School of Mathematics & Statistics, Newcastle University

Geomagnetic reversals from low order/turbulent shell models

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Graeme Sarson, David Ryan

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A low order $\alpha\omega$ model

We want to model the statistics of geomagnetic reversals on long timescales, beyond the range of detailed simulations.

We consider the toy ODE model:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\kappa S + \alpha T \ ,$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\kappa T + \omega S \ ,$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\kappa_\omega \omega + f_\omega \left[1 - \lambda_1 ST - \lambda_2 (S^2 + T^2)\right] \ .$$

For constant α, irregular reversals can already be obtained (cf. Rikitake 1958). With multiplicative noise in mind (cf. Hoyng et al. 2001), we want to couple these equations to a dynamically varying form of α. (Ryan and Sarson, *GRL*, **34**, L02307, 2007.)

Shell model of turbulence

Shell models provide a scalar analogue of the spectral Navier–Stokes equation.

The spectral domain is represented by N shells, of wavenumbers $k_n = k_0 2^n$, $n = 1, 2, \ldots, N$.

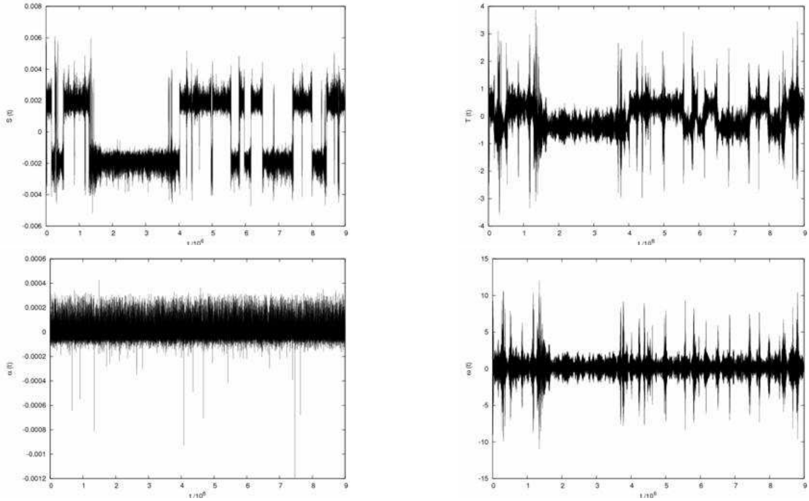
In the Gledzer–Ohkitani–Yamada (GOY) model, the complex modes u_n satisfy

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = -\nu k_n^2 u_n + f_\alpha \left[1 - \lambda_3 ST - \lambda_4 (S^2 + T^2)\right] \delta_{n,n_0}$$
$$+ik_n \left(u_{n+2}^* u_{n+1}^* - \frac{1}{4} u_{n+1}^* u_{n-1}^* + \frac{1}{8} u_{n-1}^* u_{n-2}^* \right) .$$

We base our α-effect on the shell-model helicity, H:

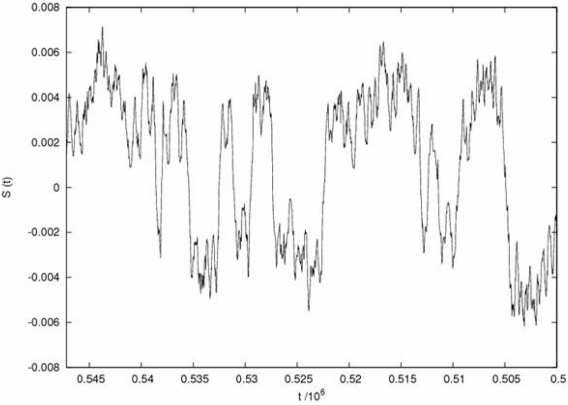
$$\alpha \sim -\frac{1}{3}\,\tau H\,, \qquad \Longrightarrow \qquad \alpha \sim -\frac{1}{3}\,\sum_{n=1}^{N} (-1)^n \left| u_n \right| \,.$$

Typical behaviour: S, T, α, ω



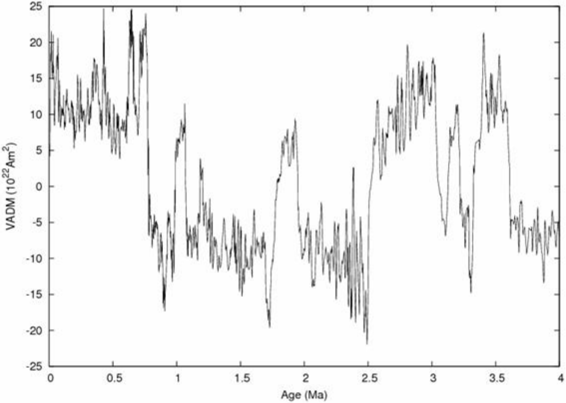
Synthetic Axial Dipole Moments (SADMs)

The ‘secular variation’ obtained in S can be surprisingly Earth-like.



Virtual Axial Dipole Moments (VADMs)

Similar 'saw-tooth' behaviour is observed in paleomagnetic VADMS (Valet & Meynadier 1993; Meynadier et al. 1994).



Reversal statistics

We compare the GO96 (Gradstein & Ogg 1996) reversal data with ca. 40 standard probability distribution functions (exponential, Gamma, etc.).

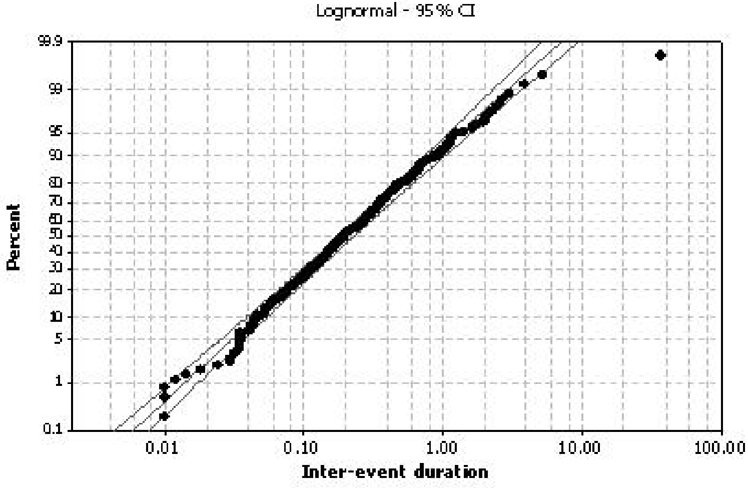
Anderson–Darling, Kolmogorov–Smirnov and χ^2 tests confirm that the inter-reversal durations are well fit by a lognormal distribution (and also by other 'heavy-tailed' distributions).

Lognormal and loglogistic distributions also give the best fits to the *synthetic* (model) reversal data.

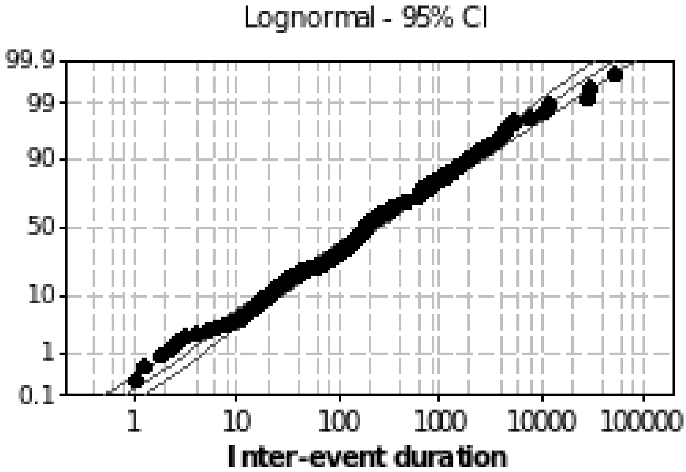
The *p*-values of these fits would not be rejected at any normal level of significance.

Synthetic reversals from *other* low order models in the literature are also best fit by lognormal and loglogistic distributions, but the fits are less significant.

Lognormal fit: GO96 data



Lognormal fit: synthetic data



Analysis of low order model behaviour

Analysis of our (de-coupled) low order model, for $\alpha = \alpha_c$ constant, suggests mechanisms for the variations observed.
Here there are three fixed points in (S, T, ω) space:

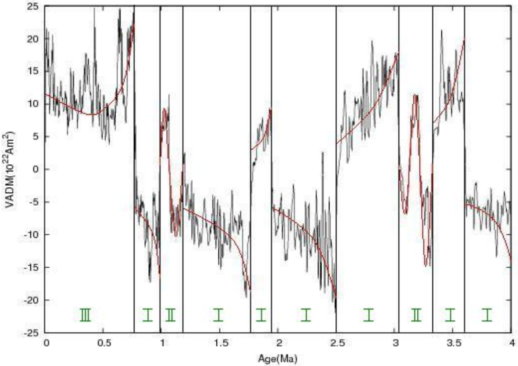
$$F^0 = (0, 0, f_\omega/\kappa_\omega) \ ,$$

$$F^\pm = (\pm\frac{\alpha_c}{\kappa}\sqrt{\frac{\kappa f_\omega \alpha_c - \kappa_\omega \kappa^3}{f_\omega \lambda_1 \alpha_c^2}}, \pm\sqrt{\frac{\kappa f_\omega \alpha_c - \kappa_\omega \kappa^3}{f_\omega \lambda_1 \alpha_c^2}}, \frac{\kappa^2}{\alpha_c}) \ .$$

- For $\alpha_c < 0$, F^0 is a stable spiral;
- For $0 \leq \alpha_c \leq \kappa^2 \kappa_\omega / f_\omega$, F^0 is a stable node;
- For $\kappa^2 \kappa_\omega / f_\omega < \alpha_c \leq \kappa / f_\omega (\kappa_\omega^2/4 + \kappa_\omega \kappa)$, $F^\pm$ are stable nodes (locally);
- For $\alpha_c > \kappa / f_\omega (\kappa_\omega^2/4 + \kappa_\omega \kappa)$, $F^\pm$ are stable spirals (locally);
- For $\alpha_c > \alpha_A$, $F^\pm$ are globally unstable to a chaotic attractor, allowing reversals.

Interpretation of VADM reversals?

Fluctuations in α (and of S, T and $\omega\ldots$) suggest tentative identification of different types of reversals,



We need some more quantitative analysis, however.

Phase space reconstruction: SINT 2000 data

We construct the delay vectors

$$V_i = \{S_i, S_{i+\tau}, S_{i+2\tau}, ..., S_{i+(m-1)\tau}\} \ ,$$

for embedding dimension m and delay τ. (Ryan and Sarson, *EPL*, **83**, 49001, 2008.)

For the SINT 2000 paleointensity data (Valet et al. 2005):

- the mutual information method suggests an optimum $\tau = 13$;
- the method of false nearest neighbours suggests an embedding dimension in the range $m = 5$–7.

Phase space reconstruction: SINT 2000 data

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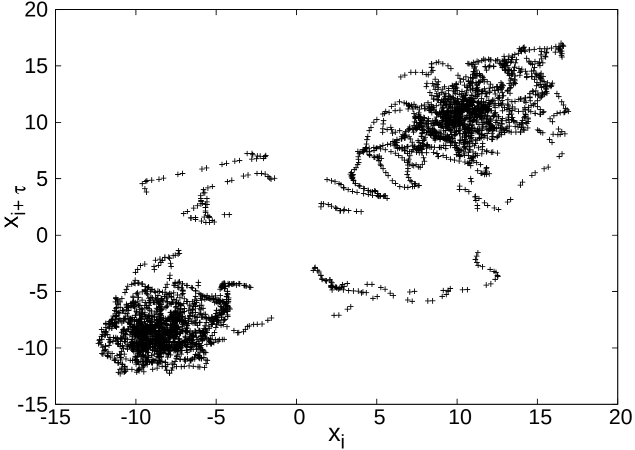
for embedding dimension m and delay τ. (Ryan and Sarson, *EPL*, **83**, 49001, 2008.)

We can analyse these embedded vectors for characteristics of deterministic chaos:

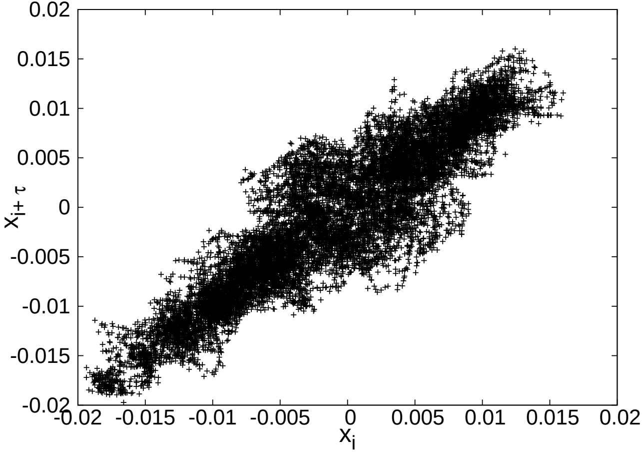
- the determinism test of Kaplan and Glass (1992) gives $\Lambda = 0.87$;
- various algorithms (Wolf et al. 1985; Rosenstein et al. 1994; Kantz 1994) give a maximum Lyapunov exponent of 11 Ma^{-1};
- the scale dependent Lyapunov exponent (SDLE) behaviour is consistent with intermittent chaos (Gao et al. 2006).

Phase portrait: SINT 2000 data

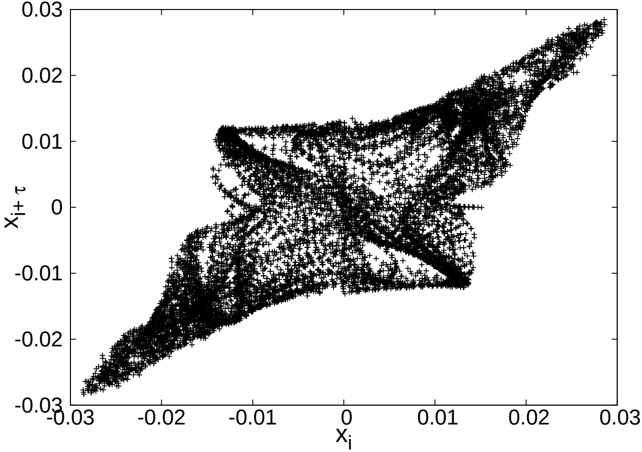
The embedded data-sets arguably exhibit similar attractors.



Phase portrait: synthetic data (turbulent shell α)



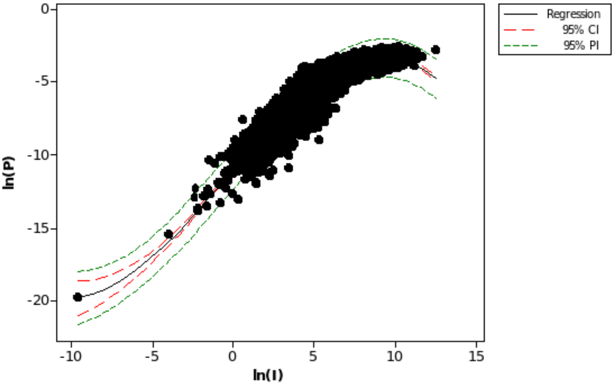
Phase portrait: Synthetic data (quenched stochastic α)



Peak Intensity-Duration Relation: synthetic data

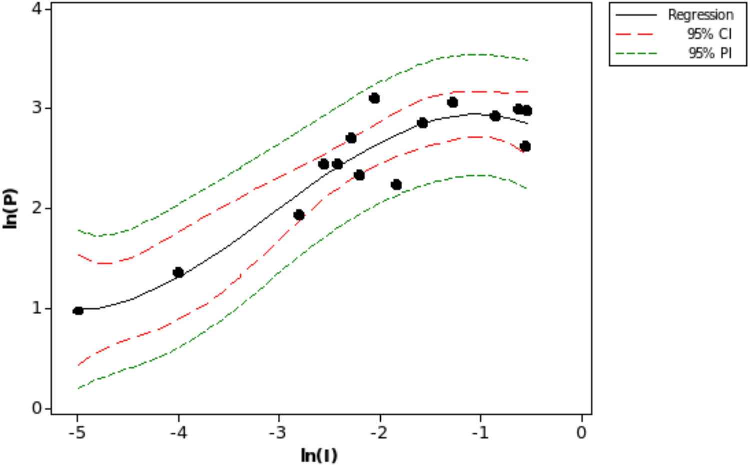
For our model, peak field intensity P and inter-reversal duration I data are well fit by the relation

$$\ln P = C_0 + C_1 \ln I + C_2 \ln^2 I + C_3 \ln^3 I \; .$$



Intensity-Duration relation: Valet & Meynadier (1993) data

Such a relation is also plausible for the VADM data.



Conclusions

- Observed reversal inter-event times, and VADM variations, are well-fit by lognormal distributions.
- Our coupled low order/turbulent shell model reproduces similar distributions.
- The paleomagnetic data is itself surprisingly well modelled as a low dimensional system, and shows evidence of intermittent deterministic chaos.
- Our model reversals exhibit a characteristic variation of peak-field intensity with chron duration, and the paleomagnetic data are consistent with a similar variation.