(s) = c_1c_2\sin(s) + c_1c_3\cos(s) +$$
$$c_2c_4\tan(s) + c_3c_2s^2\sqrt{1-s^2}(5+11s^2)$$

gives 4 homogeneous constraints

➔ trivial solution.

Counting constraints II

Consider expanding radially in spherical Bessel functions suitable for electrically insulating exterior. $j_l(k_n r)$

Each contribution to

$$\mathcal{T}(s) \equiv \int_{C(s)} ([\nabla \times \mathbf{B}] \times \mathbf{B})_\phi \, s \, d\phi \, dz$$

is some independent algebraic form, requiring the full N^2 set of constraints - worst case scenario

N^2 constraints, N degrees of freedom.

➔ No solution.

Nonlinearity

Despite reduction to finite number of constraints (even using polynomials), still have quadratic nonlinearity:

$$\mathcal{T}(s) \equiv \int_{C(s)} ([\nabla \times \mathbf{B}] \times \mathbf{B})_\phi \, s \, d\phi \, dz = s^2\sqrt{1-s^2}\left(A_0 + A_1 s^2 + A_2 s^4 + \cdots\right)$$

$$A_i = Q(a_{lmn}, b_{lmn})$$

Q is sparse, can be exploited to find exact Taylor states.

A Suite of exact Taylor States in a full sphere

- Take observed poloidal magnetic field up to L=3; extend into core using simple profile. Toroidal field is unconstrained by observation.

- Expand toroidal field in 4 low degree axisymmetric basis functions.

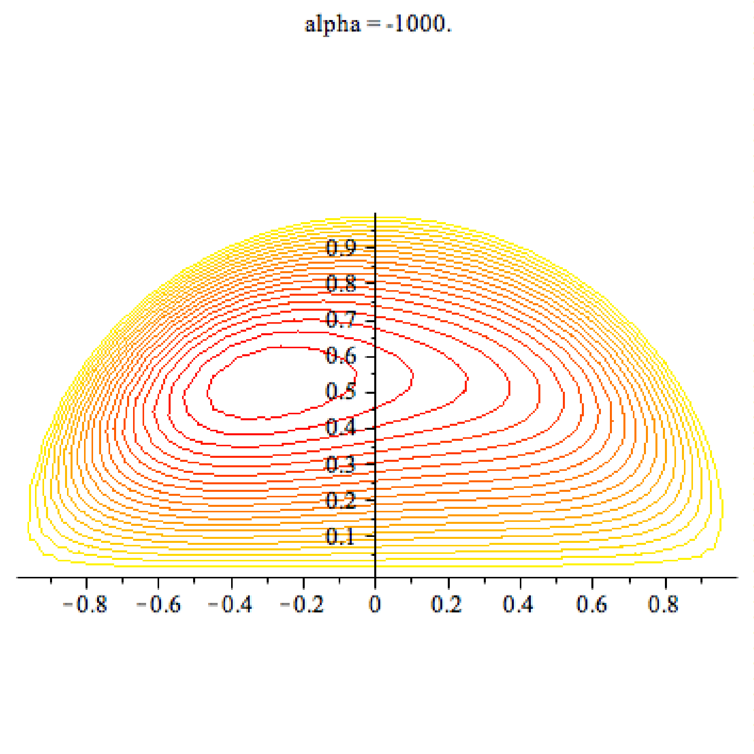
In this case:

No quadratic coupling between toroidal terms

(not immediately obvious, but true); hence problem is linear.

- 3 constraints (3 terms in series).
- Linear problem for toroidal field with 4 unknown coefficients.
- One parameter family of solutions.

A Suite of exact Taylor States



Contours of B_{φ}

Conclusions

- Finitely truncated field, Taylor's constraint amounts to a finite set of conditions.

 - Exact number depends on radial basis.

 - Similar (in number) to matching to an electrically insulating exterior.

- Can find exact Taylor states - although not dynamo generated.

- Look for general characteristics of TS.

- Investigate extremal models e.g. TS of least energy or dissipation consistent with observations.

Preprint at http://igpppublic.ucsd.edu/~plivermore/