

Seismology Lectures

- Lecture 1 Equations and Waves (BR)
- Lecture 2 Surface waves (GM)
- Lecture 3 Geophysical Inverse Problems (GM)
- Lecture 4 Receiver functions (AS)
- Lecture 5 Array Methods (AL)
- Lecture 6 Transition zone in the context (AD)

Equations and Waves

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Equations and waves

- (Stress and strain)
- Equations of motion
- Ray theory
- Surface waves
- Free oscillations
- Attenuation
- Anisotropy
- Current challenges

Elasticity: Linear stress-strain relationship

- Most generally:

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl}$$

where c_{ijkl} is the elastic tensor.

- 21 independent elements (symmetries)
- If properties are the same in all directions: material is *isotropic* – there are only 2 independent elements, λ and μ (Lamé Parameters)

Isotropic medium

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl})$$

The stress-strain relation can then be written:

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij}$$

μ is shear modulus

Bulk modulus is $\kappa = \lambda + \frac{2}{3}\mu$

P velocity is

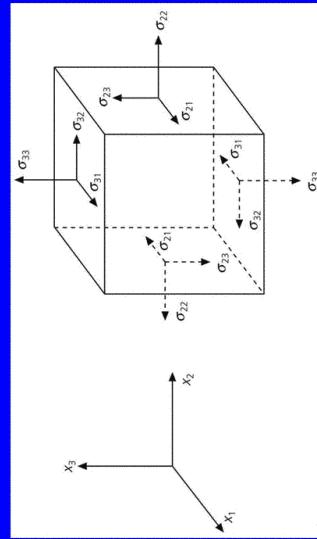
$$\alpha = \sqrt{\left(\frac{\lambda + 2\mu}{\rho}\right)}$$

S velocity:

$$\beta = \sqrt{\left(\frac{\mu}{\rho}\right)}$$

Lamé parameters and bulk modulus have the same units as stress (Pa)

Seismic wave equation



- We apply $F = m\mathbf{a}$
- If stress field is homogeneous: no net force: Need spatial gradients in the stress field.
- Force on plane normal to x_1 :

$$F(x_1) = \frac{\partial}{\partial x_1} \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{bmatrix} dx_1 dx_2 dx_3$$

Total force will then be:

$$F_i = \partial_j \sigma_{ij} dx_1 dx_2 dx_3$$

there may also be a body force on the cube F :

$$F_i = f_i dx_1 dx_2 dx_3$$

Mass is $m = \rho dx_1 dx_2 dx_3$, hence:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \partial_j \sigma_{ij} + f_i$$

Body forces: generally source term and gravity term

Homogeneous wave equation (no body forces)

- From substitutions, we obtain:

$$\rho \frac{\partial^2 u}{\partial t^2} = \nabla \lambda (\nabla \cdot u) + \nabla \mu \cdot [\nabla u + \nabla u^T] + (\lambda + 2\mu) \nabla \nabla \cdot u - \mu \nabla \times \nabla \times u$$

- First two terms contain gradients of Lamé parameters which are non zero when material is inhomogeneous – complicate the equations
- 1) If structure only function of depth \rightarrow series of homogeneous layers. Solutions linked through the computation of reflection and transmission coefficients:
 - Homogeneous layer methods*
- 2) gradient terms vary as $1/\omega$, where ω is frequency \rightarrow neglect them at high frequency:
 - ray methods* \rightarrow "infinite frequency approximation"

- In a homogeneous medium:
- Taking divergence and curl and remembering

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu) \nabla \nabla \cdot u - \mu \nabla \times \nabla \times u$$

$$\nabla \cdot (\nabla \times \Psi) = 0:$$

$$\frac{\partial^2 (\nabla \cdot u)}{\partial t^2} = \frac{\lambda + 2\mu}{\rho} \nabla^2 (\nabla \cdot u)$$

$$\frac{\partial^2 (\nabla \times u)}{\partial t^2} = \alpha^2 \nabla^2 (\nabla \times u)$$

$$\frac{\partial^2 (\nabla \times u)}{\partial t^2} = \beta^2 \nabla^2 (\nabla \times u)$$

Plane waves

- **Plane wave:**
 - Displacement varies only in the direction of propagation
 - Most generally, for propagation in the direction \mathbf{u} :

$$u(\mathbf{x}, t) = f(t - \mathbf{s} \cdot \mathbf{x})$$

– Harmonic waves:

$$\mathbf{u} = \mathbf{A}(\omega) e^{i(\omega t - \mathbf{k} \cdot \mathbf{x})}$$

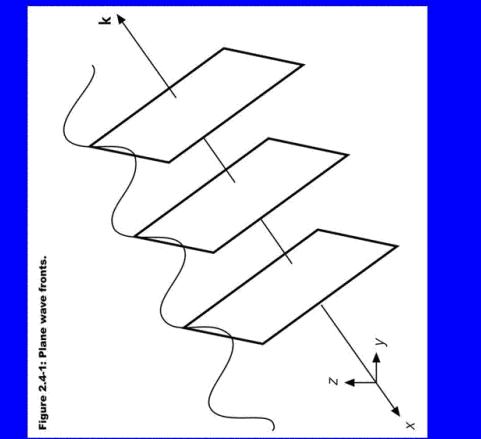
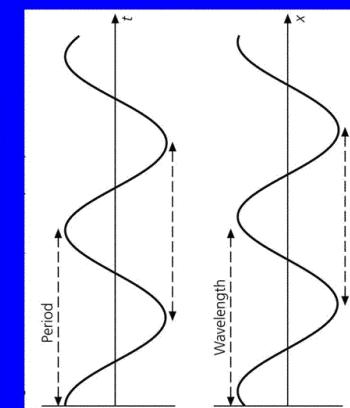
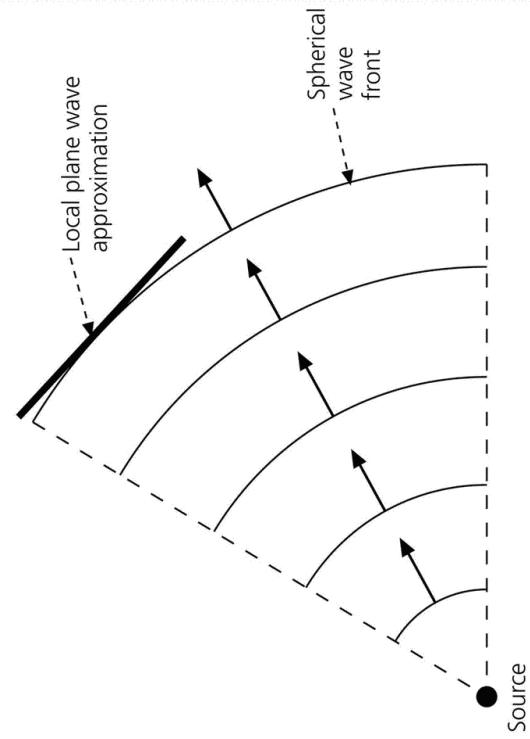


Figure 2.4-1: Plane wave fronts.

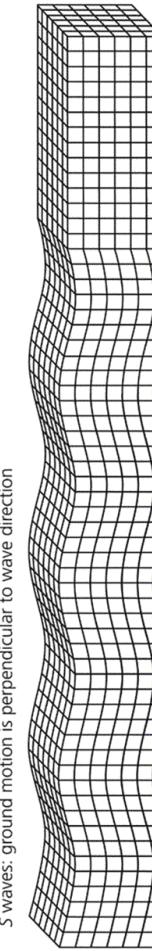
\mathbf{s} is slowness vector $\mathbf{s} = \lambda \mathbf{s}/c$
 \mathbf{k} is wavenumber vector $\mathbf{k} = \omega \mathbf{s}$

Figure 2.4-2: Approximation of a spherical wave front as plane waves.



Polarisation of P and S waves

S waves: ground motion is perpendicular to wave direction



P waves: ground motion is parallel to wave direction

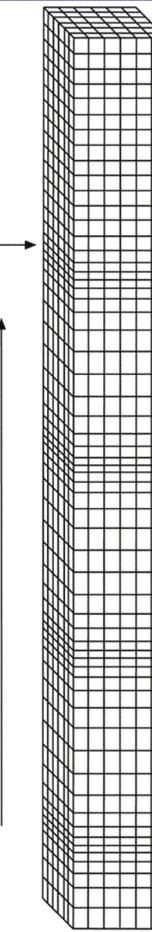
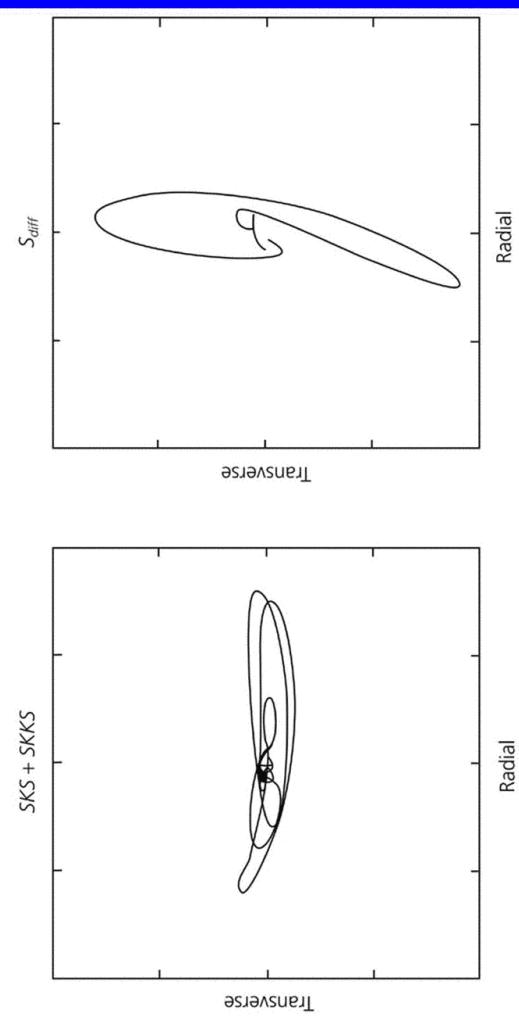
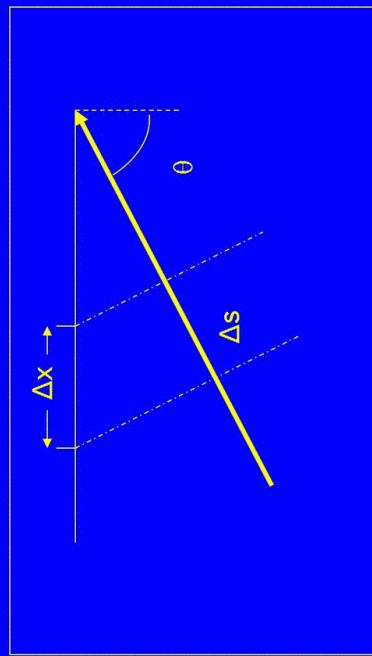


Figure 2.4-6: Particle motion plots for SKS+SKKS and S_{diff} .



Ray theory

- **Simple and fast:**
 - Used extensively in earthquake location, focal mechanisms, inversion for structure in crust and mantle
- **Shortcomings**
 - High frequency approximation: fails at long periods
 - Does not predict non geometrical effects i.e. diffracted waves, head waves
 - Limitations in predicting effects of heterogeneity on waveforms



Define incidence angle θ , then:

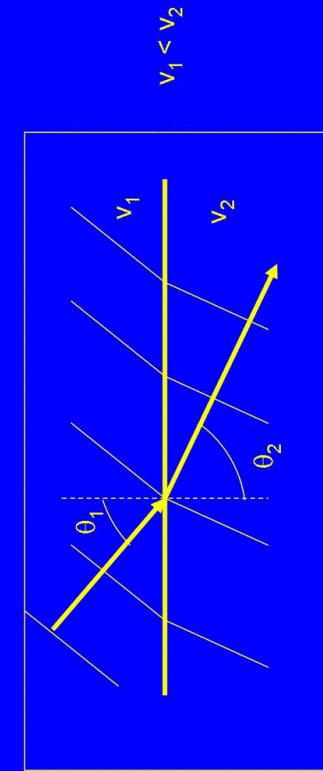
$$\Delta s = v\Delta t = \Delta x \sin\theta$$

or:

$$\frac{\Delta t}{\Delta x} = \frac{\sin\theta}{v} = p$$

p "ray parameter" – apparent horizontal slowness

At interface between two media:



Ray angle at the interface must change to preserve the timing of the wavefronts across the interface. Snell's law:

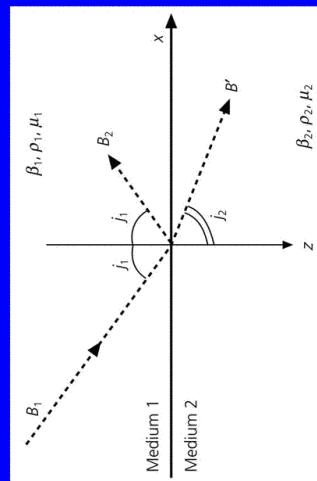
$$p_1 \sin \theta_1 = p_2 \sin \theta_2$$

Ray parameter is a constant of the ray

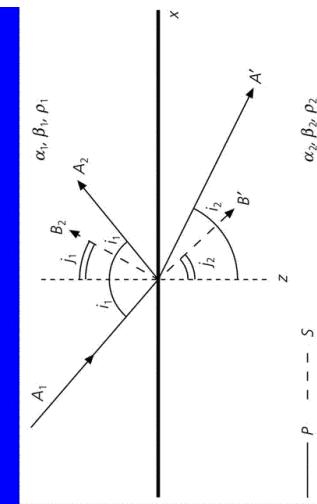
Ray "bottoms" for a "critical" incidence angle; no transmitted wave for larger incidence angles (post-critical or total reflection)

In general, some of the energy is transmitted, some reflected, and, in the P-SV case, some converted

SH case

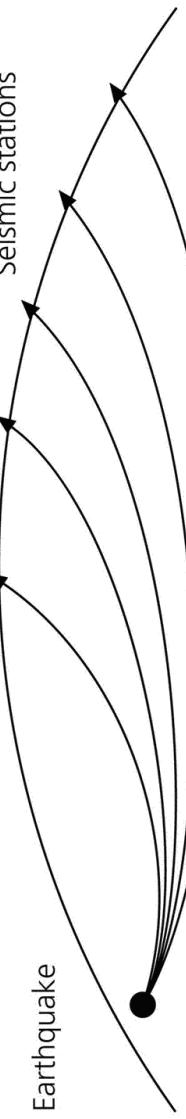


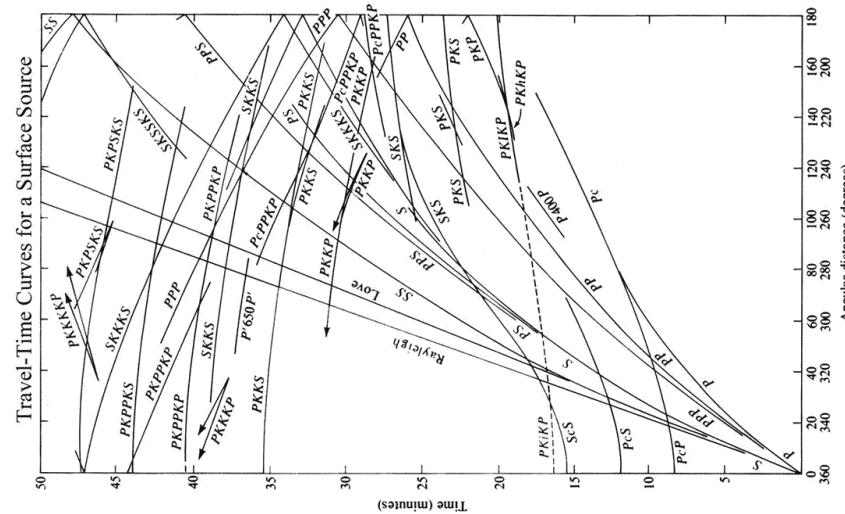
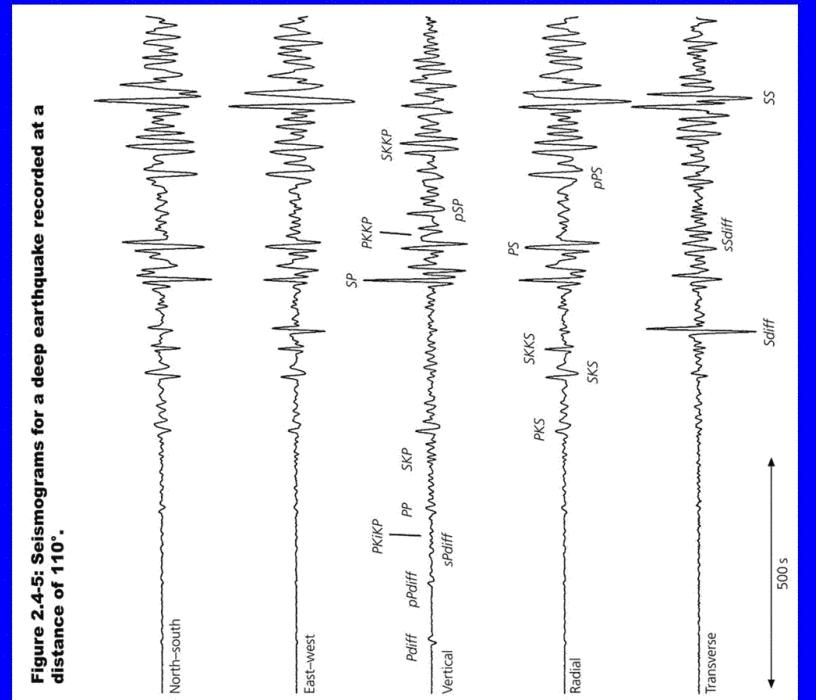
P-SV case

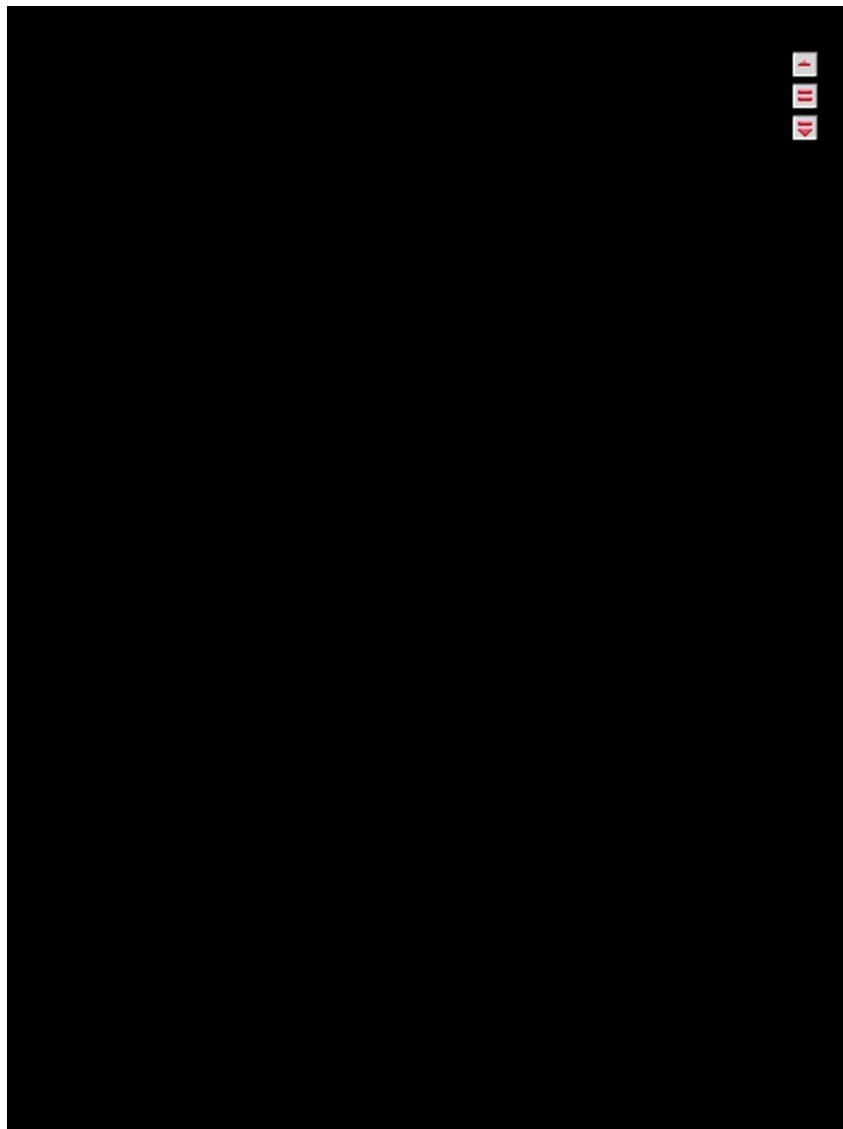


Angles of reflection/transmission depend only on velocities
Amplitudes depend on impedance (ρv)

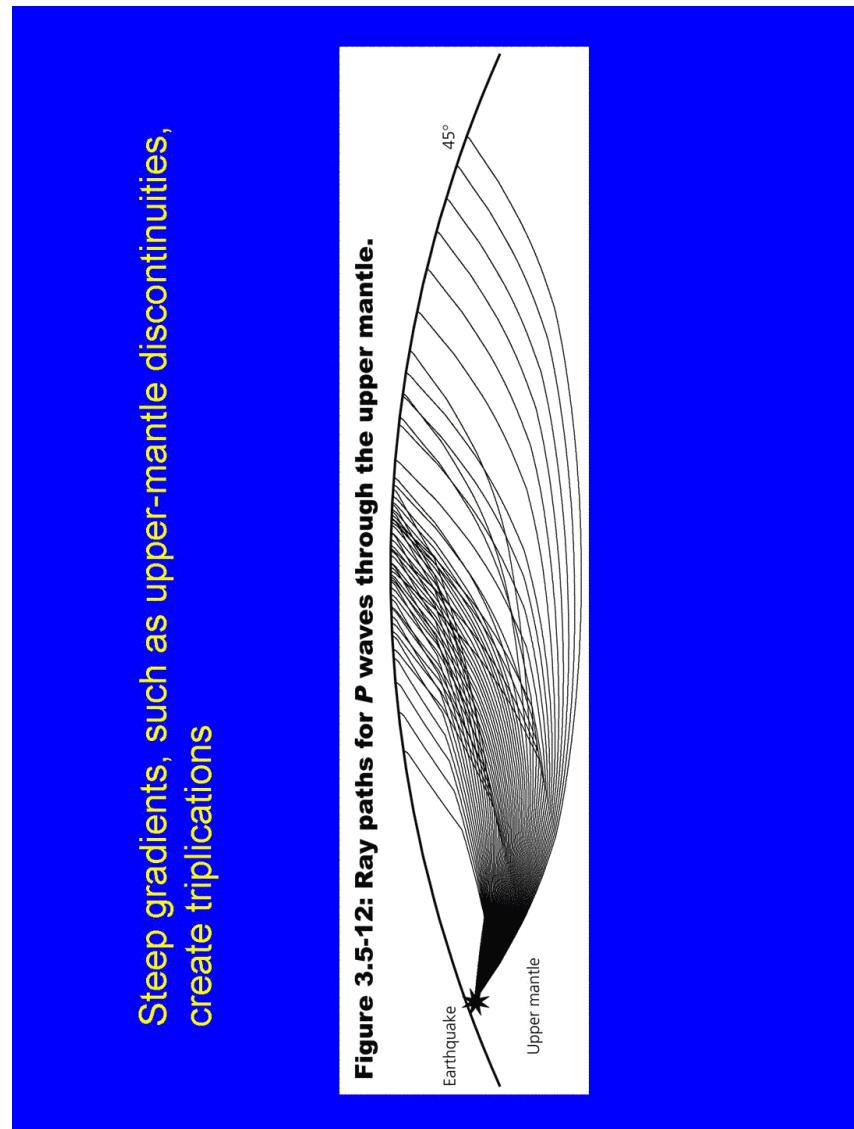
Ray paths in spherical geometry, when velocity increases with depth



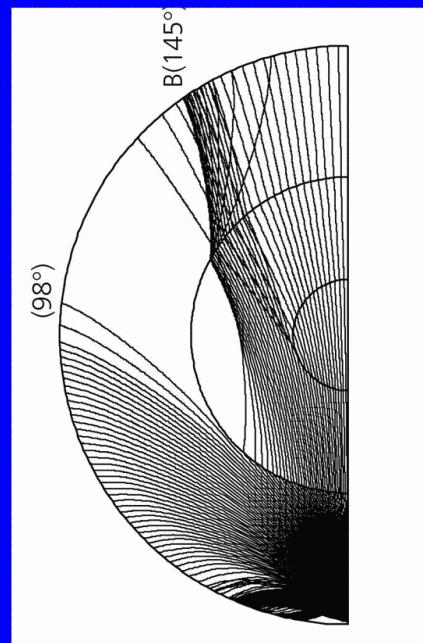


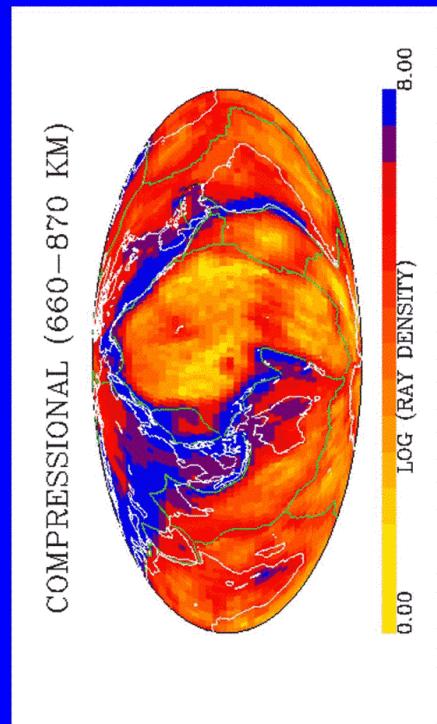


Steep gradients, such as upper-mantle discontinuities, create triplications



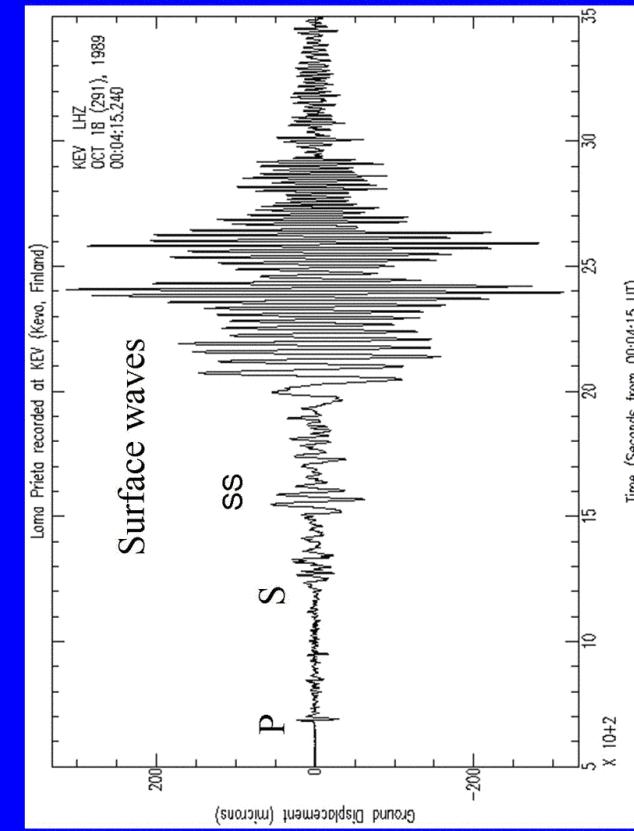
Low velocity layers create shadow zones





Vasco and
Johnson, 1998

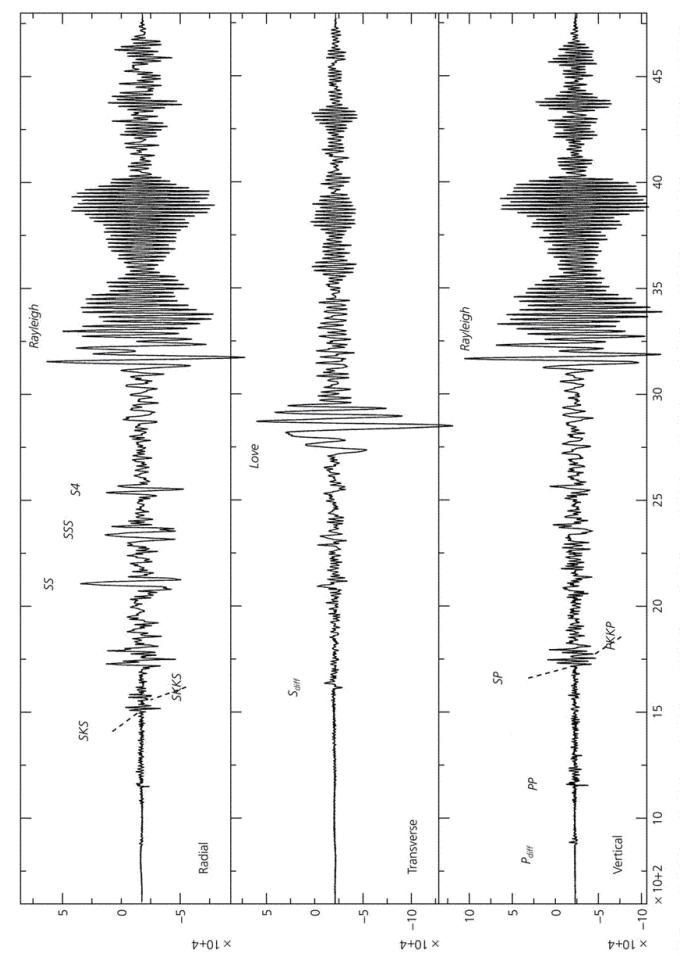
Most seismograms look like this:



Surface waves

- Arise from interaction of body waves with free surface.
- Energy confined near the surface
- *Rayleigh waves*: interference between P and SV waves
 - exist because of free surface
- *Love waves*: interference of multiple S reflections.
 - Require increase of velocity with depth
- *Surface waves are dispersive*: velocity depends on frequency

Figure 2.7-1: Seismograms recorded at a distance of 110°, showing surface waves.



Surface wave dispersion

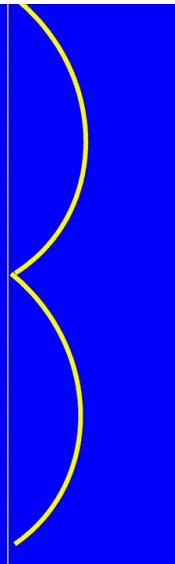
- Slowness along surface : p
- If surface bounces are separated by distance $X(p)$, then travel time along the surface between bounce points is $pX(p)$, and phase delay is $\omega p X(p)$.
- Travel time along the ray path is $T(p)$, phase delay is $\omega T(p) - \pi/2$
- Constructive interference if:

$$\omega p X(p) = \omega(T)(p) - \frac{\pi}{2} - n2\pi$$

where n is an integer. Hence:

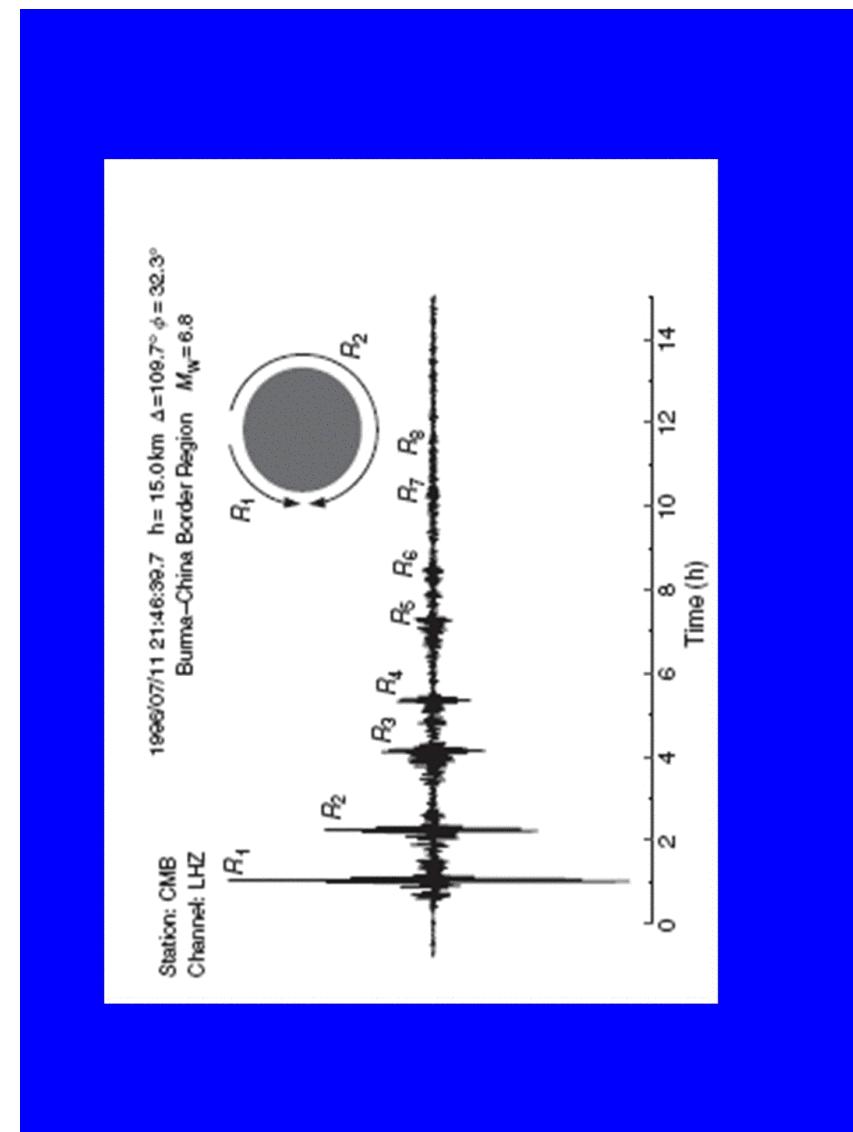
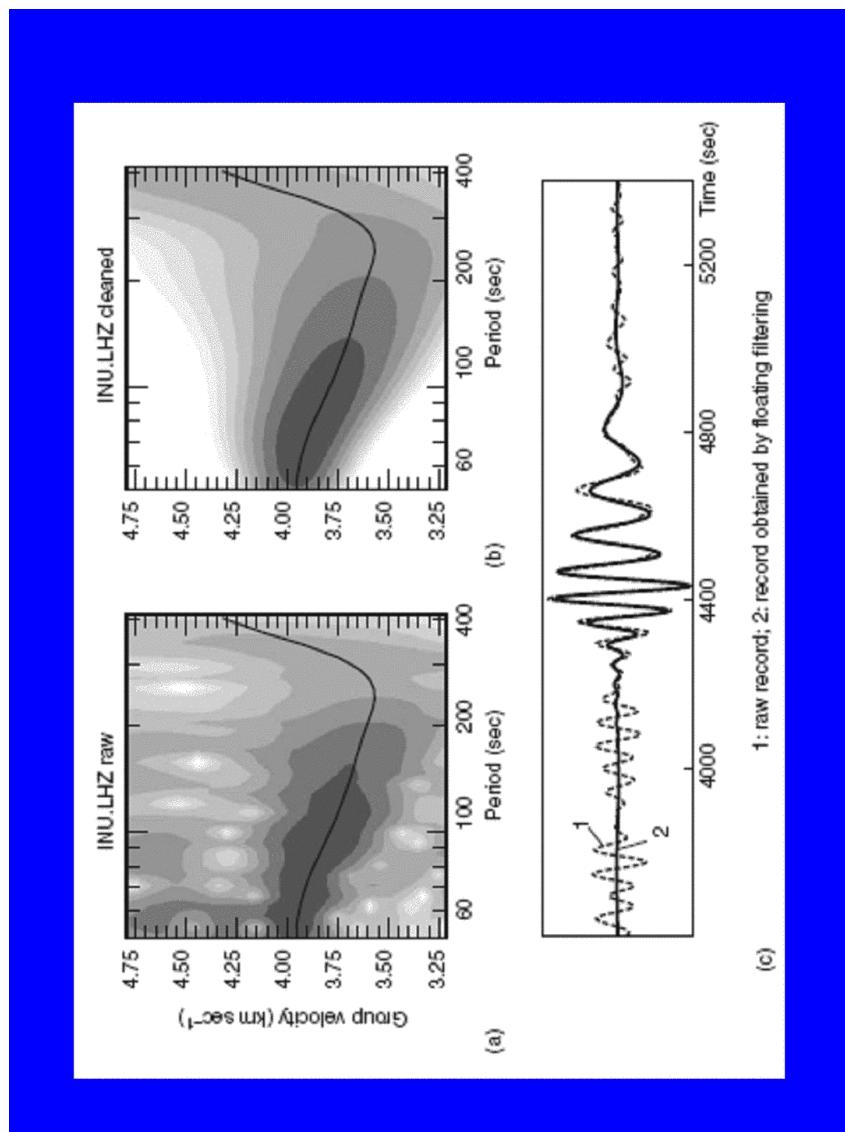
$$\omega = \frac{n2\pi + \pi/2}{T(p) - pX(p)} = \frac{n2\pi + \pi/2}{\tau(p)}$$

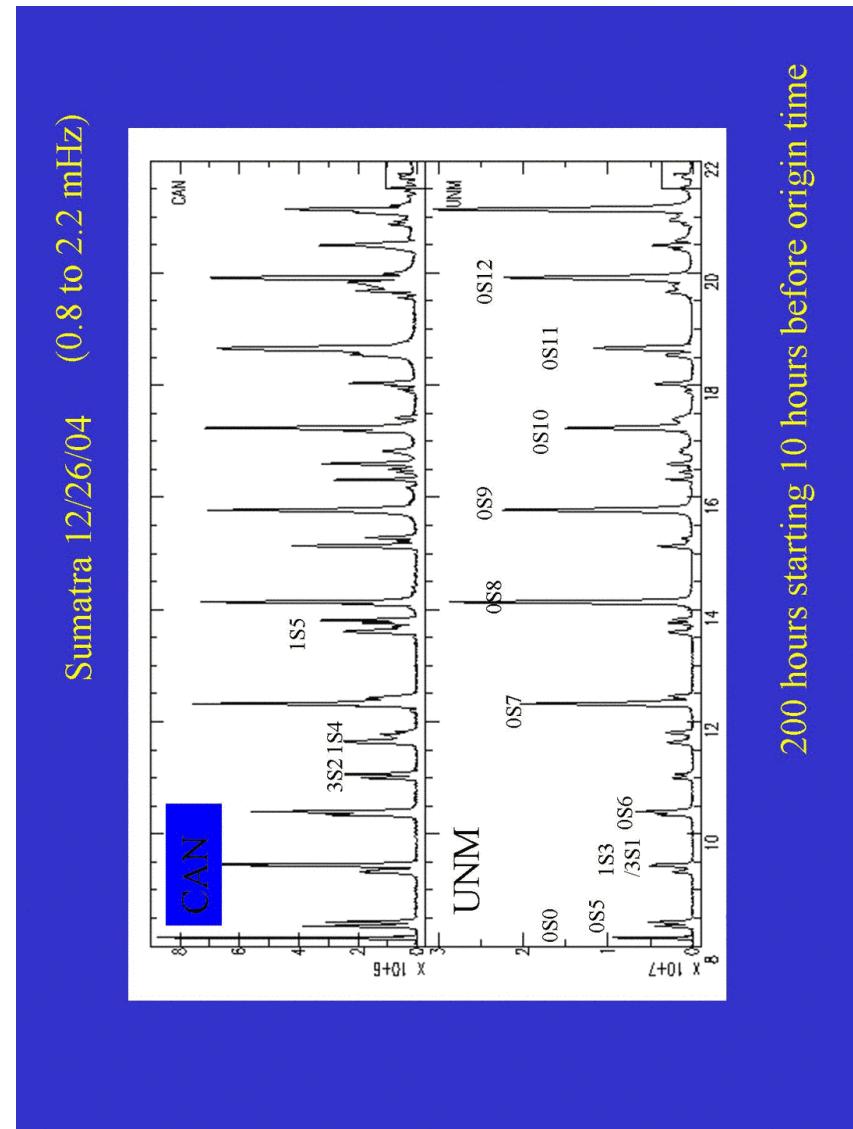
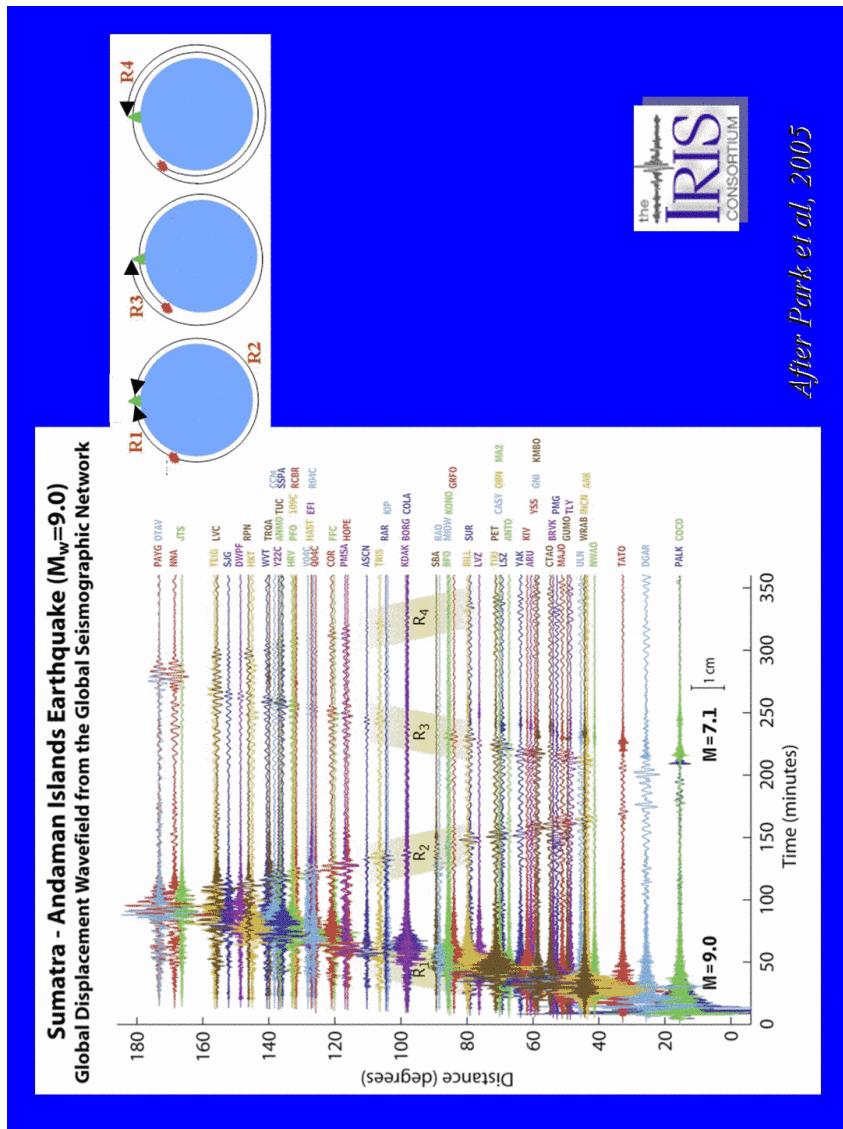
This defines $c=1/p$ as a function of ω : *dispersion relation*



- C is the “phase velocity”: velocity at which the picks and troughs travel
- Energy travels with “group velocity” (along the actual ray paths):

$$U = \frac{X(p)}{T(p)}$$





200 hours starting 10 hours before origin time

Free Oscillations

$$\rho_0 \frac{\partial^2 \mathbf{s}}{\partial t^2} = \mathbf{L}(\mathbf{s}) + \mathbf{f}$$

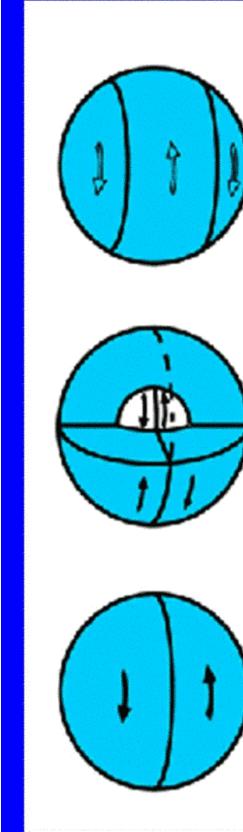
The k^{th} free oscillation $\mathbf{s} = \mathbf{s}_k e^{i\omega_k t}$ satisfies:

$$\mathbf{L}(\mathbf{s}_k) + \rho_0 \omega_k^2 \mathbf{s}_k = 0$$

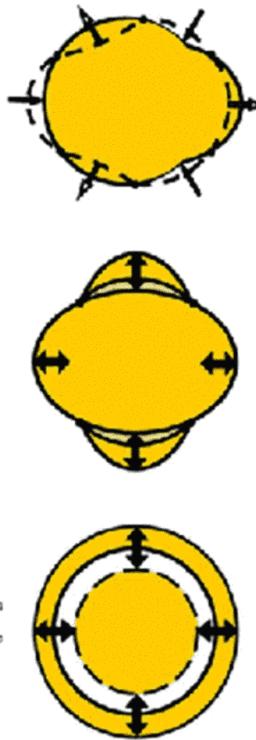
SNREI model; Solutions of the form

$$\mathbf{s}_k = [\hat{\mathbf{r}}_k U(r) Y_l^m(\theta, \phi) +_k V(r) \nabla_1 Y_l^m(\theta, \phi) -_k W(r) \hat{\mathbf{r}} \times \nabla_1 Y_l^m(\theta, \phi)] e^{i\omega_k t}$$

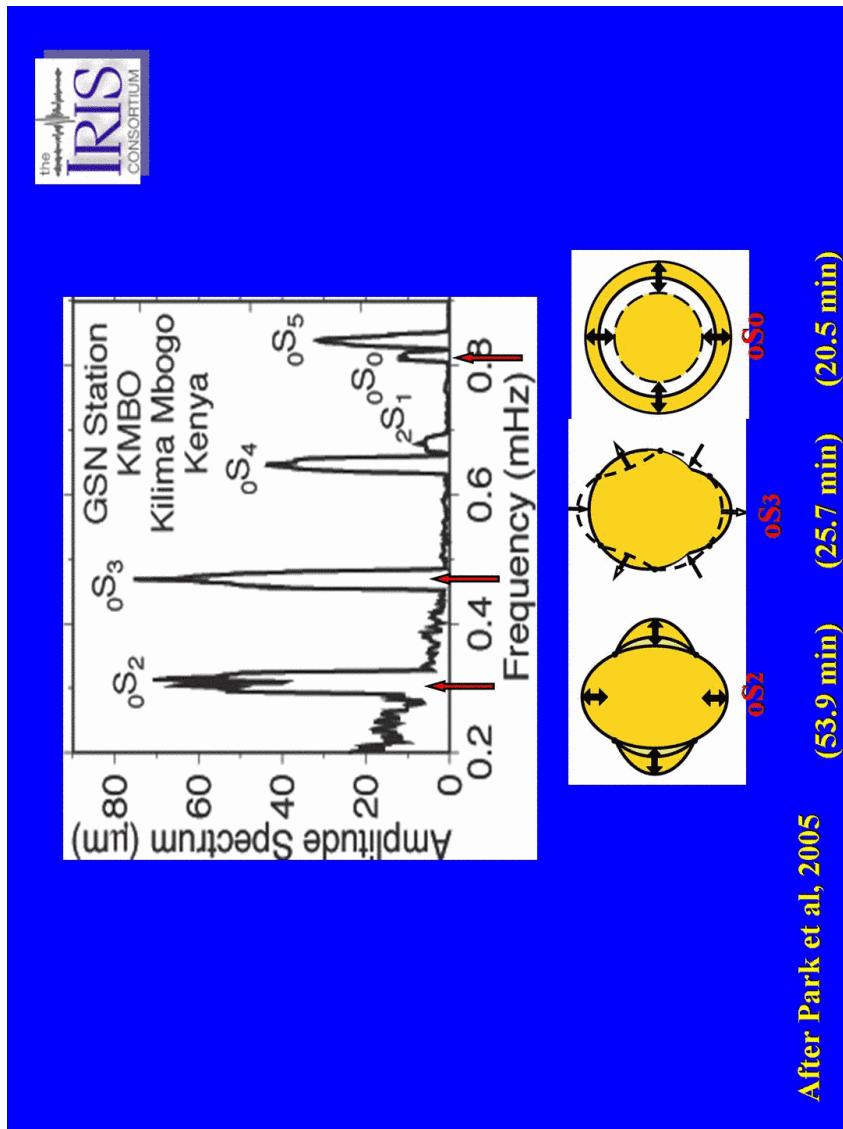
$$k = (\ell, m, n)$$



Toroidal modes ${}_0T_2$ (44.2 min), ${}_1T_2$ (12.6 min)
and ${}_0T_3$ (28.4 min)



Spheroidal modes ${}_0S_0$ (20.5 min), ${}_0S_2$ (53.9 min)
and ${}_0S_3$ (25.7 min)



Seismograms by mode summation

$$\rho_0 \frac{\partial^2 s}{\partial t^2} = L(s) + f$$

The k 'th free oscillation $s = s_k e^{i\omega_k t}$ satisfies:

$$L(s_k) + \rho_0 \omega_k^2 s_k = 0$$

L is a self-adjoint operator:

- completeness

$$s = \sum_k a_k(t) s_k(r)$$

$$\int_V \rho_0 s_{k'}^* s_k dV = \delta_{kk'}$$

Substitution in eq. (1) gives:

$$\rho_0 \sum_k \frac{\partial^2 a_k}{\partial t^2} \mathbf{s}_k = \sum_k a_k(t) \mathbf{L}(\mathbf{s}_k) + \mathbf{f}$$

Since \mathbf{s}_k is a mode:

$$\sum_k \left[\rho_0 \omega_k^2 a_k(t) \mathbf{s}_k + \rho_0 \frac{\partial^2 a_k}{\partial t^2} \mathbf{s}_k \right] = \mathbf{f}$$

Multiply by \mathbf{s}_k^* and integrate over the volume V of the earth

$$\omega_k^2 a_k(t) + \frac{\partial^2 a_k}{\partial t^2} = \int_V \mathbf{s}_k^* \cdot \mathbf{f}(t) dV = F_k(t)$$

$$\omega_k^2 a_k(t) + \frac{\partial^2 a_k}{\partial t^2} = \int_V \mathbf{s}_k^* \cdot \mathbf{f}(t) dV = F_k(t)$$

Assuming F_k represents integration over volume of the source:

$$a_k(t) = \frac{1}{\omega_k} \int_{-\infty}^t \sin [\omega_k(t-t')] F_k(t') dt'$$

Or, after integration by parts:

$$\begin{aligned} a_k(t) &= -\frac{1}{\omega_k^2} \int_{-\infty}^t \frac{d}{dt'} [1 - \cos(\omega_k(t-t'))] F_k(t') dt' \\ &= \frac{1}{\omega_k^2} \int_{-\infty}^{\infty} [1 - \cos(\omega_k(t-t'))] \frac{\partial}{\partial t'} F_k(t') dt' \end{aligned}$$

We now assume that the motion is zero before $t=0$ and define

$$C_k(t) = [1 - \cos(\omega_k t)]H(t)$$

Where $H(t)$ is the Heaviside function. Then

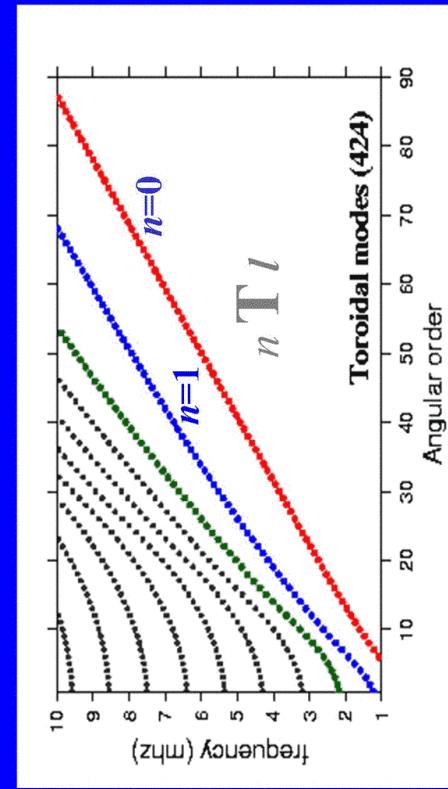
$$\begin{aligned} a_k(t) &= \frac{1}{\omega_k^2} \int_{-\infty}^{\infty} C_k(t-t') \frac{\partial}{\partial t'} F_k(t') dt' \\ &= \frac{1}{\omega_k^2} C_k(t) * \frac{\partial}{\partial t} F_k(t) \end{aligned}$$

The solution for the displacement then has the form:

$$\mathbf{s}(\mathbf{r}, \mathbf{r}_0, t) = \sum_k \frac{1}{\omega_k^2} \mathbf{s}_k(\mathbf{r}) \frac{\partial}{\partial t} F_k(\mathbf{r}_0, t) * C_k(t)$$

Spheroidal modes : Vertical & Radial component

Toroidal modes : Transverse component

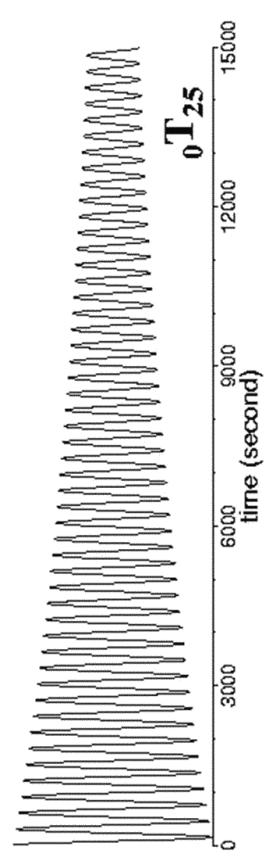


l : angular order, horizontal nodal planes

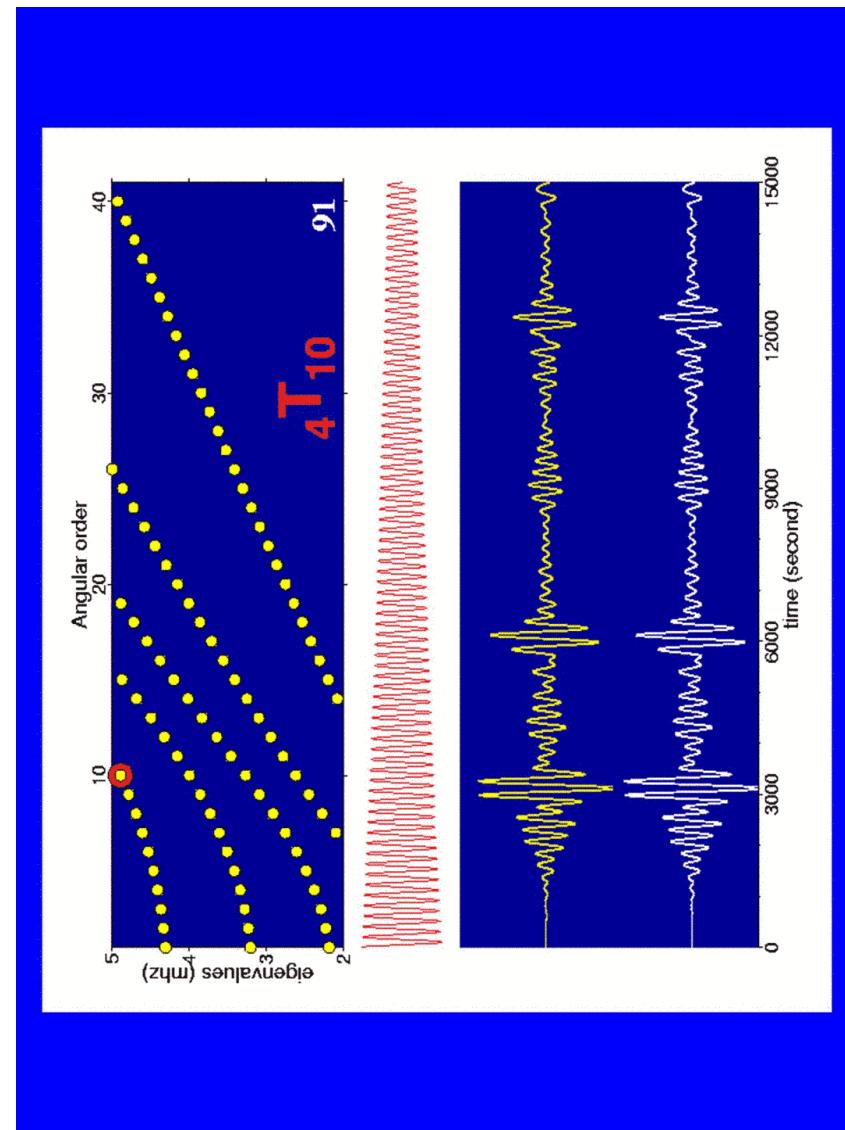
n : overtone number, vertical nodes

Normal mode summation – 1D

$$U(t) = \sum A_k \exp(i\omega_k t) \exp(-\omega_k t/2Q_k)$$

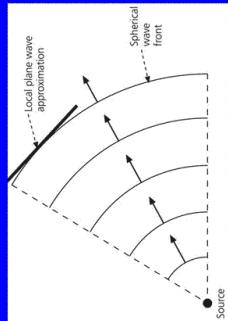


A : excitation
 ω : eigen-frequency
 Q : Quality factor (attenuation)



Wave attenuation

- Amplitudes of seismic waves are affected by:
 - Geometrical spreading
 - Scattering/focusing (total energy in the wavefield conserved)
 - Intrinsic attenuation (energy loss due to friction on anelastic processes).



Intrinsic attenuation is described by the "quality factor" Q

$$\frac{1}{Q(\omega)} = -\frac{\Delta E}{2\pi E}$$

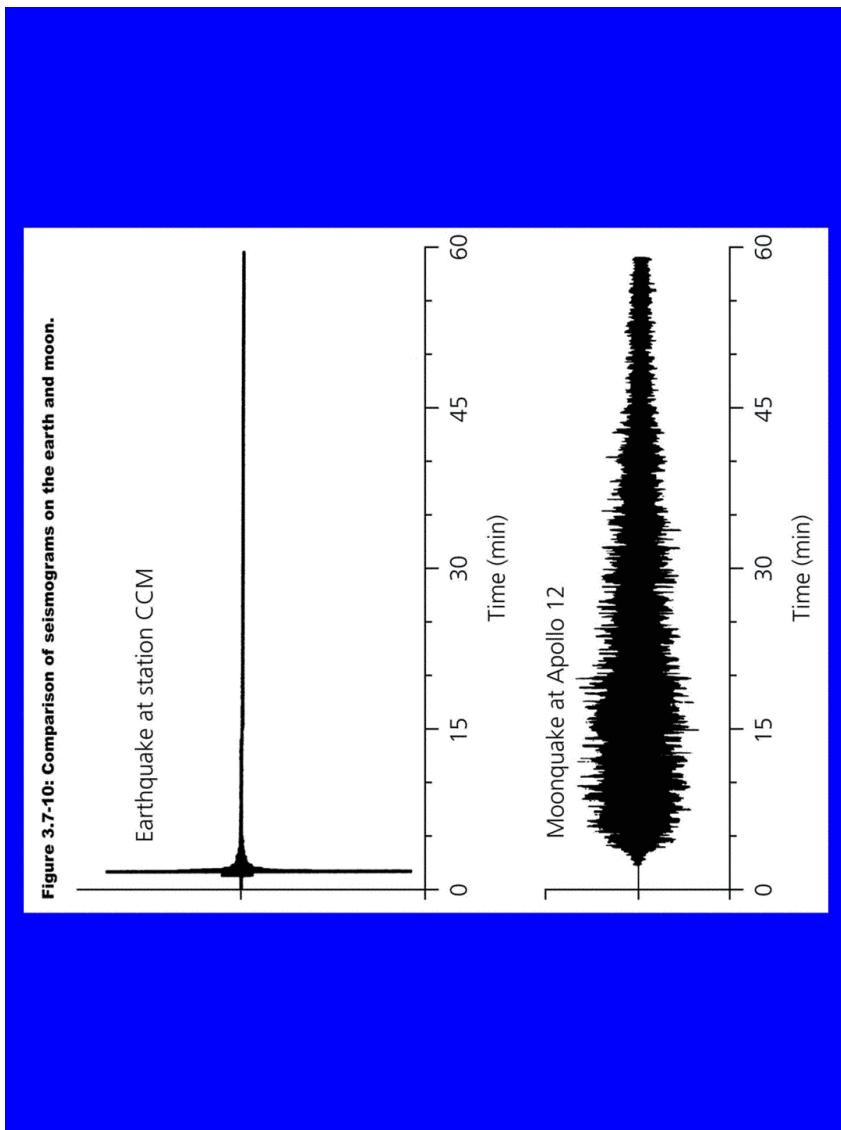
E = peak strain energy
 ΔE = energy loss per cycle

$$A(x, t) = A_0 e^{-\omega x / 2cQ} e^{-i\omega(t-x/c)}$$

In ray theory:

$$t^* = \int_{\text{path}} \frac{dt}{Q(\mathbf{r})} \quad \text{hence}$$

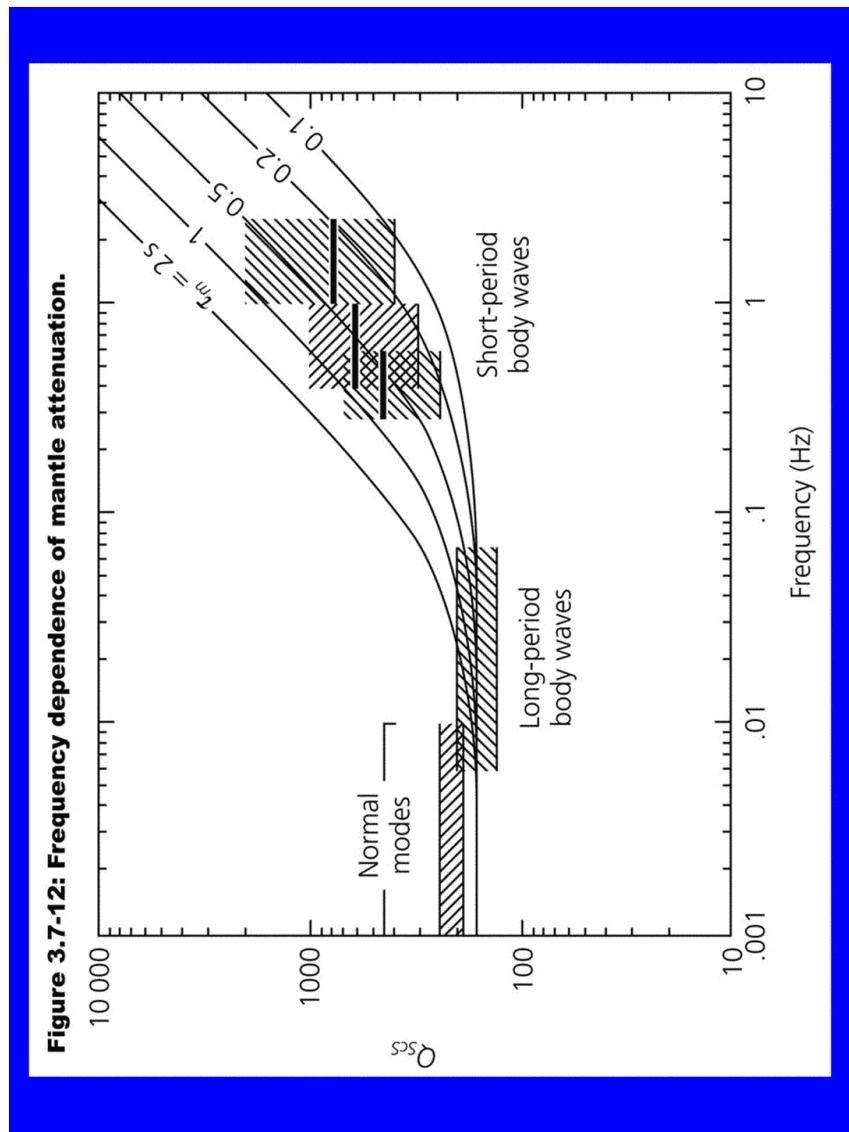
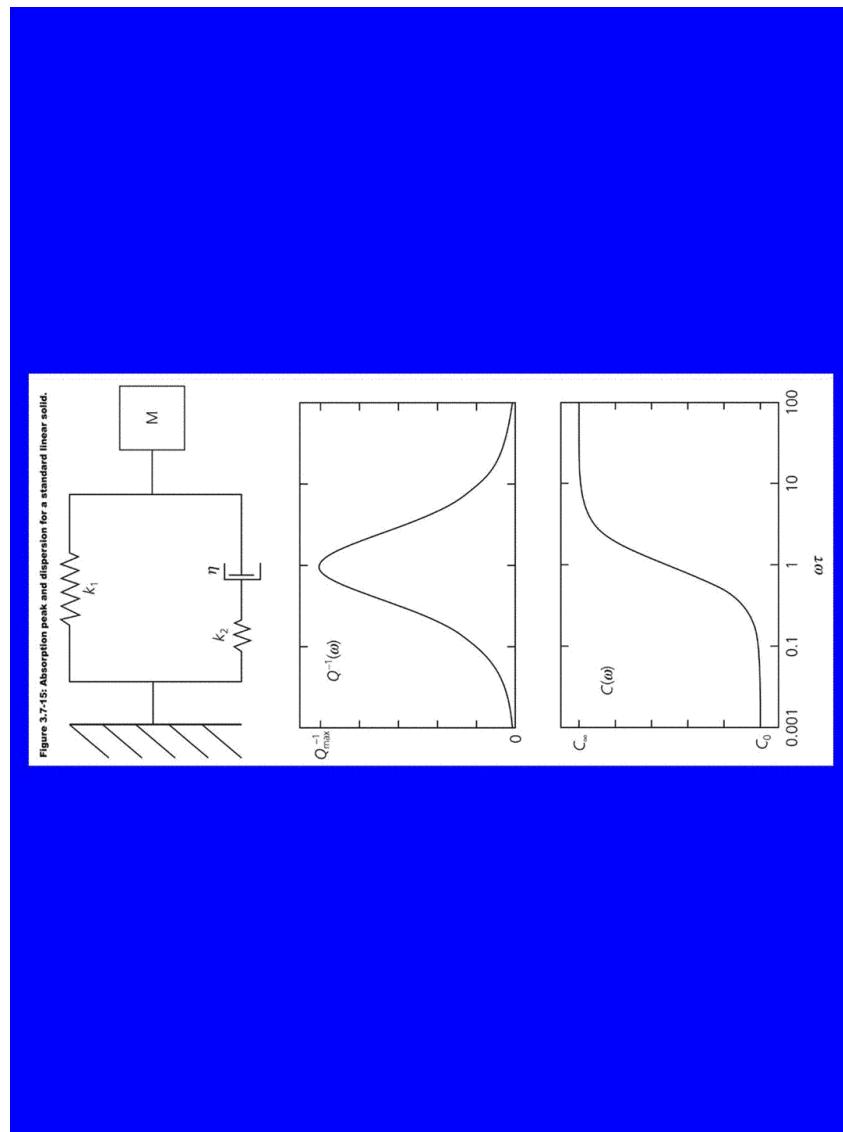
$$A(\omega) = A_0(\omega) e^{-\omega t^*/2}$$

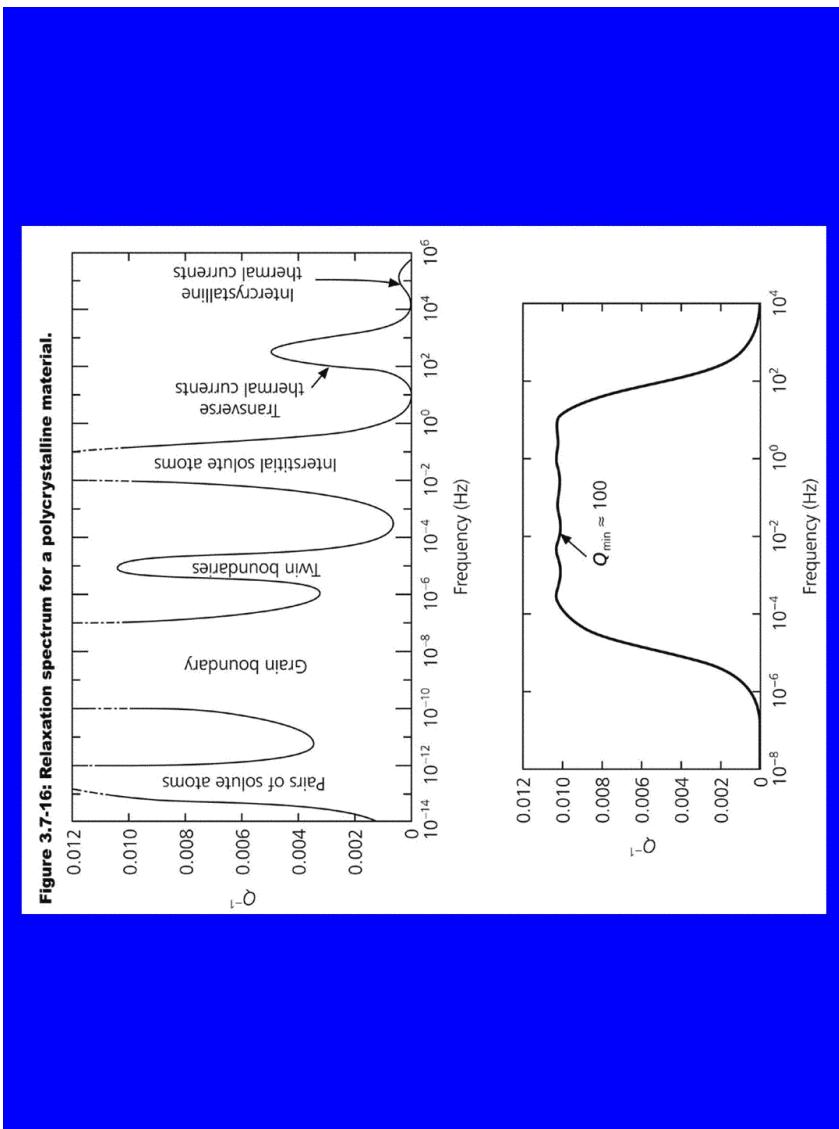


- Dispersion due to attenuation:
 - Causality
 - Physical mechanisms for attenuation

In a frequency band where Q is constant:

$$c(\omega) = c(\omega_0) \left(1 + \frac{1}{\pi Q} \ln \frac{\omega}{\omega_0} \right)$$

Figure 3.7-12: Frequency dependence of mantle attenuation.**Figure 3.7-15: Absorption peak and dispersion for a standard linear solid.**



- P and S wave attenuation are related to shear and bulk attenuation through:

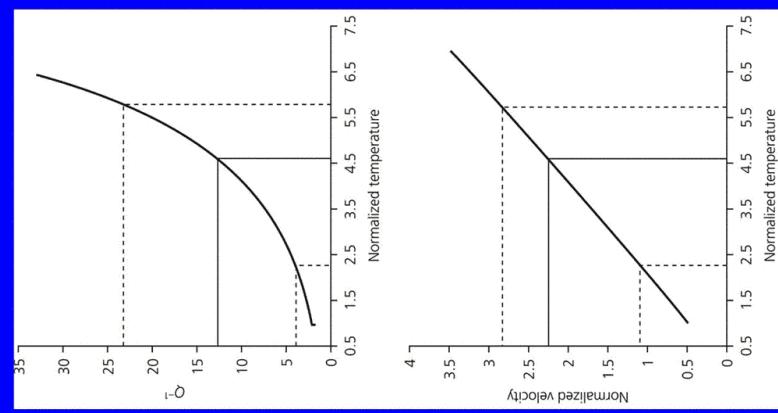
$$Q_\alpha^{-1} = L Q_\mu^{-1} + (1 - L) Q_\kappa^{-1}$$

$$Q_\beta^{-1} = Q_\mu^{-1}$$

- where:

$$L = (4/3)(\beta/\alpha)^2$$

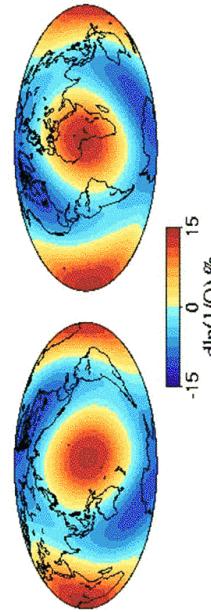
Effects of temperature on attenuation and velocity



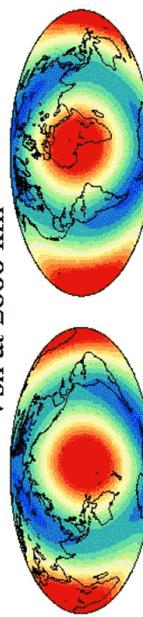
attenuation

velocity

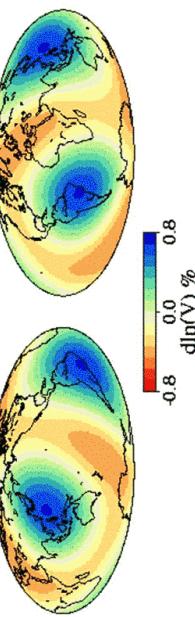
QRLW8 at 500 km

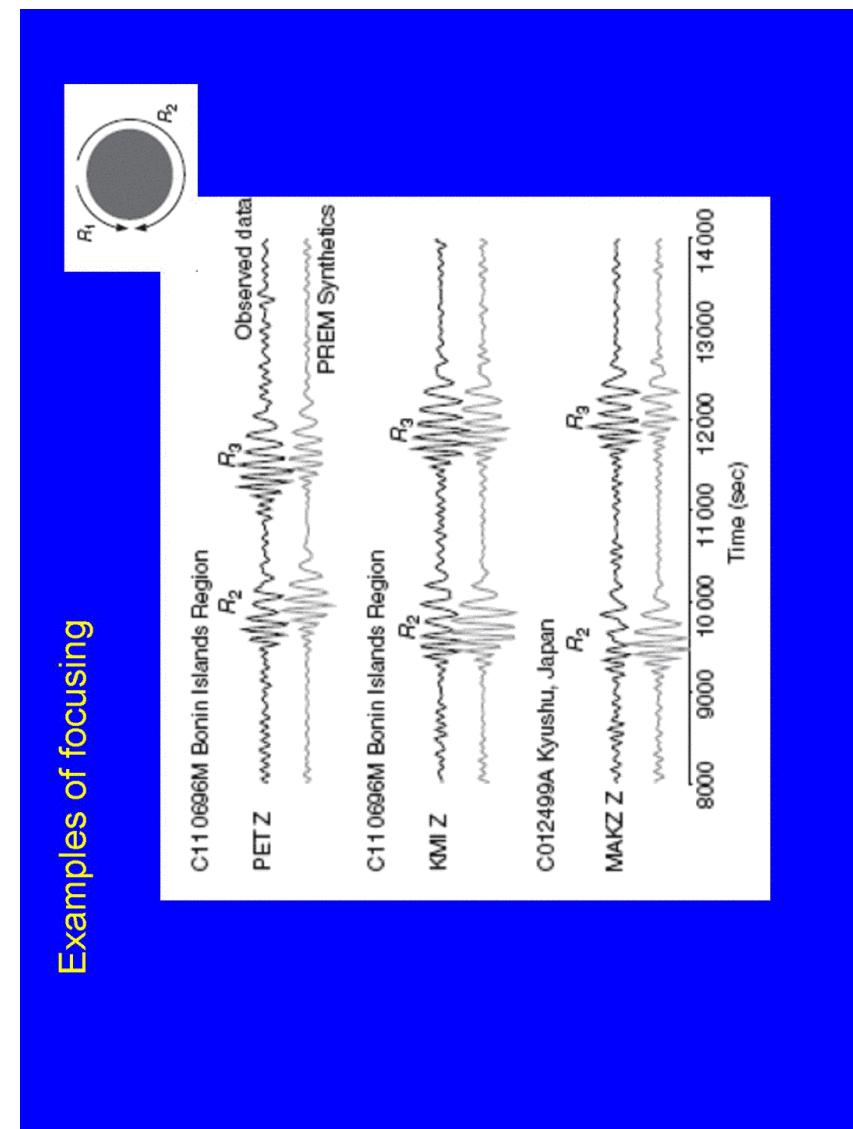
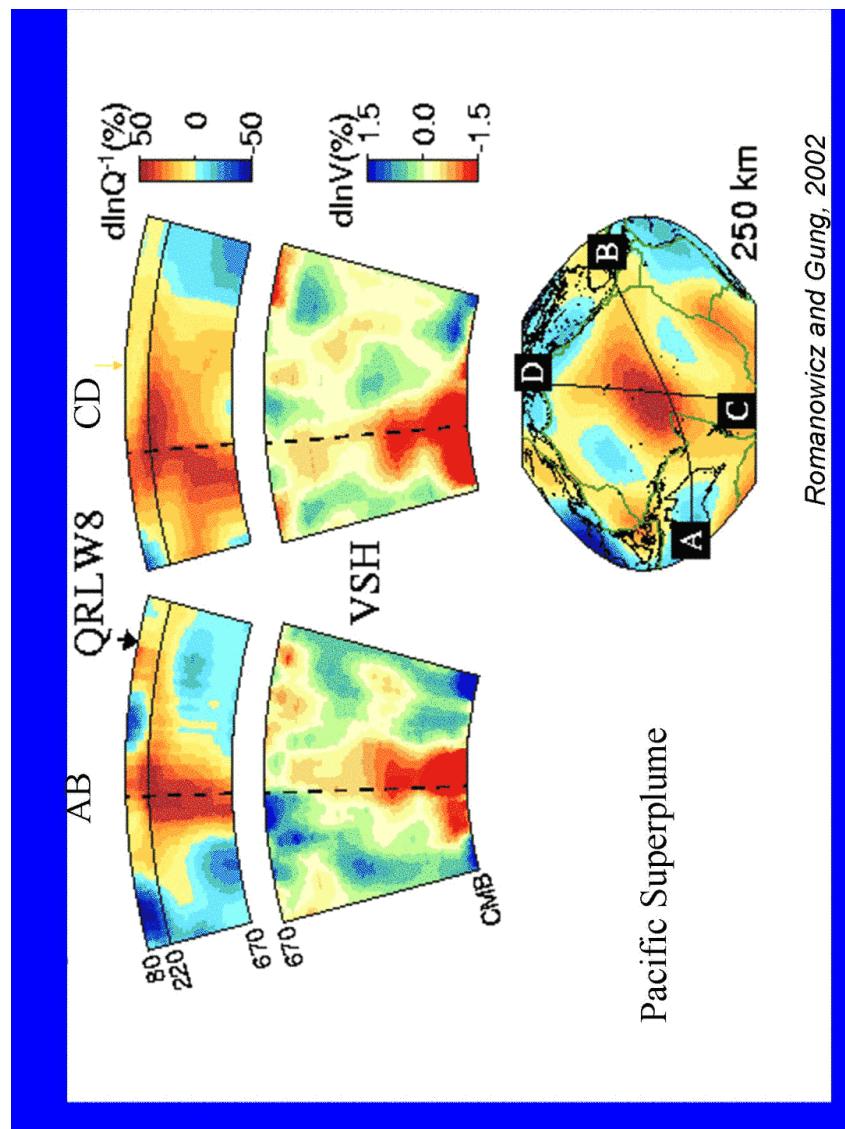


Vsh at 2600 km

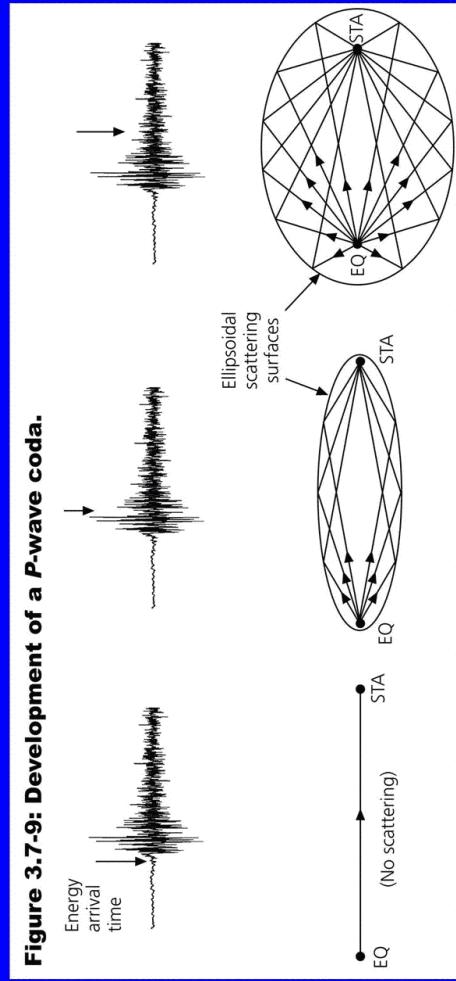


Vsh at 500 km





Scattering



Anisotropy

- In general elastic properties of a material vary with orientation

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl}$$

- Anisotropy causes wave splitting

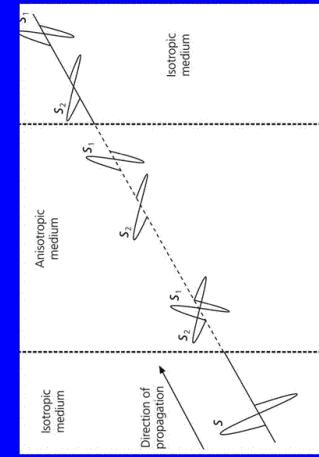
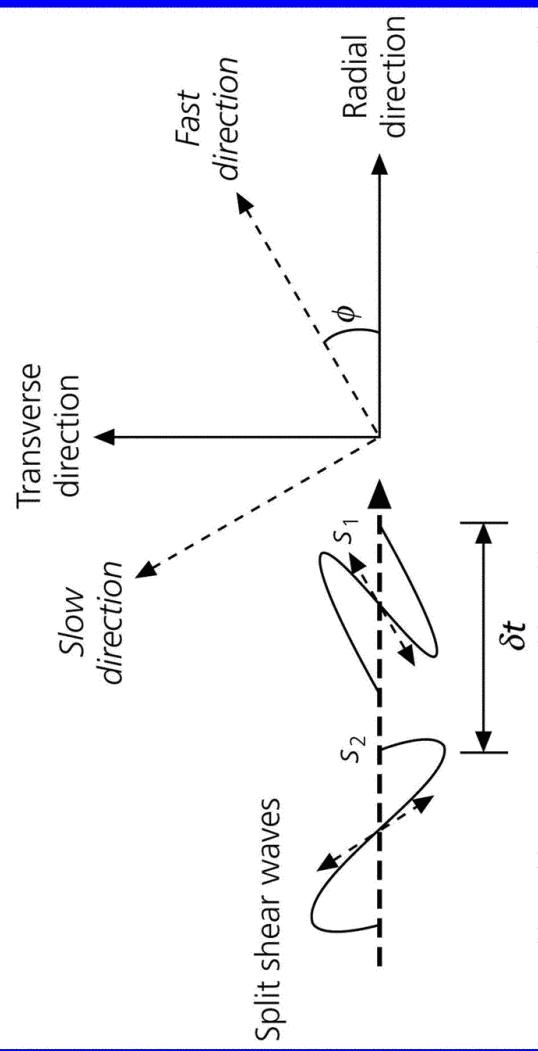
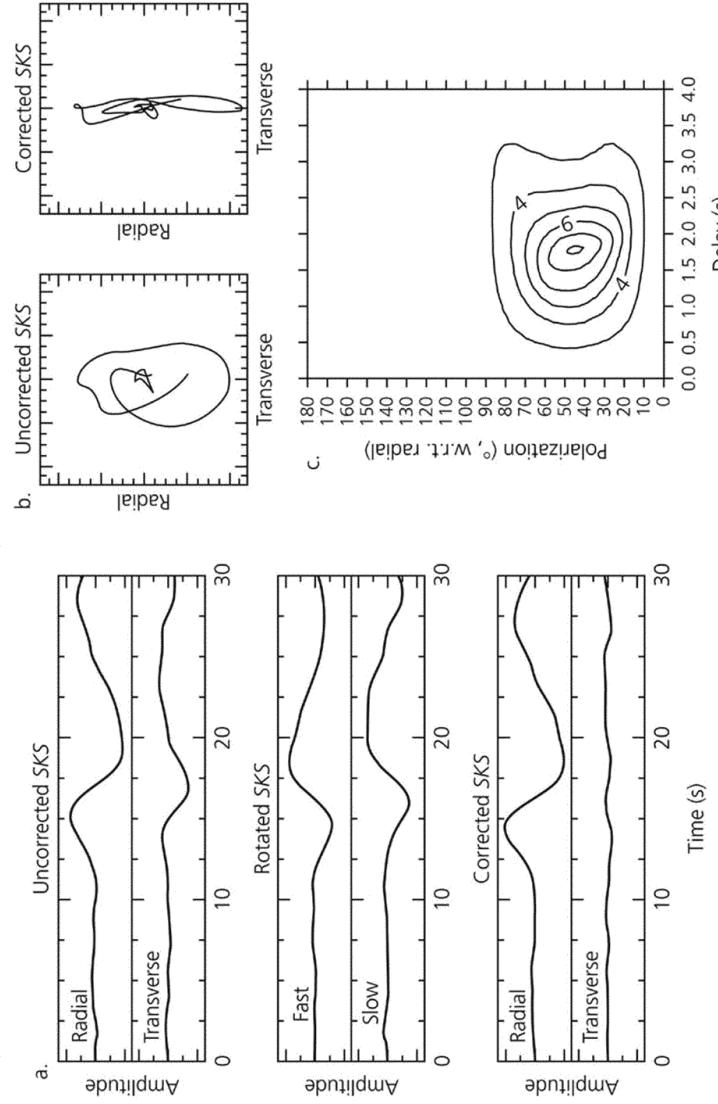
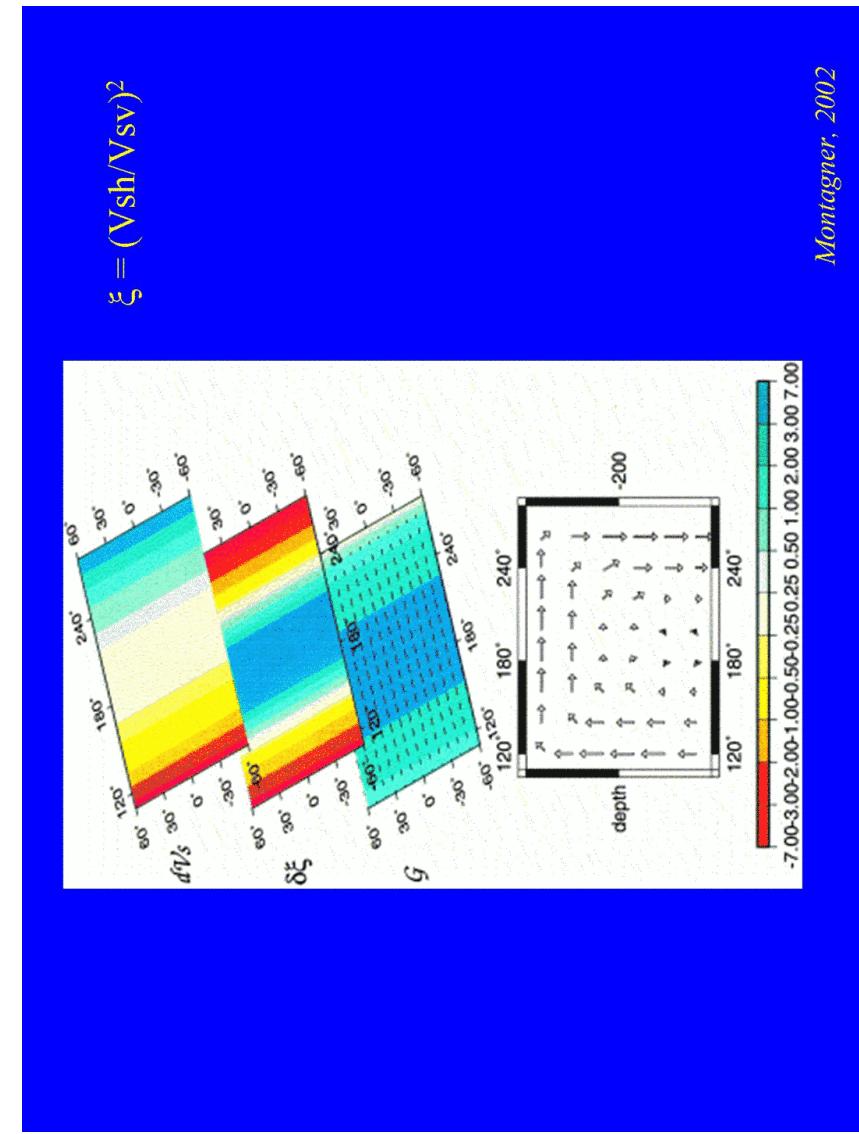
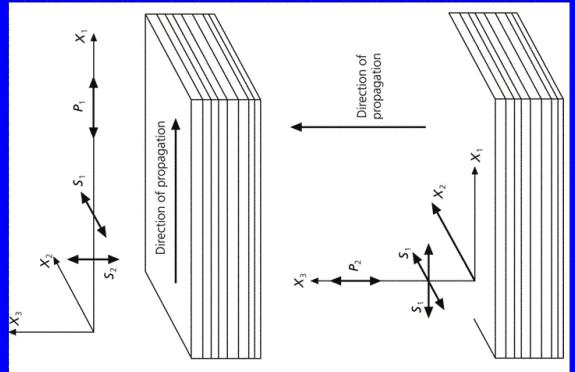
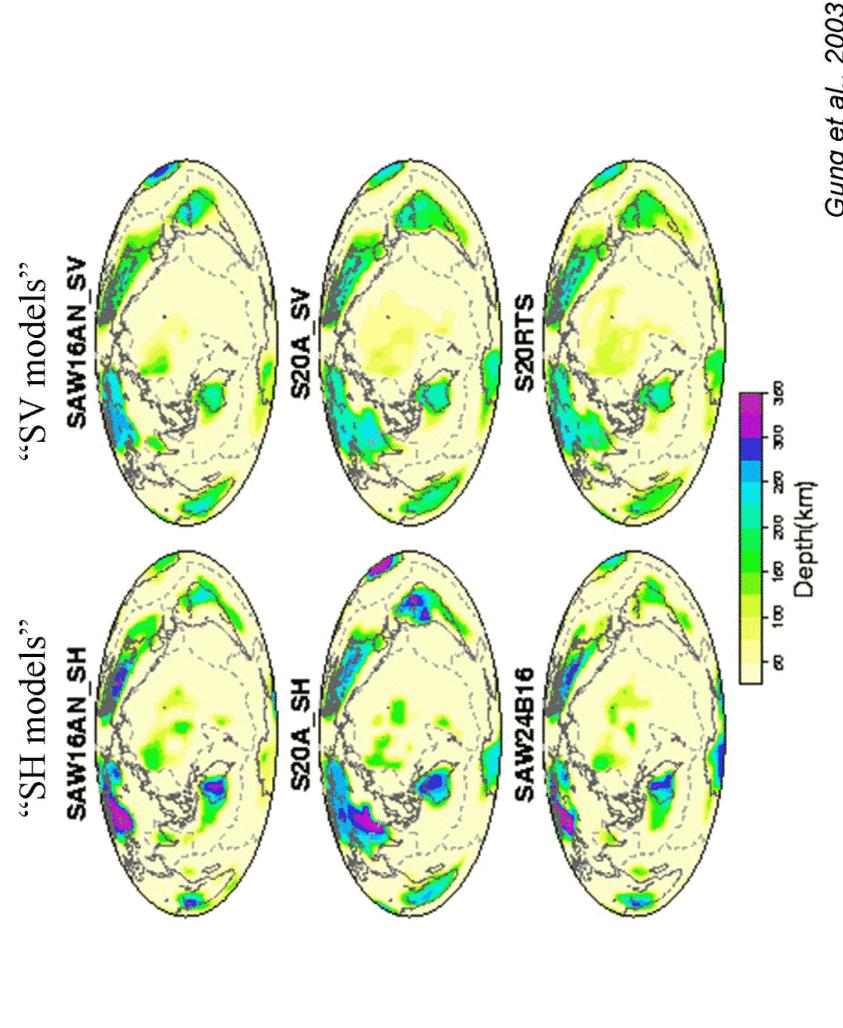


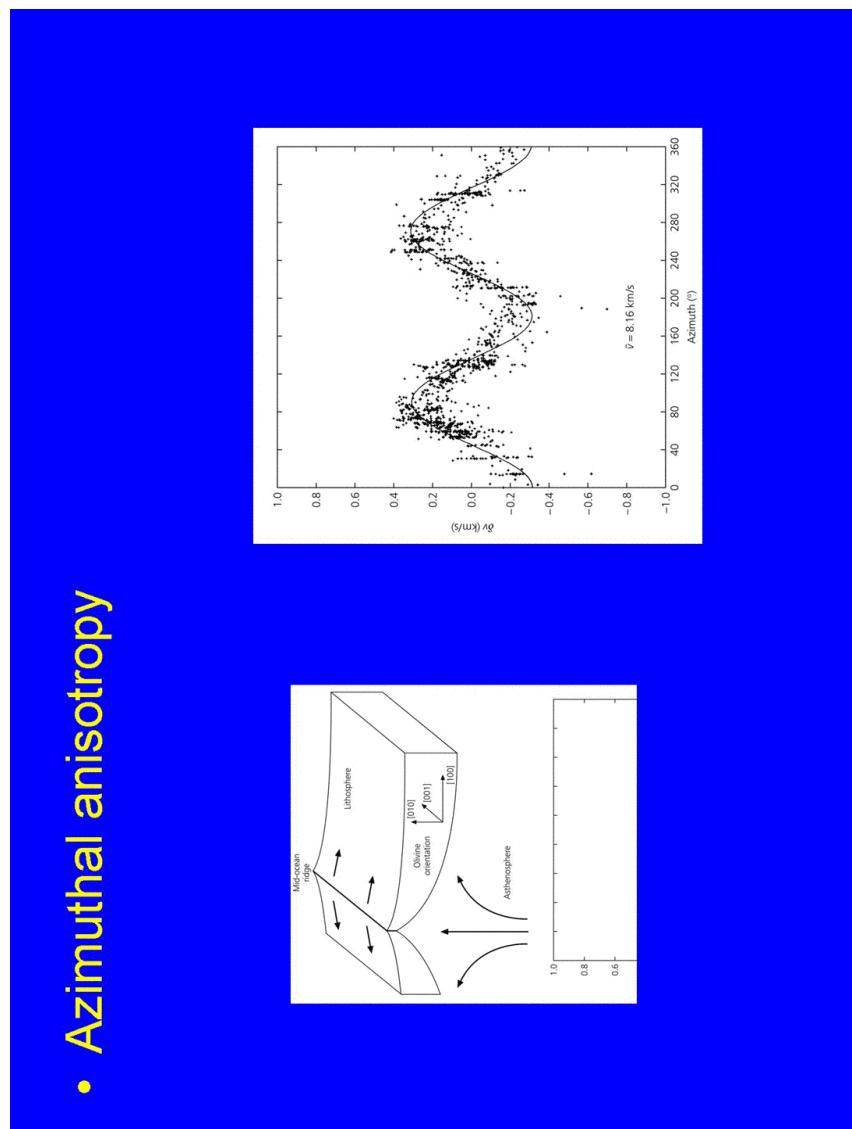
Figure 3.6-6: Orientation of split shear waves.**Figure 3.6-7: Example of shear wave splitting of SKS waves.**

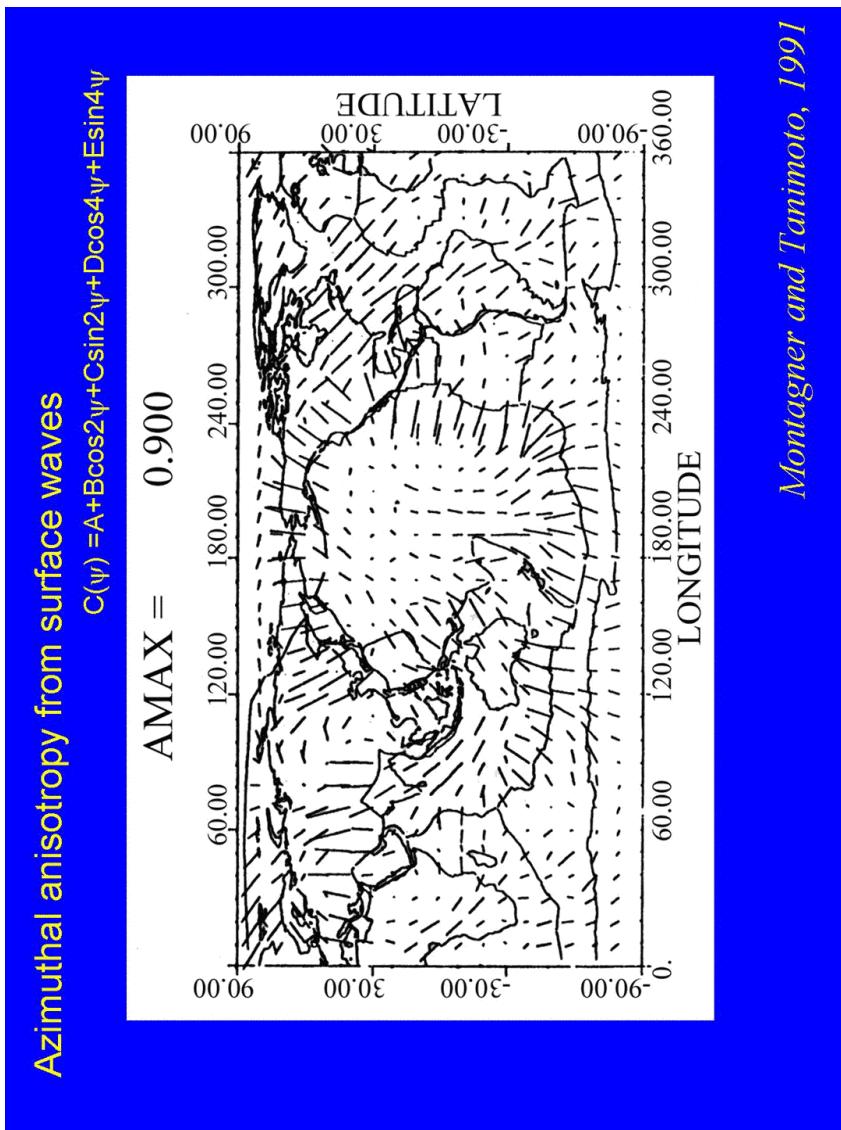
- Simplified models of anisotropy
 - Transverse isotropy / radial anisotropy
 - 5 independent elastic coefficients A,C,F,L,N
- $$- L = \rho V_{sv}^2$$
- $$- N = \rho V_{sh}^2$$
- $$- C = \rho V_{pv}^2$$
- $$- A = \rho V_{ph}^2$$
- $$- \eta = F/(A-2L)$$





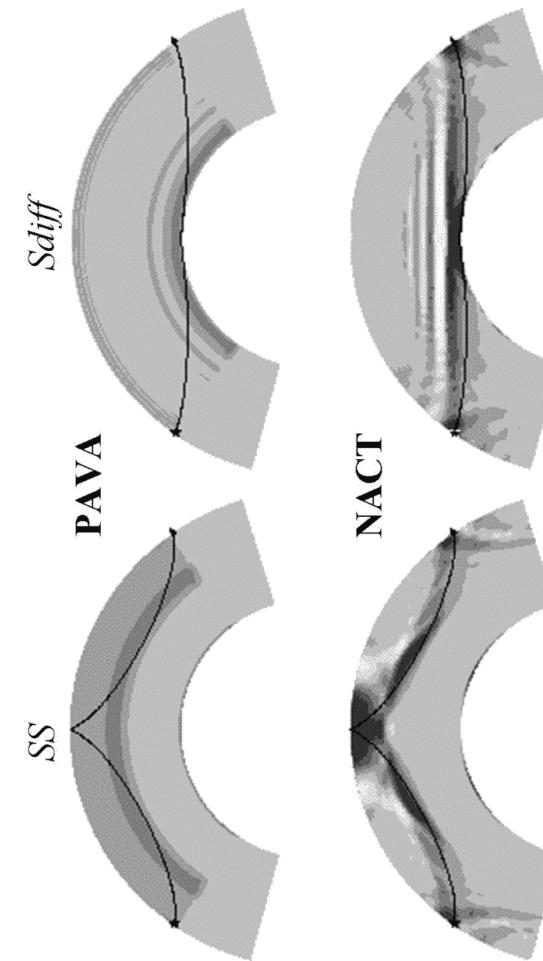
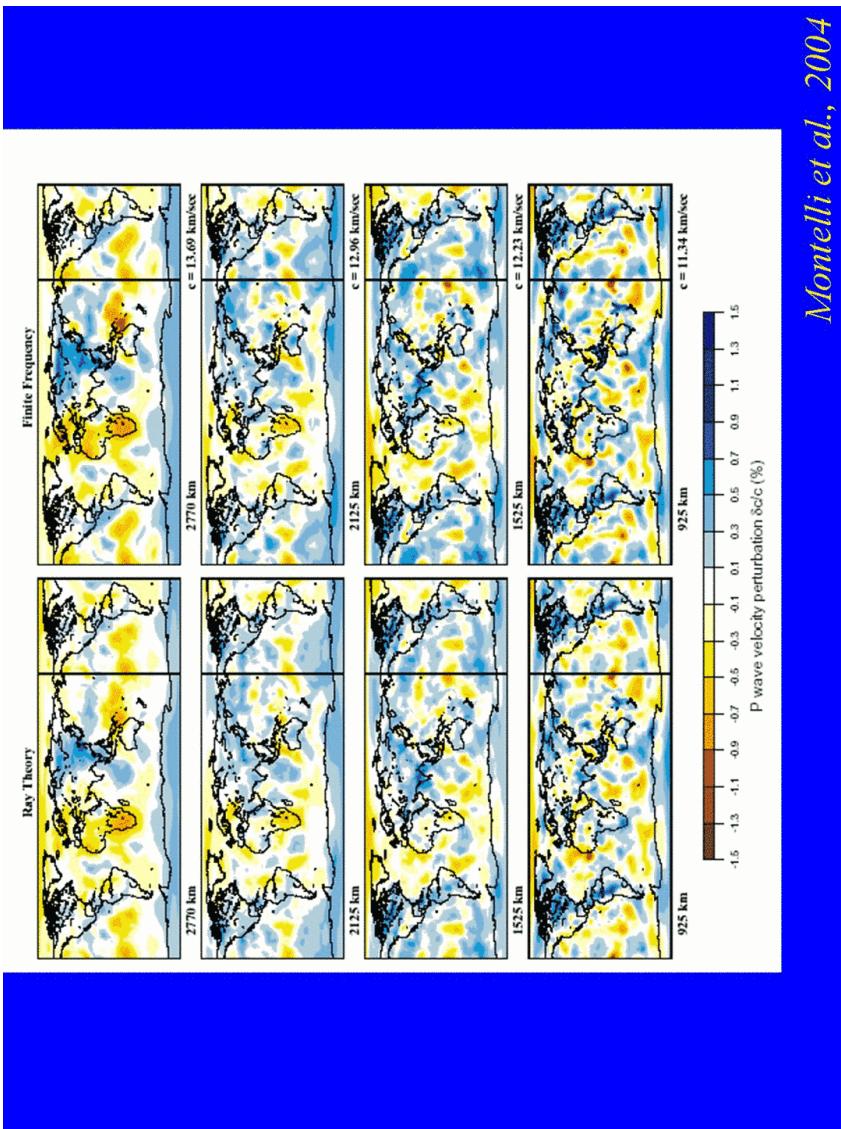
- Azimuthal anisotropy

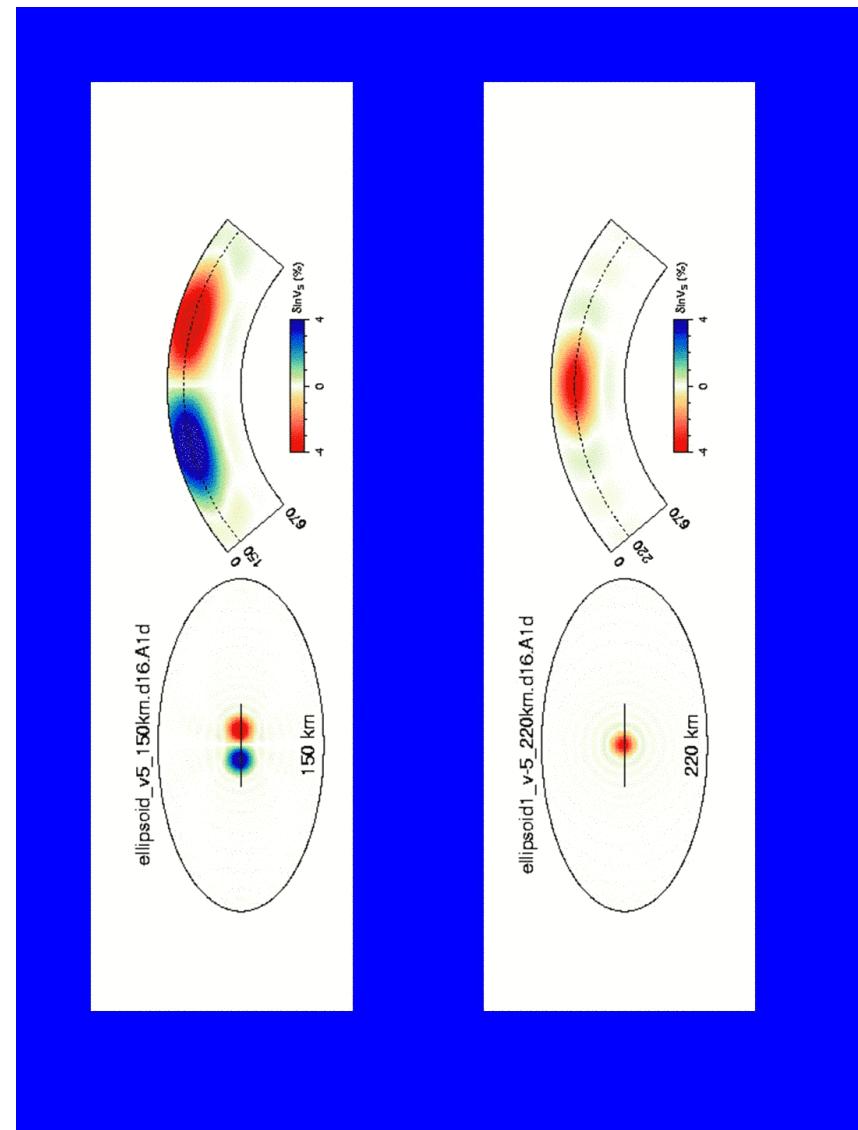
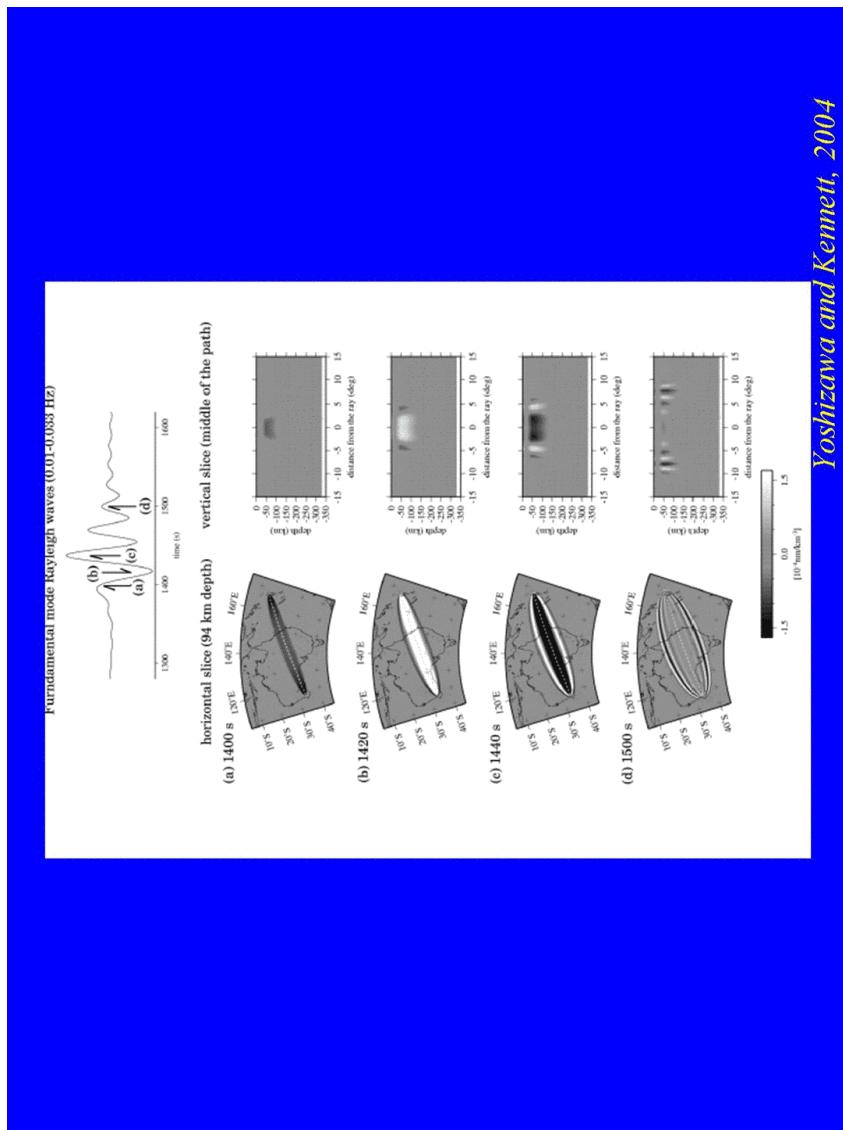




Current challenges

- Ray theory: infinite frequency approximation
- Normal modes “path average approximation”:
1 D kernels
- How do we use the complete information contained in seismic waveforms?





- Path Average Approximation
 - 1D kernels
 - Along branch coupling
 - zeroth order asymptotics
- NACT
 - 2D kernels (in vertical plane)
 - across branch coupling
 - zeroth order asymptotics
- NACT + focusing
 - 2.5D kernels (off plane effects included)
 - order 1/l asymptotics
- Full 1st Order Born Approximation
 - 3D kernels
 - single scattering

