Mean-field magnetohydrodynamics:
TEST-FIELD METHOD
for calculating the coefficients which determine the mean electromotive force

$$\mathcal{E} = \overline{u \times b}$$

Karl-Heinz Rädler
with
Axel Brandenburg
Matthias Rheinhardt
Kandaswamy Subramanian

Mean-field magnetohydrodynamics

Consider a problem for which

$$\partial_t \boldsymbol{B} - \boldsymbol{\nabla} \times (\boldsymbol{U} \times \boldsymbol{B}) - \eta \boldsymbol{\nabla}^2 \overline{\boldsymbol{B}} = \boldsymbol{0}$$

$\boldsymbol{U}$ given directly
or by momentum balance etc.

and a mean-field theory seems suitable

$$\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{b}, \quad \boldsymbol{U} = \overline{\boldsymbol{U}} + \boldsymbol{u}$$

$$\partial_t \overline{\boldsymbol{B}} - \boldsymbol{\nabla} \times (\overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}} + \boldsymbol{\mathcal{E}}) - \eta \boldsymbol{\nabla}^2 \overline{\boldsymbol{B}} = \boldsymbol{0}$$

$\boldsymbol{\mathcal{E}} = \overline{\boldsymbol{u} \times \boldsymbol{b}}$ mean electromotive force
due to fluctuations

Mean-field magnetohydrodynamics

$$\boldsymbol{\mathcal{E}} = \overline{\boldsymbol{u} \times \boldsymbol{b}}$$

$$\partial_t \boldsymbol{b} - \boldsymbol{\nabla} \times (\overline{\boldsymbol{U}} \times \boldsymbol{b} + \boldsymbol{G}) - \eta \boldsymbol{\nabla}^2 \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \overline{\boldsymbol{B}})$$

$$\boldsymbol{G} = \boldsymbol{u} \times \boldsymbol{b} - \overline{\boldsymbol{u} \times \boldsymbol{b}} \quad (= (\boldsymbol{u} \times \boldsymbol{b})')$$

$\Rightarrow$ $\boldsymbol{b}$ is a functional of $\boldsymbol{u}$, $\overline{\boldsymbol{U}}$ and $\overline{\boldsymbol{B}}$, which is linear in $\overline{\boldsymbol{B}}$

$\Rightarrow$ $\boldsymbol{b} = \boldsymbol{b}^{(0)} + \boldsymbol{b}^{(\overline{B})}$

$\Rightarrow$ $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^{(0)} + \boldsymbol{\mathcal{E}}^{(\overline{B})}$, with $\boldsymbol{\mathcal{E}}^{(0)}$ independent of $\overline{\boldsymbol{B}}$
and $\boldsymbol{\mathcal{E}}^{(\overline{B})}$ linear and homogeneous in $\overline{\boldsymbol{B}}$

Mean-field magnetohydrodynamics

$$\mathcal{E}_i(\boldsymbol{x},t) = \mathcal{E}_i^{(0)}(\boldsymbol{x},t) + \int_0^\infty \int_\infty K_{ij}(\boldsymbol{x},t;\boldsymbol{\xi},\tau)\overline{B}_j(\boldsymbol{x}+\boldsymbol{\xi},t-\tau)\mathrm{d}^3\xi\,\mathrm{d}\tau$$

K_{ij} depends on $\boldsymbol{u}$ and $\overline{\boldsymbol{U}}$ only (which may depend on $\overline{\boldsymbol{B}}$),
vanishes for large $|\boldsymbol{\xi}|$ and τ.

$\boldsymbol{\mathcal{E}}$ at $(\boldsymbol{x},t)$ depends on the behavior of $\overline{\boldsymbol{B}}$
in some surroundings of $(\boldsymbol{x},t)$ only.

Mean-field magnetohydrodynamics

$$\mathcal{E}_i(\boldsymbol{x},t) = \mathcal{E}_i^{(0)}(\boldsymbol{x},t)$$

$$+ \int_0^\infty \int_\infty K_{ij}(\boldsymbol{x},t;\boldsymbol{\xi},\tau)\overline{B}_j(\boldsymbol{x}+\boldsymbol{\xi},t-\tau)\mathrm{d}^3\xi\,\mathrm{d}\tau$$

$$\overline{B}_j(\boldsymbol{x}+\boldsymbol{\xi},t-\tau) = \overline{B}_j(\boldsymbol{x},t) + \frac{\partial\overline{B}_j(\boldsymbol{x},t)}{\partial x_k}\xi_k - \frac{\partial\overline{B}_j(\boldsymbol{x},t)}{\partial t}\tau - \cdots$$

$$\mathcal{E}_i = \mathcal{E}_i^{(0)} + a_{ij}\overline{B}_j + b_{ijk}\frac{\partial\overline{B}_j}{\partial x_k} + b_{ij}\frac{\partial\overline{B}_j}{\partial t} + \cdots$$

$$a_{ij} = \int_0^\infty \int_\infty K_{ij}(\boldsymbol{x},t;\boldsymbol{\xi},\tau)\mathrm{d}^3\xi\mathrm{d}\tau$$ etc.

Mean-field magnetohydrodynamics

$$\mathcal{E}_i = \mathcal{E}_i^{(0)} + a_{ij}\overline{B}_j + b_{ijk}\frac{\partial \overline{B}_j}{\partial x_k} + b_{ij}\frac{\partial \overline{B}_j}{\partial t} + \cdots$$

The frequently used ''ansatz''

$$\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\frac{\partial \overline{B}_j}{\partial x_k}$$

or

$$\boldsymbol{\mathcal{E}} = \alpha\overline{\boldsymbol{B}} - \beta(\boldsymbol{\nabla} \times \overline{\boldsymbol{B}})$$

is an approximation.

Its applicability needs to be checked in each case !

Calculation of the coefficients $a_{ij},\ b_{ijk}$

Recall $\partial_t \boldsymbol{b} - \nabla \times (\overline{\boldsymbol{U}} \times \boldsymbol{b} + \boldsymbol{G}) - \eta\nabla^2 \boldsymbol{b} = \nabla \times (\boldsymbol{u} \times \overline{\boldsymbol{B}})$

$$\boldsymbol{G} = \boldsymbol{u} \times \boldsymbol{b} - \overline{\boldsymbol{u} \times \boldsymbol{b}}$$

Second-order correlation approximation (SOCA, FOSA)

defined by $\boldsymbol{G} = 0$

E.g., homogeneous isotropic turbulence

$$\alpha = -\frac{1}{3}\int_0^\infty \int_\infty G(\boldsymbol{\xi},\tau)\overline{\boldsymbol{u}(\boldsymbol{x},t)\cdot(\nabla \times \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi},t-\tau))}\,\mathrm{d}^3\xi\mathrm{d}\tau$$

$$G(\boldsymbol{\xi},\tau) = (4\pi\eta t)^{-3/2}\exp(-\xi^2/4\eta t)$$

... high-conductivity (low η) limit $\alpha = -\frac{1}{3}\overline{\boldsymbol{u}\cdot(\nabla \times \boldsymbol{u})}\,\tau_{\mathrm{c}}$

Calculation of the coefficients $a_{ij}, \; b_{ijk}$

Recall $$\partial_t \boldsymbol{b} - \boldsymbol{\nabla} \times (\overline{\boldsymbol{U}} \times \boldsymbol{b} + \boldsymbol{G}) - \eta \boldsymbol{\nabla}^2 \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \overline{\boldsymbol{B}})$$

$$\boldsymbol{G} = \boldsymbol{u} \times \boldsymbol{b} - \overline{\boldsymbol{u} \times \boldsymbol{b}}$$

Second-order correlation approximation (SOCA, FOSA)

defined by $\boldsymbol{G} = 0$

Range of applicability in the high-conductivity (low η) limit

$$St = u_{\mathrm{c}} \tau_{\mathrm{c}} / \lambda_{\mathrm{c}} \ll 1$$!!!

Higher-order correlation approximations possible

.... but tedious

Calculation of $a_{ij},\ b_{ijk}$

TEST-FIELD METHOD

Recall $$\partial_t \boldsymbol{b} - \nabla \times (\overline{\boldsymbol{U}} \times \boldsymbol{b} + \boldsymbol{G}) - \eta \nabla^2 \boldsymbol{b} = \nabla \times (\boldsymbol{u} \times \overline{\boldsymbol{B}})$$

$$\boldsymbol{G} = \boldsymbol{u} \times \boldsymbol{b} - \overline{\boldsymbol{u} \times \boldsymbol{b}}$$

Assume (for example) $$\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\partial\overline{B}_j/\partial x_k$$

Choose (suitable) set of test-fields $\overline{\boldsymbol{B}}^{(n)}$

Calculate the corresponding $\boldsymbol{b}^{(n)}$ from

$$\partial_t \boldsymbol{b}^{(n)} - \nabla \times (\overline{\boldsymbol{U}} \times \boldsymbol{b}^{(n)} + \boldsymbol{G}^{(n)}) - \eta \nabla^2 \boldsymbol{b}^{(n)} = \nabla \times (\boldsymbol{u} \times \overline{\boldsymbol{B}}^{(n)})$$

TEST-FIELD METHOD

Assume (for example) $\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\partial\overline{B}_j/\partial x_k$

Choose (suitable) set of test-fields $\overline{\boldsymbol{B}}^{(n)}$

Calculate the corresponding $\boldsymbol{b}^{(n)}$ from

$$\partial_t \boldsymbol{b}^{(n)} - \nabla \times (\overline{\boldsymbol{U}} \times \boldsymbol{b}^{(n)} + \boldsymbol{G}^{(n)}) - \eta\nabla^2 \boldsymbol{b}^{(n)} = \nabla \times (\boldsymbol{u} \times \overline{\boldsymbol{B}}^{(n)})$$

Calculate the $\boldsymbol{\mathcal{E}}^{(n)} = \overline{\boldsymbol{u} \times \boldsymbol{b}^{(n)}}$

Solve $a_{ij}\overline{B}_j^{(n)} + b_{ijk}\partial\overline{B}_j^{(n)}/\partial x_k = \mathcal{E}_i^{(n)}$

to obtain $a_{ij},\ b_{ijk}$

TEST-FIELD METHOD

developed

in papers by Schrinner, Rädler, Schmitt, Rheinhardt, Christensen, e.g., GAFD 101(2007) 81-116

(magnetoconvection, geodynamo)

applied

Sur et al. MNRAS 385 (2008) L15

(alpha and magnetic diffusivity in isotropic turbulence)

Brandenburg et al. ApJ 676 (2008) 740

(effects of shear and rotation, shear-current dynamo)

Brandenburg et al. A&A 482 (2008) 789

(scale dependence of alpha and magnetic diffusivity)

Brandenburg et al. ApJ L submitted

(alpha and magnetic diffusivity quenching)

••••••

TEST-FIELD METHOD

$$\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\partial\overline{B}_j/\partial x_k$$

The test-fields should be linearly independent
and all higher than first-order derivatives should be small .

They need not to satisfy any boundary conditions,
and they need not to be solenoidal.

$$\boldsymbol{\mathcal{E}}^{(n)} = \overline{\boldsymbol{u} \times \boldsymbol{b}^{(n)}}$$

$$\partial_t \boldsymbol{b}^{(n)} - \nabla \times (\overline{\boldsymbol{U}} \times \boldsymbol{b}^{(n)} + \boldsymbol{G}^{(n)}) - \eta\nabla^2 \boldsymbol{b}^{(n)} = \nabla \times (\boldsymbol{u} \times \overline{\boldsymbol{B}}^{(n)})$$

$$\mathcal{E}_i^{(n)} = a_{ij}\overline{B}_j^{(n)} + b_{ijk}\partial\overline{B}_j^{(n)}/\partial x_k$$

TEST-FIELD METHOD

$$\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\partial\overline{B}_j/\partial x_k$$

The test-fields should be linearly independent
and all higher than first-order derivatives should be small .

They need not to satisfy any boundary conditions,
and they need not to be solenoidal.

The test-field method works independent
on whether U depends on B.

It is therefore suitable for investigating magnetic quenching.

The test-field method implies no approximation !

A simple case

Assume that $\overline{\boldsymbol{B}}$ does not depend on x and y. Then all first–order spatial derivatives of $\overline{\boldsymbol{B}}$ can be expressed by $\overline{\boldsymbol{J}} = \nabla \times \overline{\boldsymbol{B}}$.

$$\Rightarrow \quad \mathcal{E}_i = \alpha_{ij}\overline{B}_j - \eta_{ij}\overline{J}_j \quad (1 \le i, j \le 2)$$

Choose testfields

$$\overline{\boldsymbol{B}}^{(1c)} = B(\cos kz, 0, 0)\,, \qquad \overline{\boldsymbol{B}}^{(2c)} = B(0, \cos kz, 0)$$

$$\overline{\boldsymbol{B}}^{(1s)} = B(\sin kz, 0, 0)\,, \qquad \overline{\boldsymbol{B}}^{(2s)} = B(0, \sin kz, 0)$$

Then

$$\alpha_{ij} = B^{-1}(\mathcal{E}_i^{(jc)} \cos kz + \mathcal{E}_i^{(js)} \sin kz)$$

$$\eta_{ij} = \cdots$$

Results apply exactly in the limit $k \to 0$

Some extension

$$\mathcal{E}_i = \alpha_{ij}\overline{B}_j - \eta_{ij}\overline{J}_j$$

→

Connection no longer local

$$\mathcal{E}_i(z) = \int \left[\alpha_{ij}(\zeta)\overline{B}_j(z+\zeta) - \eta_{ij}(\zeta)\overline{J}_j(z+\zeta)\right] \mathrm{d}\zeta$$

or

$$\hat{\mathcal{E}}_i(k) = \hat{\alpha}_{ij}(k)\hat{\overline{B}}_j(k) - \hat{\eta}_{ij}(k)\hat{\overline{J}}_j(k)$$

→

$$\hat{\alpha}_{ij}(k) = B^{-1}(\mathcal{E}_i^{(jc)} \cos kz + \mathcal{E}_i^{(js)} \sin kz)$$

$$\hat{\eta}_{ij}(k) = \cdots$$

(arbitrary k)