Recent Developments in Asymptotically Good Quantum LDPC Codes

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DYNISQ22 – KITP

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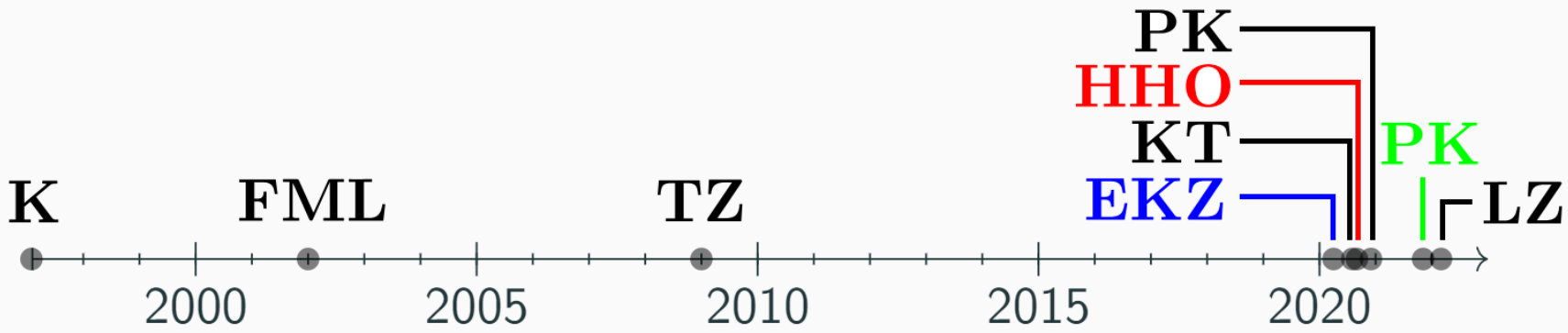
What is the fundamental limitation to efficient quantum information storage?

The Quantum LDPC Progression

k	d	Name
2	$\sqrt{n\sqrt{\log n}}$	Freedman-Meyer-Luo
n	$\sqrt{n}$	Tillich-Zémor
n	$\sqrt{n}\log n$	Evra–Kaufman–Zémor
$\sqrt{n}$	$\sqrt{n}\log^c n$	Kaufman-Tessler
$n^{3/5}/\mathrm{polylog}n$	$n^{3/5}/\mathrm{polylog}n$	Haah-Hastings-O'Donnell
$\log n$	$n/\log n$	Panteleev-Kalachev
$\Theta(n)$	$\Theta(n)$	Panteleev-Kalachev
$\Theta(n)$	$\Theta(n)$	Leverrier-Zémor



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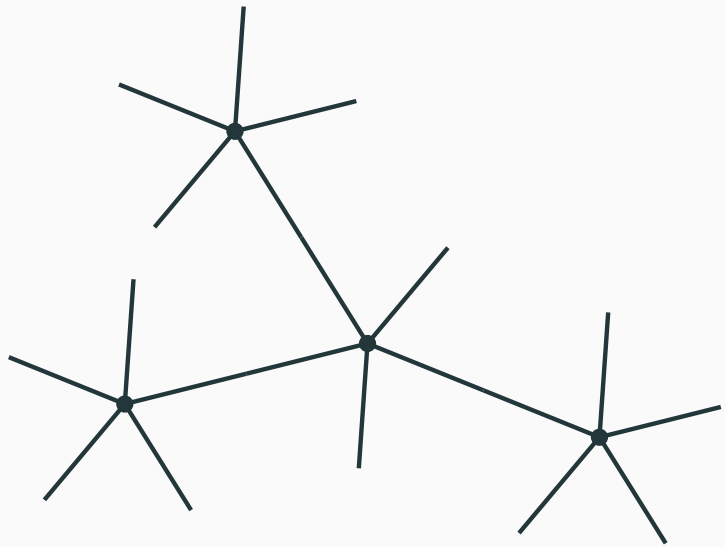
The code family is *low-density parity check* (LDPC) if every row and column of H has $O(1)$ non-zero entries.

Classical Tanner Codes

Take a Δ-regular expander graph $\Gamma = (V, E)$.



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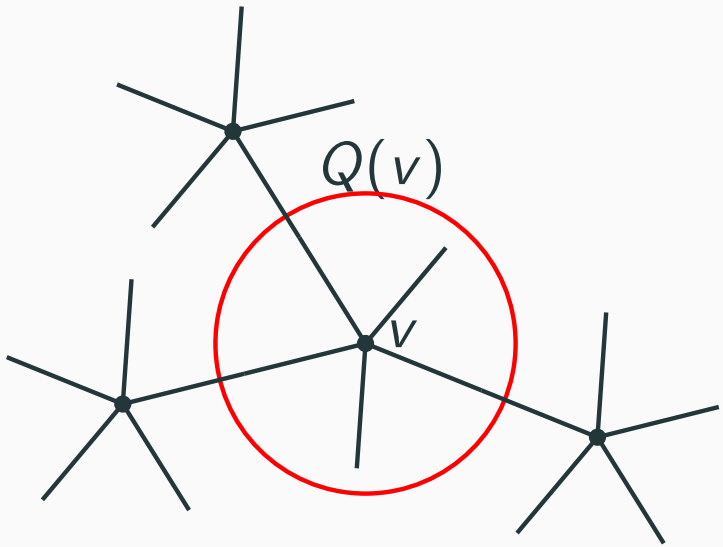

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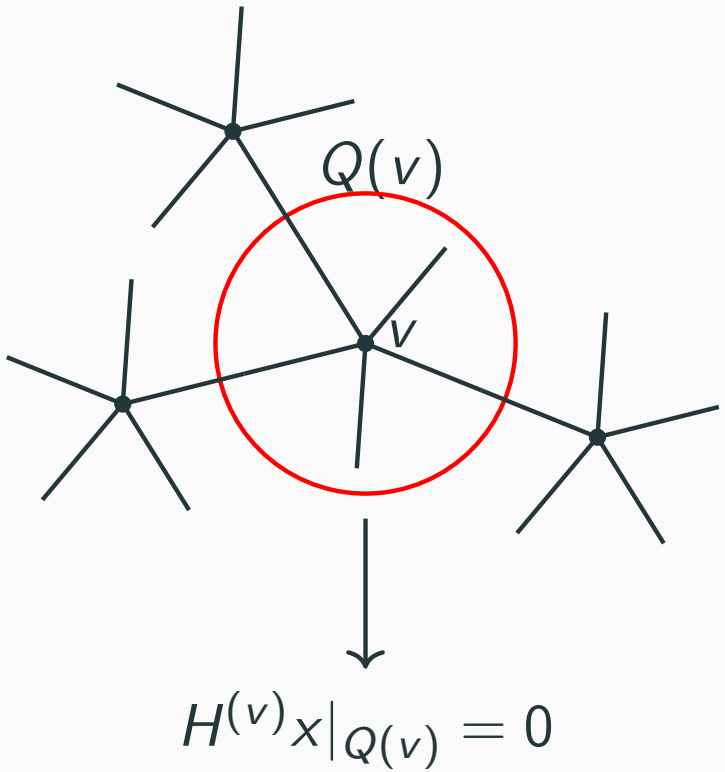
The *Tanner code* $T(\Gamma, C) \subseteq \mathbb{F}_2^E$ is defined with bits placed on the edges of Γ so that $x \in T$ if and only if $x|_{Q(v)} \in C$ for each vertex $v \in V$.



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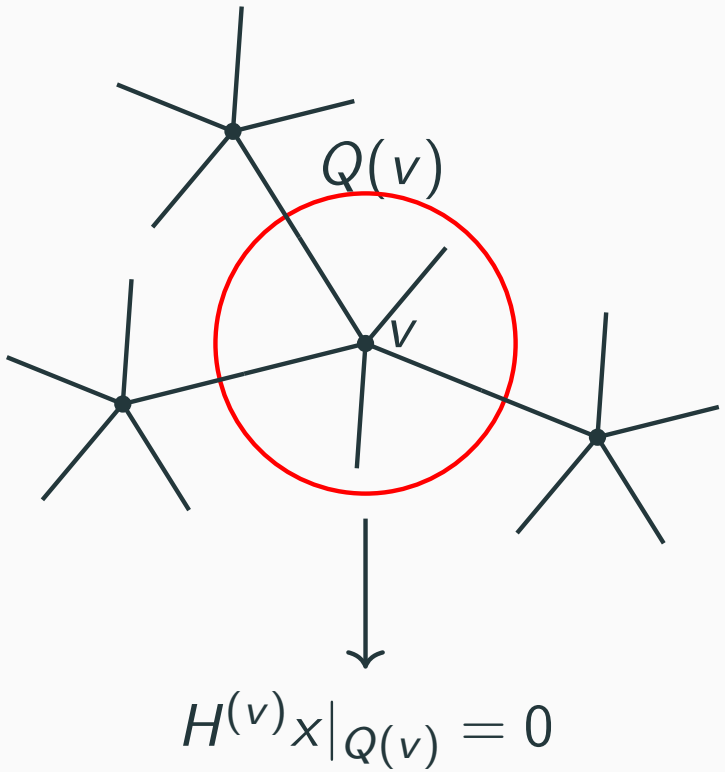
Efficiently decodable good classical LDPC code family [SS96].



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4. Sufficient expansion implies $|x| \geq \delta\Delta|V'|/2 = \Theta(|E|)$.

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The distance of the CSS code is given by

$$d_X = \min_{x \in C_Z \setminus C_X^{\perp}} |x|, \quad d_Z = \min_{x \in C_X \setminus C_Z^{\perp}} |x|, \tag{5}$$

with $d = \min(d_X, d_Z)$.

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2. Appropriate choices of local codes to ensure that the two codes compatible. This local compatibility is amplified by the expanding complex $\Gamma_1 \otimes_G \Gamma_2$ into a global compatibility.

Left-Right Cayley Complex [BE20, DELLM21]

Let G be a group. Let $A, B \subseteq G$ be two generating sets of size Δ such that $\mathrm{Cay}(A, G)$ and $\mathrm{Cay}(G, B)$ are bipartite λ-expanders.

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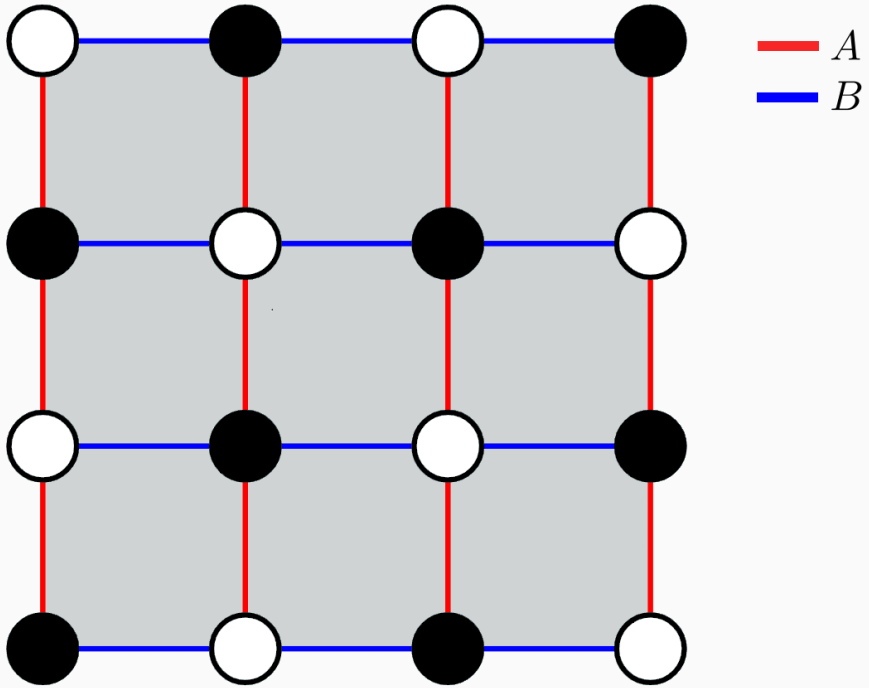
   We impose a technical restriction, called *total non-conjugacy* to ensure that faces are non-degenerate.

Quantum Tanner Codes [LZ21]



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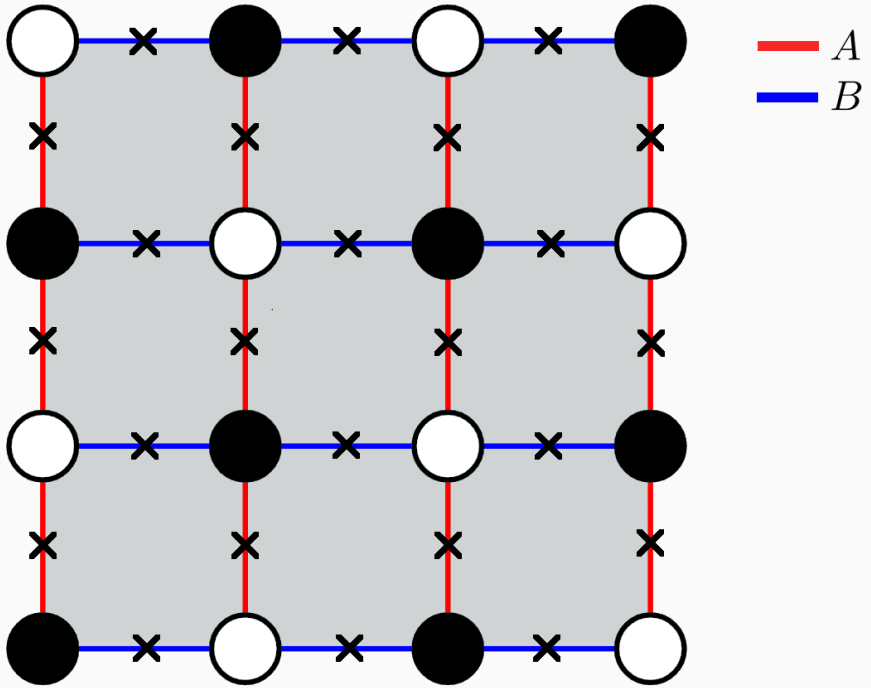

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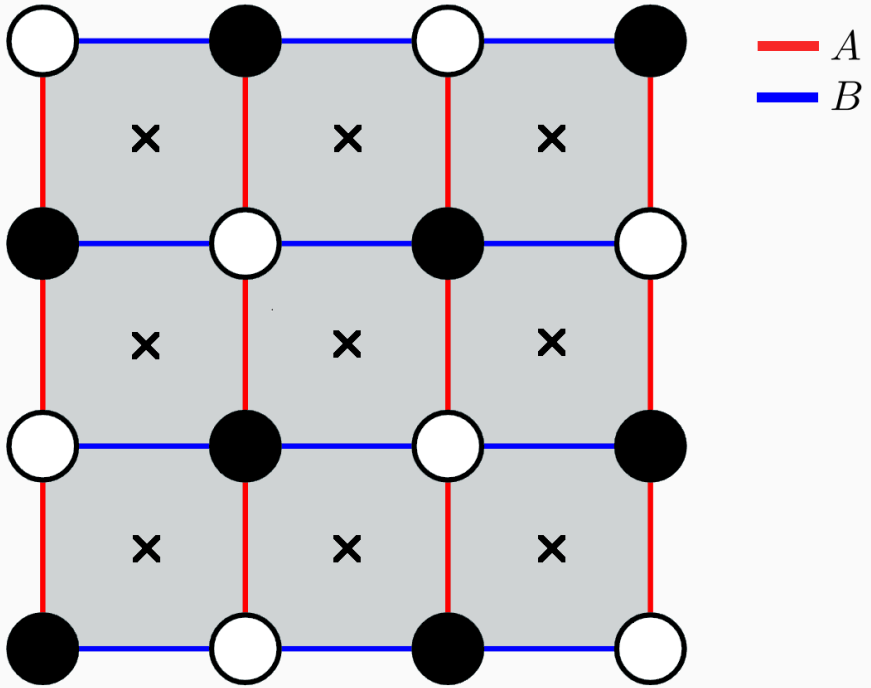

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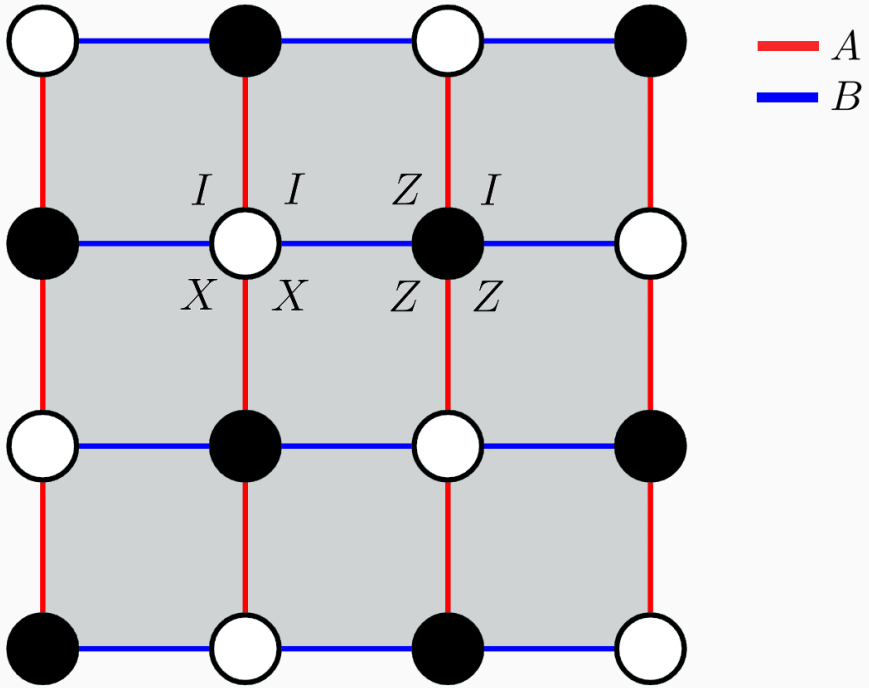
We will choose local codes C_X and C_Z such that X (resp. Z) stabilizers correspond to codewords of C_X (resp. C_Z).



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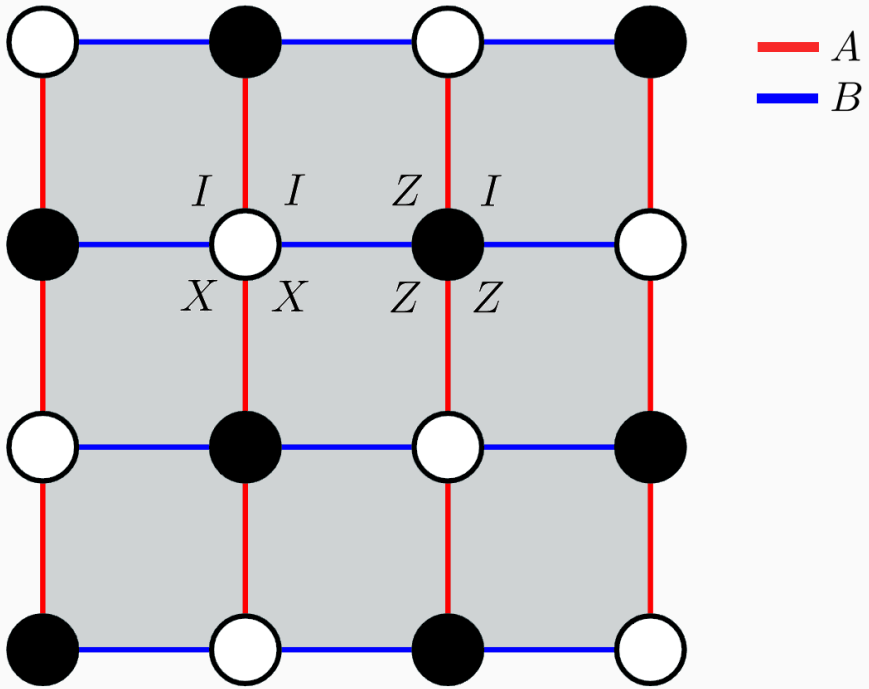

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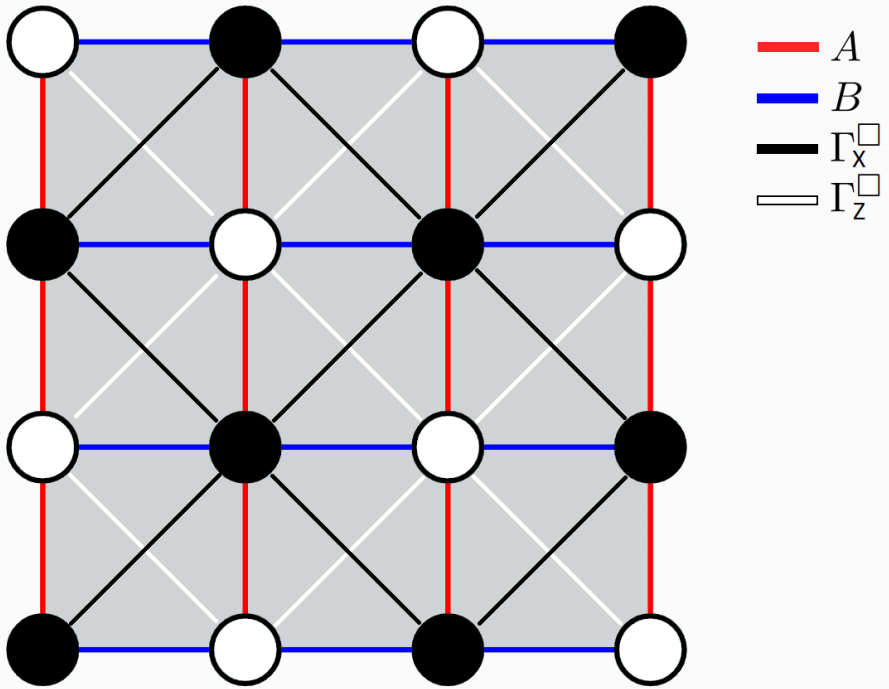

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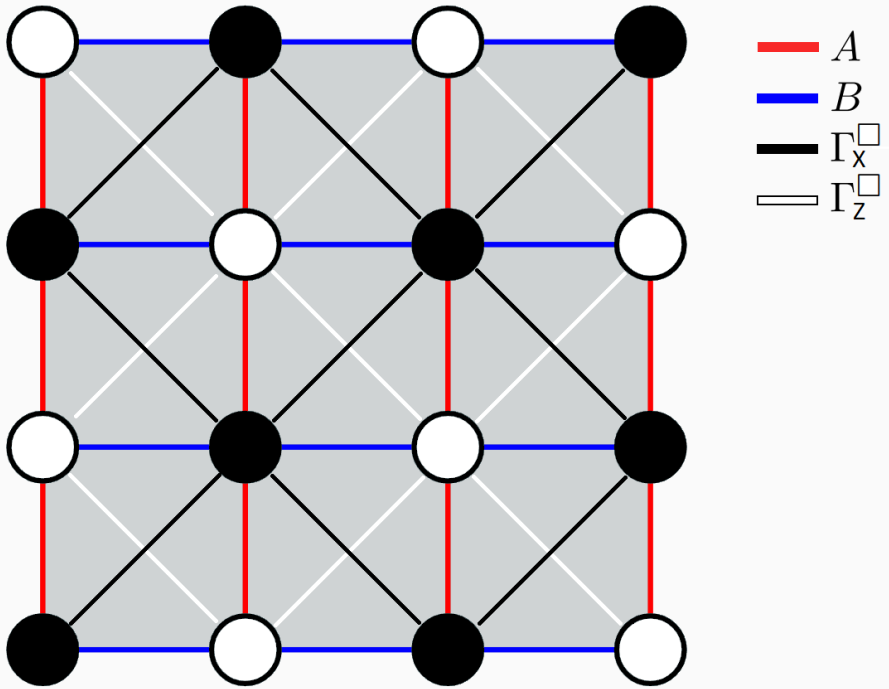
Then we have

$$T_X \equiv \ker H_X = T(\Gamma_X^{\square}, C_X^{\perp}),$$
$$T_Z \equiv \ker H_Z = T(\Gamma_Z^{\square}, C_Z^{\perp}),$$

with $\mathcal{C} = \mathrm{CSS}(T_X, T_Z)$.



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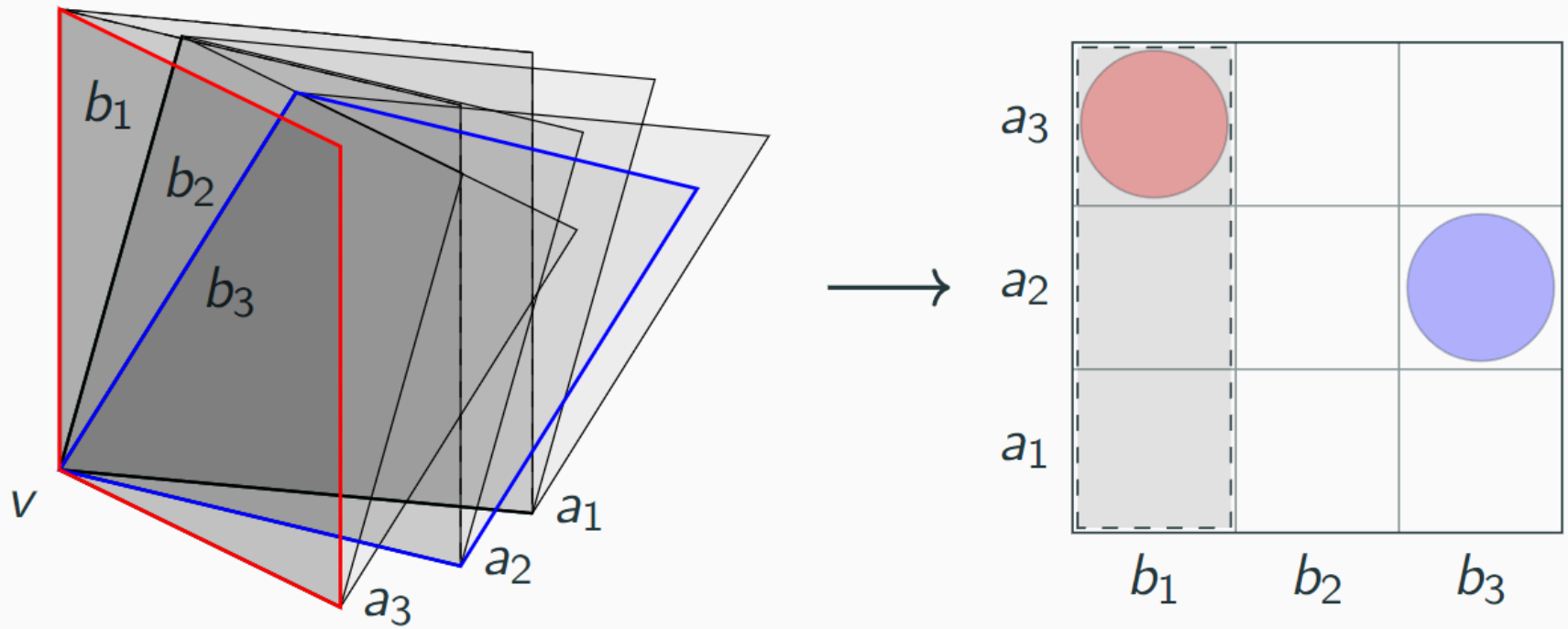

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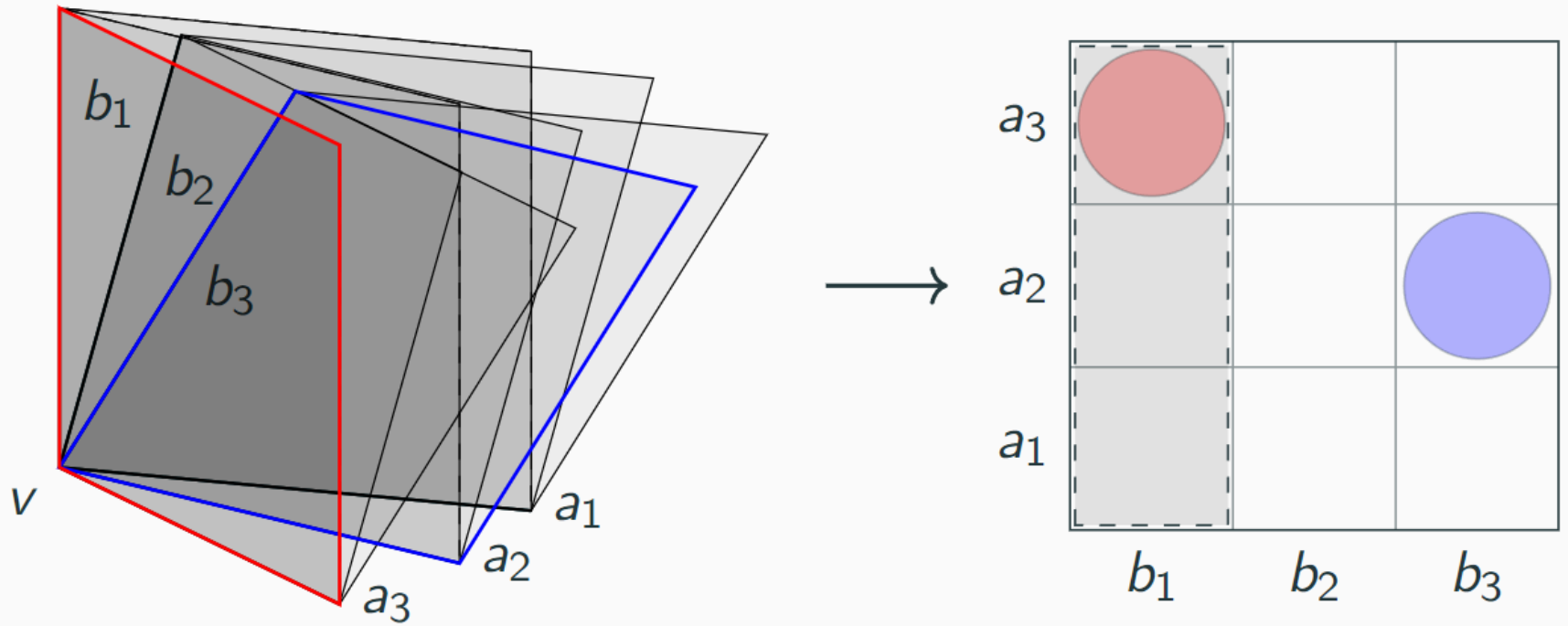
Convenient to pick the local codes as *tensor codes*: $C = C_1 \otimes C_2$.



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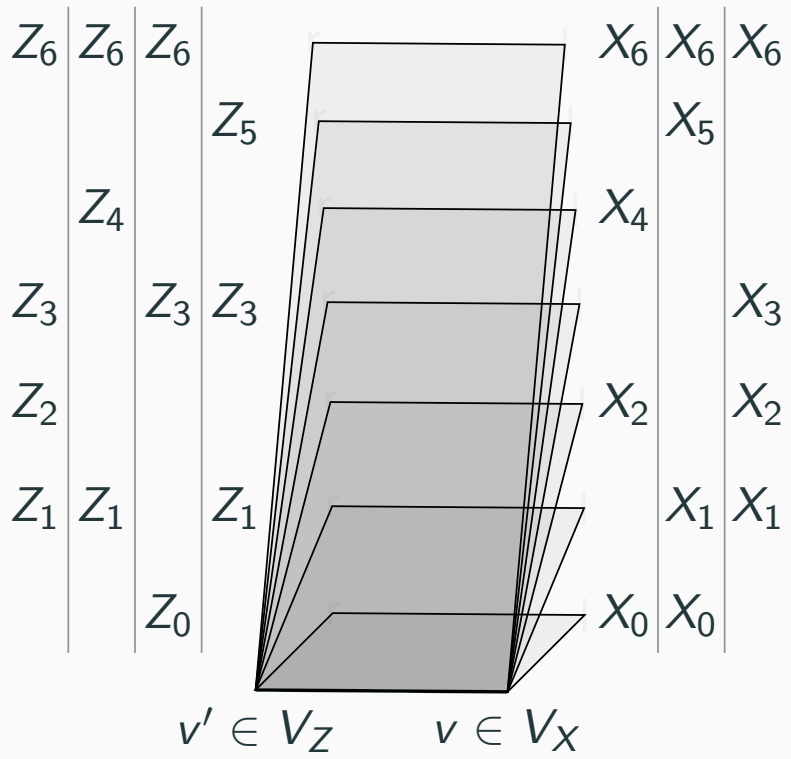
An X-stabilizer can overlap with a Z-stabilizer iff their anchoring vertices are adjacent. Local views coincide on a shared row or column.



Classical Tensor Codes

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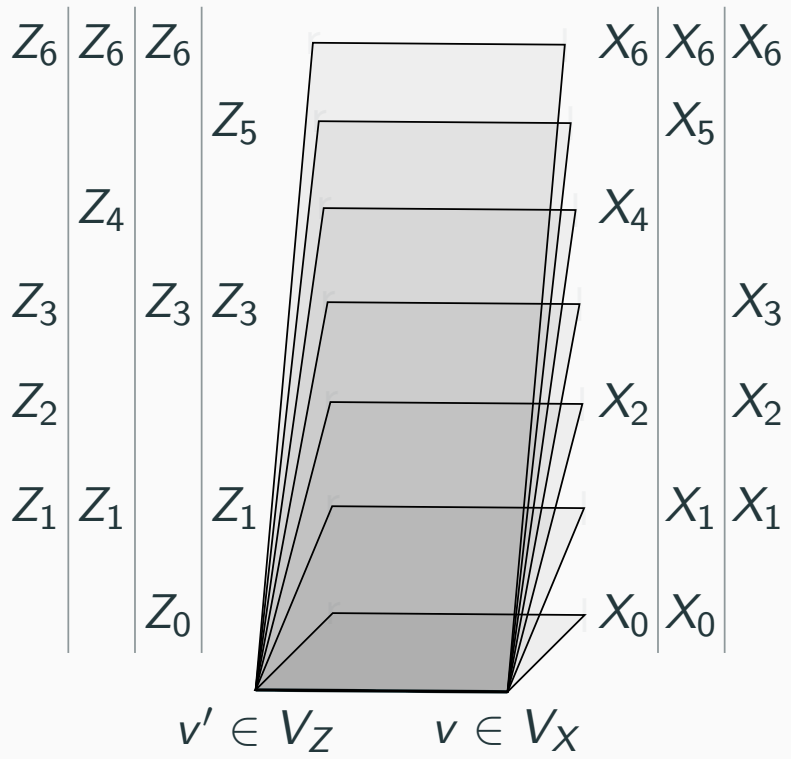
To satisfy the CSS condition, we want to choose C_X so that each row is orthogonal to C_B and each column to C_A, i.e., $C_X = C_A^\perp \otimes C_B^\perp$.



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Let's look at the constituent Tanner code T_X. A string $x \in \mathbb{F}_2^Q$ is a codeword for T_X iff

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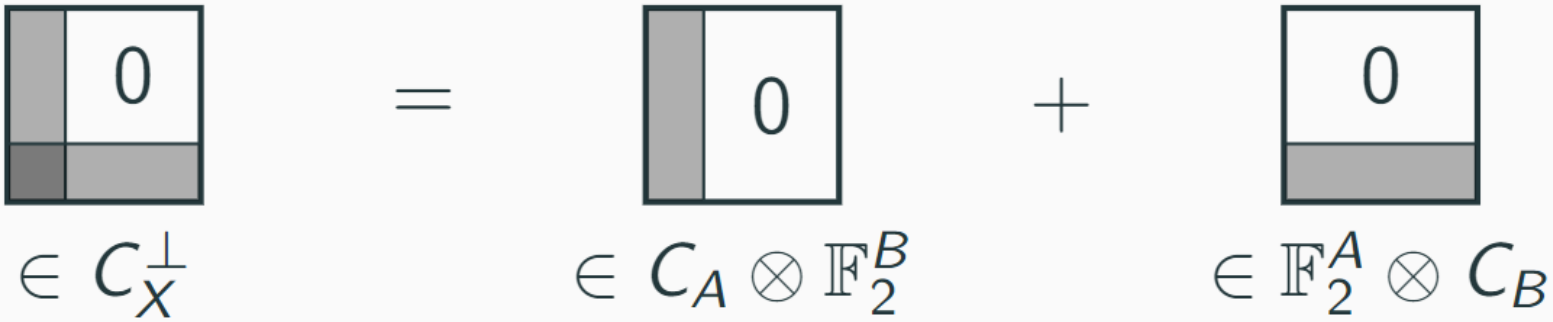

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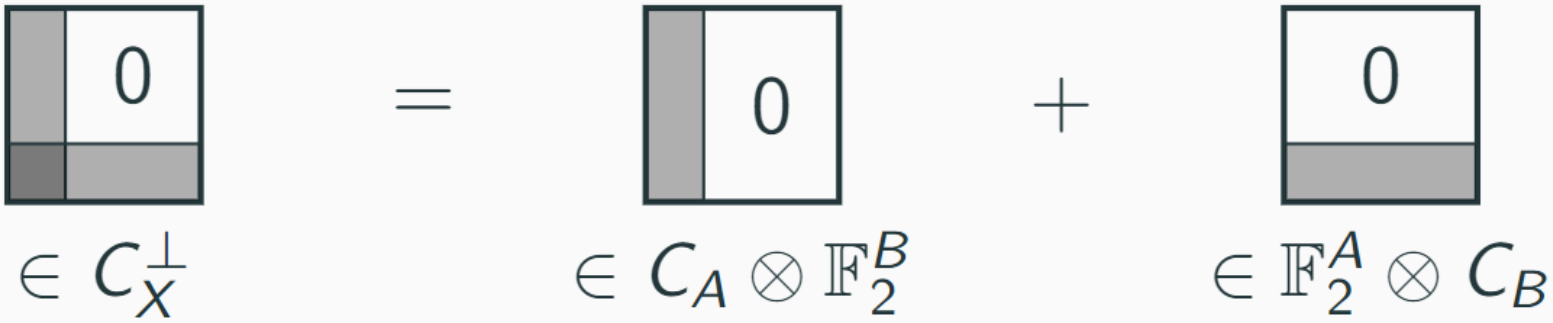

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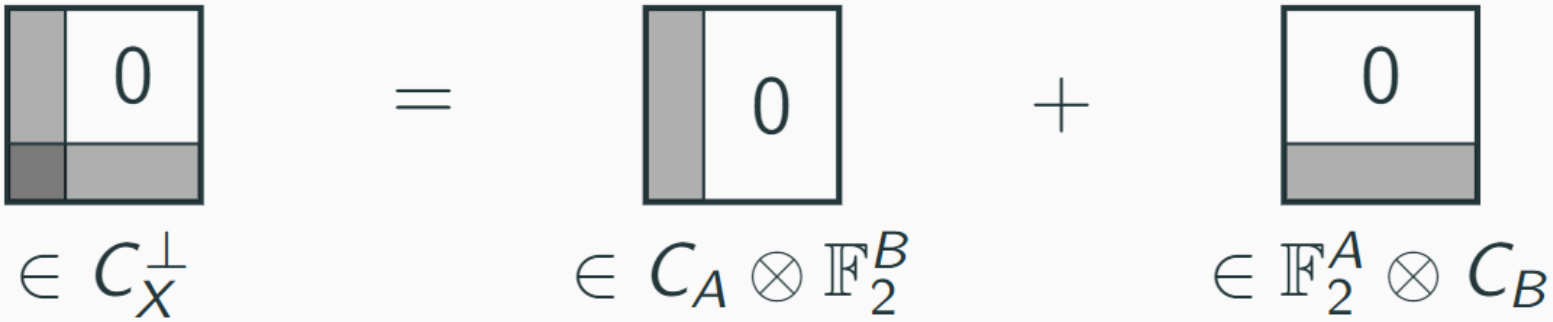

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Robust testability is a property of a pair of codes (C_A, C_B) saying that there cannot be "miraculous cancellations".

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A pair of independently randomly chosen codes (C_A, C_B) will be robustly testable with high probability [DHLV22, KP22].

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Recall minimum distance for a CSS code is

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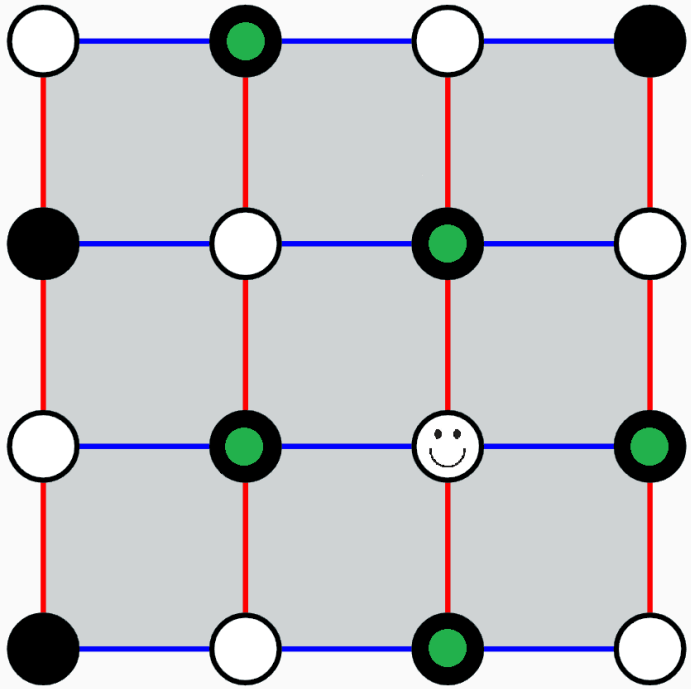
4. Expansion implies that we can find a vertex $v_0 \in \Gamma_Z^{\square}$ which is adjacent to mostly "good" local views.



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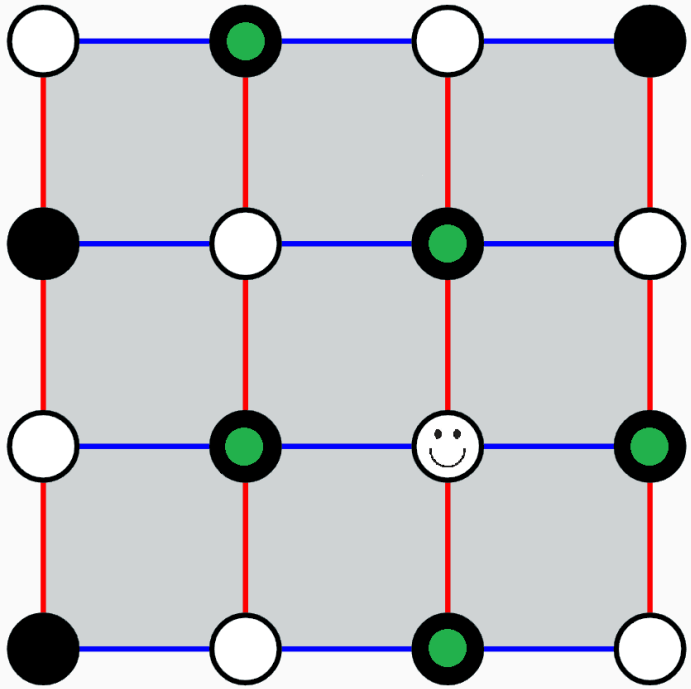

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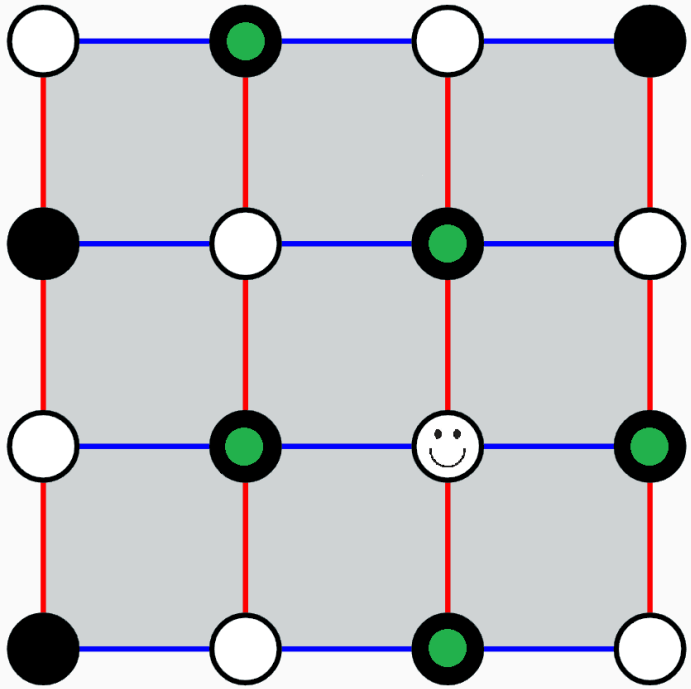
6. Iterating, it follows we can always decrease the weight of any sufficiently low weight codeword by stabilizers.



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Decoding Problem for Quantum Tanner Codes

Let $e \in \mathbb{F}_2^Q$ be a X error. Given the syndrome $\sigma = H_Z e$ as input, attempt to output a correction $f \in \mathbb{F}_2^Q$ such that $e + f \in T_X^\perp$.

Decoder for Quantum Tanner Codes

Decoding Procedure:

1. Define a "potential" or "mismatch" function U which serves as a proxy for the weight of the remaining error.

Potential Function: Let $e \in \mathbb{F}_2^Q$ be an error. The *potential* of e is defined by

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such that for each n, there exists a $O(n)$-time decoder which can correct all errors of weight

$$|e| \leq \frac{\delta n}{14\Delta^{3+\varepsilon}}.$$

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Corollary (Soundness): Let e be a correctable error. Then we have

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This is precisely soundness, so we generally expect *efficiently decodable* good qLDPC codes to be NLTS.

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