DYNAMICS OF 1D QUANTUM SYSTEMS WITH TENSOR NETWORKS

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arXiv:1410.4186
WHAT ARE TNS?

- TNS = Tensor Network States

Context: quantum many body systems
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Context: quantum many body systems

interacting with each other

Goal: describe equilibrium states

ground, thermal states
WHAT ARE TNS?

- TNS = Tensor Network States

A general state of the N-body Hilbert space has exponentially many coefficients

\[ |\Psi\rangle = \sum_{i_j} c_{i_1 \ldots i_N} |i_1 \ldots i_N\rangle \]
WHY SHOULD TNS BE USEFUL?

Which properties characterize physically interesting states?

finite range
gapped
Hamiltonians
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finite range
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states with
little entanglement

Area law
WHY SHOULD TNS BE USEFUL?

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Area law

\[ S_{A_{\text{max}}} \propto |\delta A| \]

TNS parametrize the structure of entanglement
ID SYSTEMS: MPS

- MPS = Matrix Product States

\[ |\Psi\rangle = \sum_{i_1...i_N} c_{i_1...i_N} |i_1...i_N\rangle \]
1D SYSTEMS: MPS

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1D SYSTEMS: MPS

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\[ |\Psi\rangle = \sum_{i_1...i_N} \text{tr}(A_1^{i_1} A_2^{i_2} ... A_N^{i_N}) |i_1...i_N\rangle \]

Area law by construction

number of parameters: \( NdD^2 \)
MIXED STATES / OPERATORS

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MIXED STATES / OPERATORS

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Same kind of ansatz for operators

\[ \hat{M} = \sum_{i_1,j_1 \ldots i_N,j_N} \text{tr}(M_1^{i_1j_1} M_2^{i_2j_2} \ldots M_N^{i_Nj_N}) |i_1 \ldots i_N \rangle \langle j_1 \ldots j_N| \]

Verstraete et al., PRL 2004
Pirvu et al., NJP 2010
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Bounded operator space
entanglement
entropy

Verstraete et al., PRL 2004
Pirvu et al., NJP 2010
SOME MPS PROPERTIES

good approximation of ground states

gapped finite range Hamiltonian $\Rightarrow$ area law (ground state)

Verstraete, Cirac, PRB 2006

SOME MPS PROPERTIES

good approximation of ground states

gapped finite range Hamiltonian $\Rightarrow$ area law (ground state)

Verstraete, Cirac, PRB 2006

extremely successful for GS, low energy

little entangled

White, PRL 1992
Verstraete, Porras, Cirac, PRL 2004

time evolution can be simulated too

Vidal, PRL 2003, PRL 2007
White, Feiguin, PRL 2004
Daley et al., 2004
Haegeman et al., 2011
DYNAMICS & MPS

Approximate action of local operators on MPS

time evolution

Vidal, PRL 2003, 2004
Verstraete, García-Ripoll, Cirac, PRL 2004
DYNAMICS & MPS

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time evolution

\[ U(t) \rightarrow [U(\delta)]^M \]

Vidal, PRL 2003, 2004
Verstraete, García-Ripoll, Cirac, PRL 2004
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time evolution

as local terms!

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TEBD
\textit{t-DMRG}

Vidal, PRL 2003, 2004
Verstraete, García-Ripoll, Cirac, PRL 2004
DYNAMICS & MPS

Entropy of evolved state may grow linearly

Osborne, PRL 2006
Schuch et al., NJP 2008
DYNAMICS & MPS

Entropy of evolved state may grow linearly

\[ D_{\text{max}} \]

\[ D_{\text{required}} \]

\[ D \sim e^{\alpha t} \]

Osborne, PRL 2006
Schuch et al., NJP 2008

required bond for fixed precision
ALTERNATIVELY

time dependent observables as TN

MCB, Hastings, Verstraete, Cirac, PRL 2009
Müller-Hermes, Cirac, MCB, NJP 2012
ALTERNATIVELY

time dependent observables as TN

problem is contracting the network

TN describe observables, not states

exact contraction not possible

#P complete

MCB, Hastings, Verstraete, Cirac, PRL 2009
Müller-Hermes, Cirac, MCB, NJP 2012
OBSERVABLE AS $U(t)$ 

$\sim e^{-iHt}$
OBSERVABLE AS TN

\[ \sim e^{-iHt} \]
OBSERVABLE AS TN

\[ |\Psi(t)\rangle \]
OBSERVABLE AS TN

\[ \langle \Psi(t) | O | \Psi(t) \rangle \]
OBSERVABLE AS TN

different approximate contraction strategies

standard (TEBD, tDMRG)
OBSERVABLE AS TN

different approximate contraction strategies

standard (TEBD, tDMRG)

evolved state approximated as MPS
OBSERVABLE AS TN

different approximate contraction strategies

Heisenberg picture

Hartmann (2008)
OBSERVABLE AS TN

different approximate contraction strategies

evolved operator as MPO

Heisenberg picture

Hartmann (2008)
OBSERVABLE AS TN

different approximate contraction strategies

transverse contraction, folding

MCB, Hastings, Verstraete, Cirac (2009); Muller-Hermes et al. (2012); Hastings, Mahajan (2014)
OBSERVABLE AS TN

different approximate contraction strategies

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MCB, Hastings, Verstraete, Cirac (2009); Müller-Hermes et al. (2012)
Hastings, Mahajan (2014)
APPLICATION TO THERMALIZATION

Global quench scenario: closed quantum system initialized out of equilibrium
APPLICATION TO THERMALIZATION

thermalization of infinite quantum spin chain
Appliation to Thermalization

Thermalization of infinite quantum spin chain

Fix non-integrable Hamiltonian

\[ H = - \sum_{i} \left( \sigma_{z}^{i} \sigma_{z}^{i+1} + g \sigma_{x}^{i} + h \sigma_{z}^{i} \right) \]

Varying initial state
APPLICATION TO THERMALIZATION

thermalization of infinite quantum spin chain
fix non-integrable Hamiltonian

\[ H = - \sum_i (\sigma_z^i \sigma_z^{i+1} + g \sigma_x^i + h \sigma_z^i) \]

varying initial state

compute \( \rho_N \) for small number of sites
APPLICATION TO THERMALIZATION

We observed different regimes of thermalization for the same Hamiltonian parameters

non-integrable regime

strong instantaneous state relaxes $|Y^+\rangle$
APPLICATION TO THERMALIZATION

time averaged

$\frac{d(\bar{\rho}(t), \rho_\beta)}{t}$

$| Y+\rangle$

$| X+\rangle$

$| Z+\rangle$

MCB, Cirac, Hastings, PRL 2011
Hastings, Mahajan (2014)
different perspective

What are the slowest evolving (local) operators?

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186
different perspective

What are the slowest evolving (local) operators?

\[ \frac{dA(t)}{dt} = i[H, A(t)] \]

numerical study using ED and TNS

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186
Scenario

1D non-integrable spin chain

\[ H = \sum_i \left( \sigma_z^i \sigma_z^{i+1} + g \sigma_x^i + h \sigma_z^i \right) \]

only local conserved quantity is energy density

operator acting on \( M \) central sites

slow operator: inhomogeneity of energy density

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186
Goal: minimizing $\|[H, A_M]\|$\n\n$$\left\| \frac{dA_M}{dt} \right\| = \|[A_M, H]\|$$
Goal: minimizing $\|[H, A_M]\|$

$$\|A_M(t) - A_M(0)\| \leq \chi(M)t$$

$$\|A\|_2 = \sqrt{\text{tr}(A^\dagger A)} \quad \text{Frobenius norm}$$
Goal: minimizing $\|[H, A_M]\|$

$$\|A_M(t) - A_M(0)\| \leq \chi(M)t$$

$$\|A\|_2 = \sqrt{\text{tr}(A^\dagger A)} \quad \text{Frobenius norm}$$

$$\lambda_M = \min_A \| [A_M, H] \|_2^2 \quad \frac{\| [A_M, H] \|_2^2}{\| A_M \|_2^2}$$

Physical meaning

$$\rho \sim I + \epsilon A_M \quad \text{high T state}$$

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186
Look for the $A_M$ with smallest $\| [H, A_M] \|_2$

$$H = \sum_i \left( \sigma_z^i \sigma_{z+1}^i + g\sigma_x^i + h\sigma_z^i \right)$$
Look for the $A_M$ with smallest $\|[H,A_M]\|_2$

$$H = \sum_i \left( \sigma_z^i \sigma_z^{i+1} + g \sigma_x^i + h \sigma_z^i \right)$$

energy density fluctuation $\Rightarrow$ diffusive mode

$$E_M = \sum_{n=-M/2}^{M/2} c_n h_{n,n+1}$$

$$c_n \sim \cos \frac{\pi n}{M}$$

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186
Results

there are operators slower than energy diffusion for a non-integrable system

1. exhaustive numerical search

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186
Results

There are operators slower than energy diffusion for a non-integrable system.

\[ \| [H, O_M] \|_2^2 \leq \frac{1}{M^\alpha} \]

\[ \alpha > 2 \]

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186
Results

there are operators slower than energy diffusion for a non-integrable system

I. exhaustive numerical search

II. MPO ansatz

\[ O_M = \]

variational search over MPO with bond dimension \( D \)

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186
Results

there are **MPO operators slower than energy diffusion** for a non-integrable system

\[ D = 4 \quad \alpha \approx 2 \]

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186
Results

there are MPO operators slower than energy diffusion for a non-integrable system

\[ D = 4 \quad \alpha \approx 2 \]
\[ D = 20 \quad \alpha \approx 2.1 \]
\[ D = 60 \quad \alpha \approx 2.18 \]
\[ D = 120 \quad \alpha \approx 2.3 \]

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186
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\[ \| [H, O_M] \|_2^2 \leq \frac{1}{M^\alpha} \]

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qualitatively similar for other parameters

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186
Results

there are operators slower than energy diffusion for a non-integrable system

$$H = \sum_i \left( \sigma_z^i \sigma_z^{i+1} + g \sigma_x^i + h \sigma_z^i \right)$$

look at system with no conserved local energy density: Floquet
Results

There are operators slower than energy diffusion for a non-integrable system

\[ H = \sum_i \left( \sigma_x^i \sigma_x^{i+1} + g \sigma_x^i + h \sigma_z^i \right) \]

Look at system with no conserved local energy density: Floquet

\[ U = e^{-i\tau H_x} e^{-i\tau H_z} \]

\[ e^{-i\tau H_x} \quad e^{-i\tau H_z} \]
Strategy: minimizing

\[ \lambda_M = \min_{A_M} \frac{\|[A_M, U]\|^2}{\|A_M\|^2} \]

similar physical meaning

\[ \rho \sim I + \epsilon A_M \]

bounds also thermalization time (step number)

\[ \left| \langle A_M^{(N)} \rangle - \langle A_M \rangle_\beta \right| \geq 1 - N \sqrt{\lambda_M} \]

\[ N_{th} \geq \frac{1}{\sqrt{\lambda_M}} \]
Results

1. exhaustive numerical search for $M<12$

\[ \alpha \sim -1.7 \]
Results

I. exhaustive numerical search for $M < 12$

II. explicit construction for larger $M$

$$A_M = \sum_{n=-M}^{M} c_n U^n O_0 U^{-n}$$

filtered operator
Results

I. exhaustive numerical search for $M < 12$

II. explicit construction for larger $M$

\[ A_M = \sum_{n=-M}^{M} c_n U^n O_0 U^{-n} \]

filtered operator

variational parameters

single site traceless operator

minimization requires computing $\text{tr} \left( A_M U A_M U^\dagger \right)$
Results

I. exhaustive numerical search for $M < 12$
II. explicit construction for larger $M$
Results

I. exhaustive numerical search for $M < 12$

II. explicit construction for larger $M$

$\| [H, A_M] \|_2^2 \leq \frac{1}{M^2}$

$\lambda_M$

$\alpha \sim -1.96$

$M = 10$

$M = 40$

$M = 100$
Results

more generic system without conserved local energy: random quantum circuit

only locality of evolution

but operators with even smaller commutator exist

\[ \| [H, A_M] \|_2^2 \leq \frac{1}{M^2} \]

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186
MANY BODY LOCALIZATION

Anderson localization: single particle states localized due to disorder

environment destroys localization

Interactions and disorder more interesting scenario

weak interactions $\Rightarrow$ MBL phase

Gornyi, Mirlin, Polyakov, PRL 2005
MANY BODY LOCALIZATION

Existing numerical studies use exact diagonalization to access full spectrum/very long times

limited to small systems

TN techniques to reach larger system sizes

long time evolution of mixed states

collaboration in progress with N. Yao, M. Lukin
$H = \sum (S_x^{[i]} S_x^{[i+1]} + S_y^{[i]} S_y^{[i+1]} + J S_z^{[i]} S_z^{[i+1]} + h_i S_i^z)$

at $J=0$ non-interacting XY localized for $h>0$

at $J=1$ shows MBL for $h\sim 3-3.5$

Oganesyan, Huse, PRB 2007
Pal, Huse, PRB 2010
Luitz et al., 2014
TNS/QI & MBL
(Q) INFORMATION AND MBL
(Q)INFORMATION AND MBL

known scenario: global quenches for pure states

single particle localization saturation of $S$
logarithmic growth of $S$

Bardarsson, Pollmann, Moore, PRL 2012
(Q)INFORMATION AND MBL

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Bardarsson, Pollmann, Moore, PRL 2012

\[ I(A : B) = S(A) + S(B) - S(AB) \]

measures correlations between subsystems
(Q)INFORMATION AND MBL

\[ \rho_\Phi \propto Id^{\otimes \frac{L-L_c}{2}} \otimes |\Phi\rangle\langle\Phi| \otimes Id^{\otimes \frac{L-L_c}{2}} \]

can be used to encode a qubit
\((Q)\) INFORMATION AND MBL

\[ \rho_\Phi \propto Id^\otimes \frac{L-L_c}{2} \otimes |\Phi\rangle\langle\Phi| \otimes Id^\otimes \frac{L-L_c}{2} \]

\[ |1\rangle \hspace{1cm} \text{can be used to encode a qubit} \]

\[ I(\text{central } L_c \text{ sites} : \text{edges}) \]
RECOVERY FIDELITY
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MUTUAL INFORMATION

Non-interacting

Interacting

no disorder
MUTUAL INFORMATION

Non-interacting

Interacting

thermalization

fast propagation at the beginning

medium disorder
RECOVERY FIDELITY

non interacting

look at middle 4 sites

look at middle 10 sites

no disorder
RECOVERY FIDELITY

Look at middle 4 sites

Look at middle 10 sites

Non-interacting

Distinguishability of 0, 1

Distinguishability of X+, X-

Maximum recovery fidelity

L = 4

L = 10

No disorder
RECOVERY FIDELITY

look at middle 4 sites

different behavior $Z/X$

look at middle 10 sites

interacting distinguishability of 0, 1

$L=4$
distinguishability of $X^+, X^-$

$L=10$
maximum recovery fidelity

strong disorder
CONCLUSIONS

Versatile TNS tools: can be used for out-of-equilibrium time evolution

- approximations involved
- state/operators
- more general TN contraction
- entanglement in TN

Applications to non-equilibrium
THANKS!