Anyon condensation and the color code

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rx	gx	bx
ry	gy	by
rz	gz	bz

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Outline

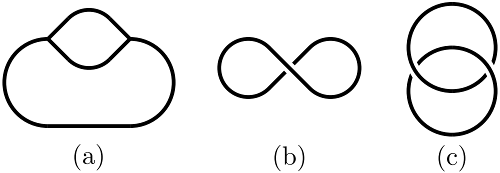
- Anyons and the vacuum
- Color code and its anyons
- Anyon condensation using the color code boson table
- Generalising Floquet codes with anyon condensation
- Constructing Floquet code boundaries with condensation

Topological quantum error-correcting codes

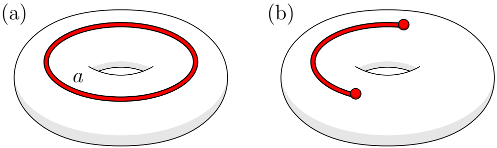
We write down a set of charge labels,
e.g., the toric code charge labels are:

$$\mathcal{C} = \{1, \quad \mathtt{e}, \quad \mathtt{m}, \quad \mathtt{f} = \mathtt{e} \times \mathtt{m}\}\,.$$

Anyon models have exchange, fusion and braid statistics



Logical operators of topological codes can be viewed as hopping operators over some manifold (or lattice of qubits)



The vacuum, charge label 1

The vacuum, charge label 1

- The vacuum has trivial exchange (i.e., bosonic):

$$R_{1,1} = 1$$

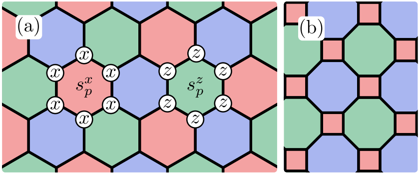
- The vacuum fuses trivially with all anyons $a \in \mathcal{C}$:

$$a \times 1 = a$$

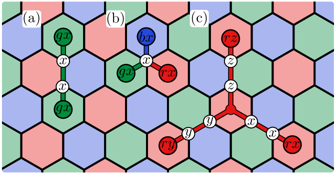
- The vacuum braids trivially with all anyons $a \in \mathcal{C}$:

$$S_{a,1} = 1$$

The color code



Stabilizer code $S_j \in \mathcal{S}$



Excitations

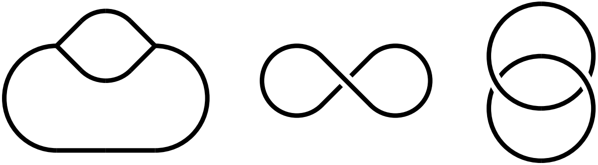
$$S_j|\psi\rangle = (+1)|\psi\rangle \qquad H = -\sum_j S_j$$

Color code bosons are labeled with:
a Pauli type $\mathtt{x}, \mathtt{y}, \mathtt{z}$
and a color $\mathtt{r}, \mathtt{g}, \mathtt{b}$
e.g.

$$\mathtt{rx}, \mathtt{gz}, \mathtt{by}, \mathtt{bz} \text{ etc. etc. etc.}$$

Bombin and Martin-Delgado, Phys. Rev. Lett. (2006)

Color code anyons



(a) (b) (c)

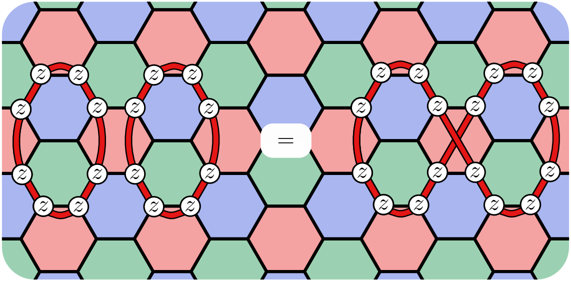
rx	gx	bx
ry	gy	by
rz	gz	bz

$\longrightarrow$

e1		1m
1m		m1

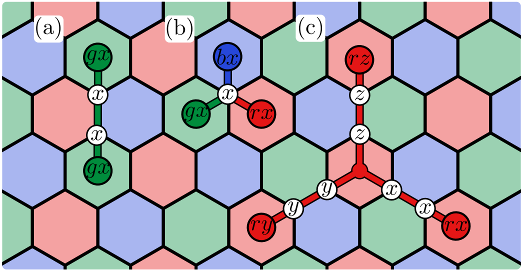
Kesselring *et al.*, Quantum (2018)
Kubica *et al.*, New J. Phys. (2015)

Color code anyons (bosonic self statistics)



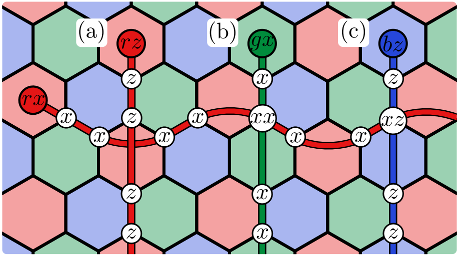
rx	gx	bx
ry	gy	by
rz	gz	bz

Color code anyons (fusion rules)



rx	gx	bx
ry	gy	by
rz	gz	bz

Color code anyons (mutual braid statistics)



rx	gx	bx
ry	gy	by
rz	gz	bz

Anyon condensation

We can condense a single boson.

●	gx	bx
ry	gy	by
rz	gz	bz

This means we identify a boson, e.g. rx with the vacuum. We write

$$\texttt{rx} \simeq 1.$$

In the boson table

$$\bullet = \text{'condensed'}$$

What does this mean with respect to the anyon model?

Remember, now charge label $1 \simeq$ `rx`

- The vacuum has trivial exchange (i.e., bosonic):

$$R_{1,1} = 1$$

- The vacuum braids trivially with all anyons $\mathrm{a} \in \mathcal{C}$:

$$S_{\mathrm{a},1} = 1$$

- The vacuum fuses trivially with all anyons $\mathrm{a} \in \mathcal{C}$:

$$\mathrm{a} \times 1 = \mathrm{a}$$

What does this mean with respect to the anyon model?

Remember, now charge label $1 \simeq \mathtt{rx}$

- The vacuum has trivial exchange (i.e., bosonic):

$$R_{\mathtt{rx},\mathtt{rx}} = 1 \qquad \checkmark$$

- The vacuum braids trivially with all anyons $\mathrm{a} \in \mathcal{C}$:

$$S_{\mathrm{a},1} = 1$$

- The vacuum fuses trivially with all anyons $\mathrm{a} \in \mathcal{C}$:

$$\mathrm{a} \times 1 = \mathrm{a}$$

What does this mean with respect to the anyon model?

Remember, now charge label $1 \simeq \texttt{rx}$

- The vacuum braids trivially with all anyons $\mathsf{a} \in \mathcal{C}$:

$$S_{\mathsf{a},\texttt{rx}} = 1$$

As $S_{\times,\texttt{rx}} \neq 1$, we have $\times$ anyons are now forbidden. They are 'confined'.

●	gx	bx
ry	×	×
rz	×	×

● = 'condensed'
× = 'confined'

What does this mean with respect to the anyon model?

Remember, now charge label $1 \simeq \mathtt{rx}$

- The vacuum braids trivially with all anyons $\mathtt{a} \in \mathcal{C}$:

$$S_{\mathtt{a},\mathtt{rx}} = 1$$

for

$$\mathtt{a} = \mathtt{ry}, \mathtt{rz}, \mathtt{gx}, \mathtt{bx}.$$

$$\checkmark$$

Now, $\mathtt{ry}$, $\mathtt{rz}$, $\mathtt{gx}$, $\mathtt{bx}$, remain ‘deconfined’.

●	gx	bx
ry	×	×
rz	×	×

What does this mean with respect to the anyon model?

Remember, now charge label $1 \simeq \texttt{rx}$

- The vacuum fuses trivially with all anyons $\texttt{a} \in \mathcal{C}$:

$$\texttt{a} \times 1 = \texttt{a}$$

What does this mean with respect to the anyon model?

Remember, now charge label $1 \simeq \texttt{rx}$

- The vacuum fuses trivially with all anyons $\texttt{a} \in \mathcal{C}$:

$$\texttt{a} \times \texttt{rx} = \texttt{a},$$

In the parent model we have:

$$\texttt{bx} \times \texttt{rx} = \texttt{gx}$$

But, for condensed $\texttt{rx}$ we need

$$\texttt{bx} \times \texttt{rx} \doteq \texttt{bx}.$$

For the above two relationships to hold we find the identification:

$$\texttt{gx} \simeq \texttt{bx},$$

Likewise

$$\texttt{ry} \simeq \texttt{rz}$$

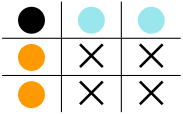
by the same argument.

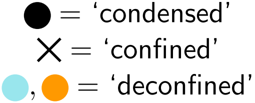
What does this mean with respect to the anyon model?

Remember, now charge label $1 \simeq \mathtt{rx}$

rx	gx	bx
ry	gy	by
rz	gz	bz

$\longrightarrow$





where now the following identifications are made

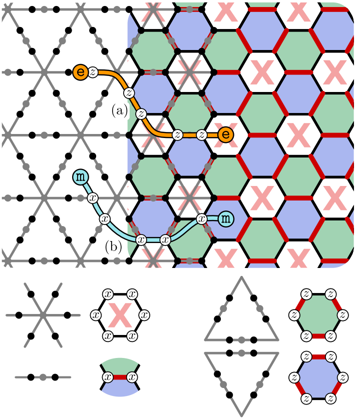
$$\mathtt{ry} \simeq \mathtt{rz} \simeq$$ 

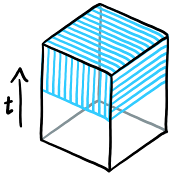
and

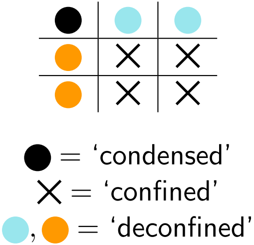
$$\mathtt{gx} \simeq \mathtt{bx} \simeq$$

due to condensation. ✓

Condensing a single boson creates a toric code phase



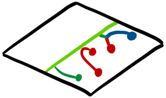




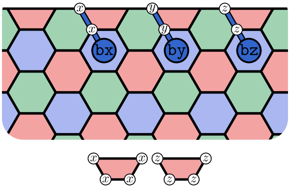
Physically, in the code picture we condense the `rx` charges by measuring the red XX edge terms.

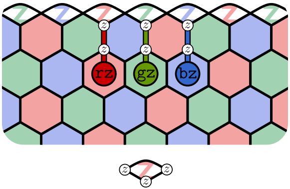
In the Hamiltonian picture we might add hopping terms V for `rx` charges $H = H_0 + V$

Lagrangian subgroups and color code boundaries (maximal condensation)



rx	gx	bx
ry	gy	by
rz	gz	bz





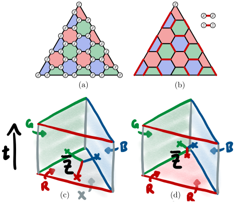
×	×	●
×	×	●
×	×	●

×	×	×
×	×	×
●	●	●

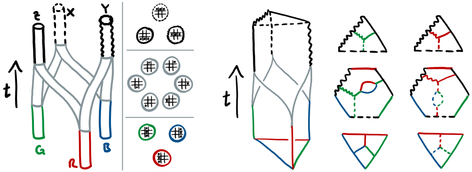
Levin, Phys. Rev. X (2013)
Kesselring et al., Quantum **2**, 101 (2018)

Applications of color-code anyon condensation

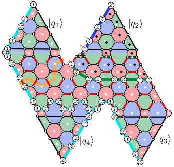
Initialisation and readout

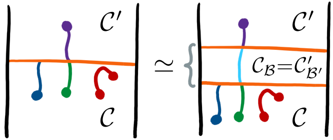


new types of code deformation



General theory for semi-transparent domain walls



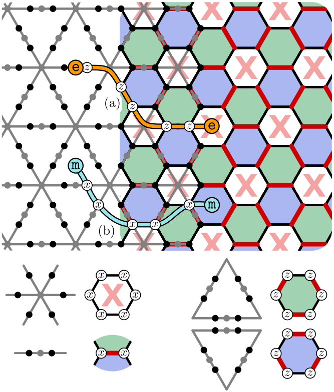


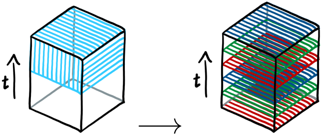
Color code lattice surgery

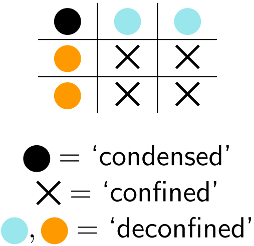
For references on color code lattice surgery see:
Landahl and Ryan-Anderson, arXiv:1407.5103 (2014)
Thomsen *et al.* arXiv:2201.07806 (2022)

Floquet codes

Hastings and Haah recently proposed Floquet codes, such as the honeycomb code, where syndromes are read out using a sequence of two-body measurements.



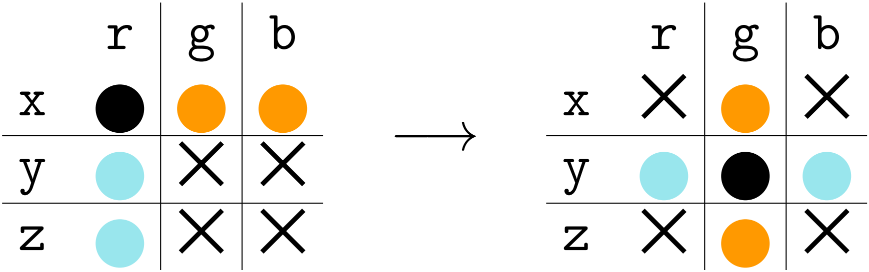




Hastings and Haah, Quantum (2021)

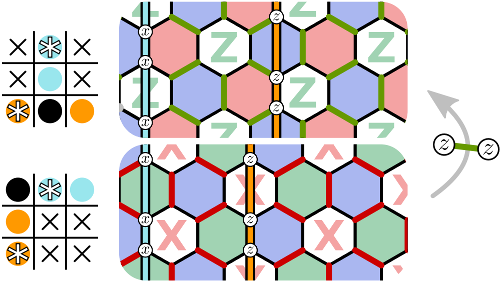
Floquet code transformations

Floquet code transformations can be viewed as a condensation operation on a confined boson.



Floquet code transformations

A single transformation from the `rx` condensed code to the gz condensed code

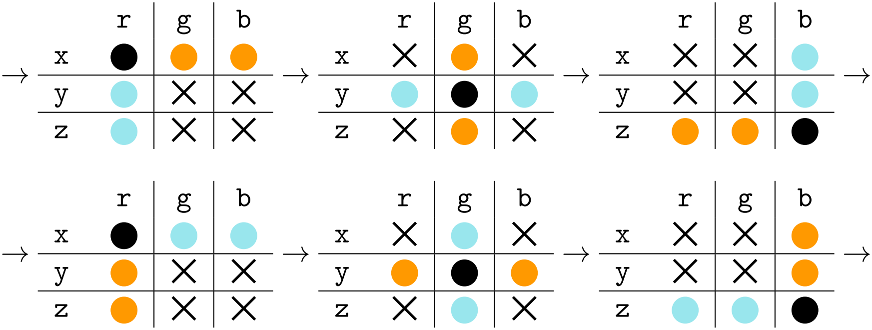


Over the transformation:

- Green Pauli-X stabilizers are turned off
- Red Pauli-Z stabilizers are initialised
- Blue Pauli-Z stabilizers are measured
- A pair of deconfined charges are preserved over the transformation
- Up to a color and Pauli label, the code is preserved

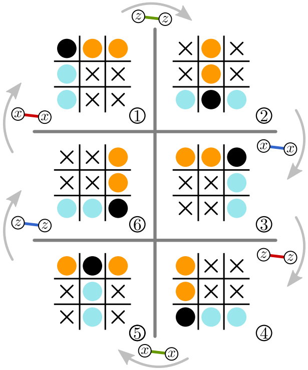
The honeycomb code

The honeycomb code cyles between a $\rightarrow$ rx $\rightarrow$ gy $\rightarrow$ bz $\rightarrow$ condensate



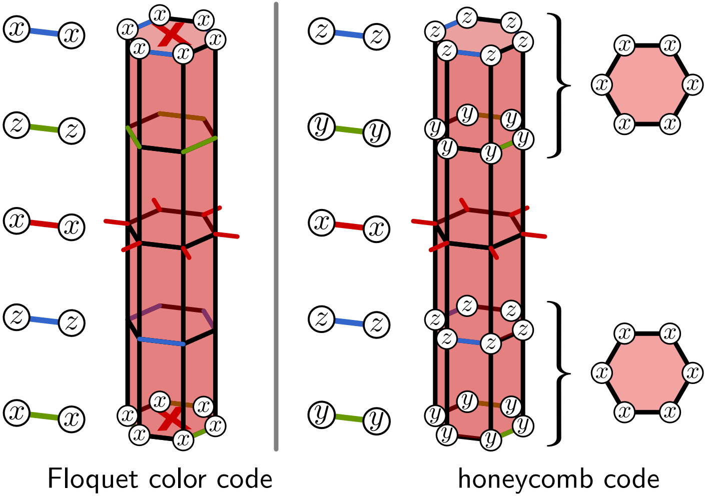
The Floquet color code

We propose a CSS Floquet code, that we call the Floquet color code, by changing the condensation sequence.



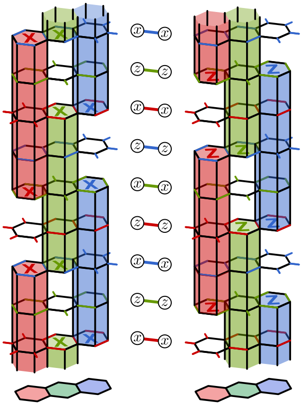
Detection cells - stabilizers in spacetime

Measurements on each hexagon are compared over time to detect errors.



Detection cells in spacetime

In the Floquet color code, we measure Pauli-X and Pauli-Z detection cells on all hexagons.

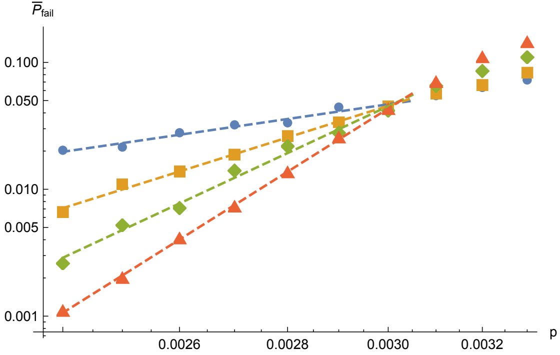


However, we also TURN OFF stabilizers.

Expressed as a subsystem code, the Floquet color code has no local stabilizers

A numerical threshold for circuit-level noise

We simulated the Floquet color code for standard circuit level noise.

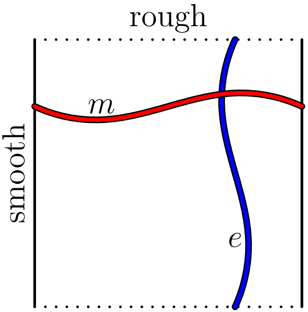


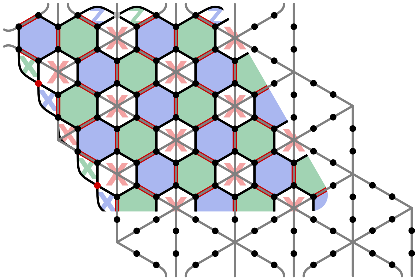
The threshold is competitive with the honeycomb code.

Gidney, Newman, McEwen, arXiv:2202.11845 (2022)

Boundaries in the condensation picture

The condensation picture shows us a constructive way to write down boundaries





A planar implementation of a Floquet code requires rough and smooth boundaries.

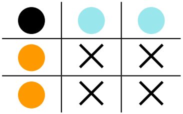
Dennis *et al.* J. Math. Phys. (2001)
Haah and Hastings, Quantum (2022)

Boundaries from the parent model

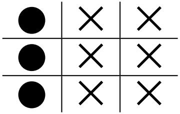
We condense the `rx` boson of the color code

rx	gx	bx
ry	gy	by
rz	gz	bz

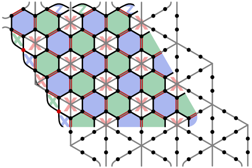
$\longrightarrow$

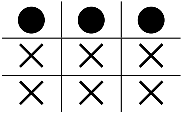


The parent rough boundary should condense red charges and the parent smooth boundary should condense Pauli-X charges.

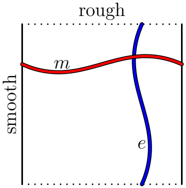


red parent boundary



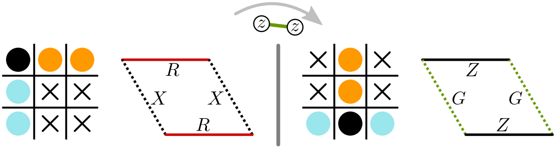


Pauli-X parent boundary



Boundary transformations

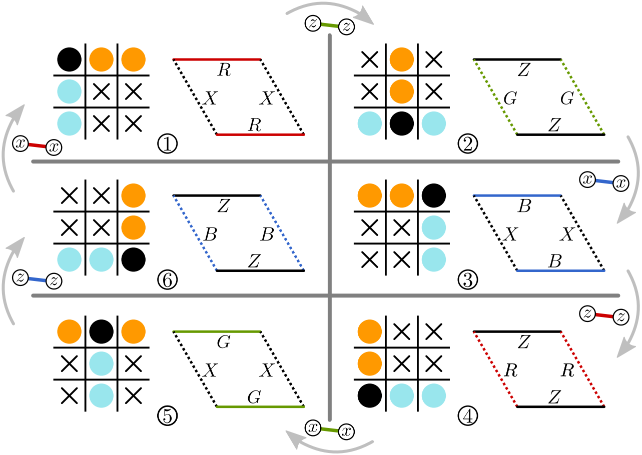
The condensation picture dictates how the boundaries must transform.



And the new boundaries can also be obtained from the parent color code theory.

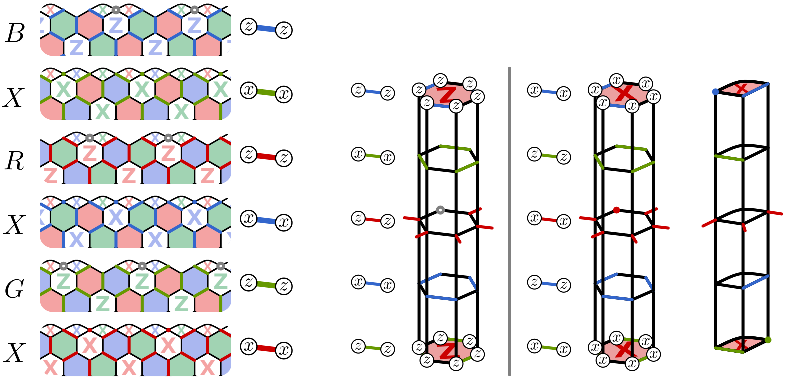
Boundary transformations for the Floquet color code

The transformations can be extended to a full period of the Floquet color code

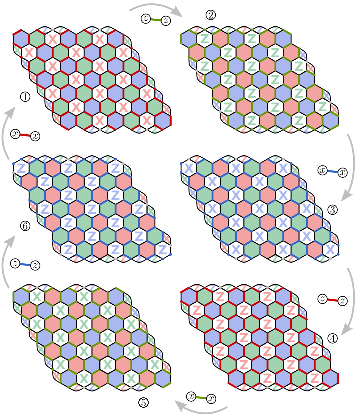


Microscopic boundary details

We used the macroscopic theory to obtain a microscopic measurement pattern.



Microscopic boundary details



We can find boundary configurations that do not drift.

Gidney, Newman, McEwen, arXiv:2202.11845 (2022)
Paetznick *et al.* arXiv:2202.11829 (2022)

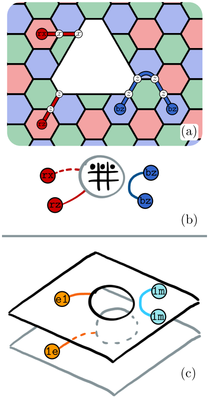
Outlook

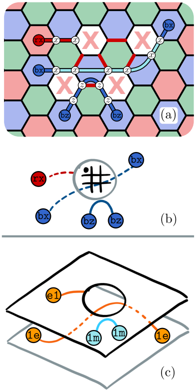
- Anyon condensation is a helpful picture to obtain and generalise Floquet codes from a parent theory.
- From our picture we can see that:
  - We can produce a new CSS Floquet code
  - We can constructively write down boundary stabilizers
  - Periodicity is not essential to Floquet codes
  - We do not need a constant local stabilizer group among checks
- Can we generalise Floquet codes by choosing exotic parent theories with other non-trivial condensate phases?

Unfolding punctures and semi punctures

We can ‘unfold’ the color code into two copies of the toric code

	r	g	b
x	e1		1e
y			
z	1m		m1

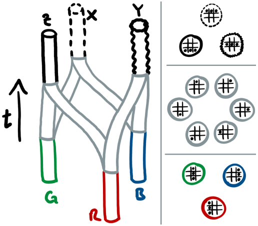


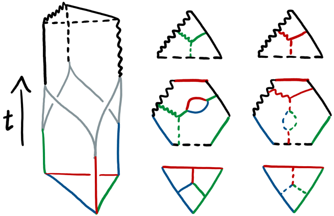


Some types of semi puncture are easily understood in the unfolded

Unfolding punctures and semi punctures

But the color code picture readily shows us new types of code deformations.





Punctures can be deformed into pairs of semi punctures.

This gives us some new identities for different logical encodings.