On the (unmetaphorical) entanglement gap in CFTs (and related topics)

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Closing the (metaphorical) Entanglement Gap, KITP, June 2015

literal?

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This talk is going be mainly about Rényi entropies in some simple geometries, mainly CFTs in 1+1 dimensions (but not entirely)

Bipartite entanglement $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Whole system in pure state (usually ground state) $\rho = |0\rangle\langle 0|$

Reduced density matrix $\rho_A = \mathrm{Tr}_{\mathcal{H}_B}\,\rho \equiv e^{-2\pi K_A}$

Entanglement hamiltonian K_A, eigenvalues $E_0 < E_1 \leq E_2 \leq \ldots$

Rényi entropies $S_A^{(n)} = (1-n)^{-1} \log \mathrm{Tr}\, e^{-2\pi n K_A}$

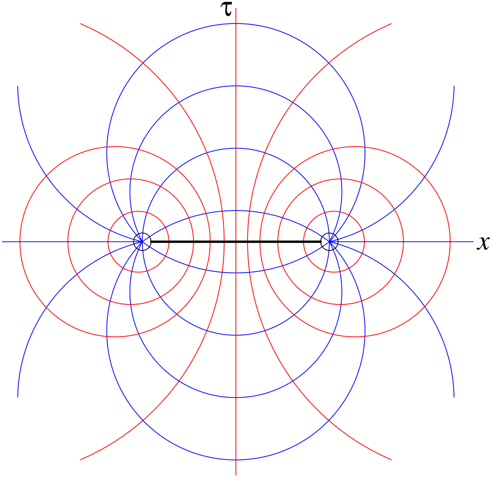
- how does E_0 behave?
- what can we say about the entanglement gaps $E_j - E_0$ and their degeneracies?
- what is the general form of the density of states of K_A?
- [what can we say about (correlation functions in) eigenstates of K_A?]

In principle the Rényi entropies determine the entanglement spectrum, and but as we'll see it's mainly the behaviors at large and small n which are important

As usual, we can say much more in $d =$1+1 dimensions than for $d > 2$, and mainly for relativistic field theories.

In fact, in 1+1 dimensions several different cases can be unified in one formula.

1. Single interval $(-R, R)$ in a 1+1-dimensional CFT

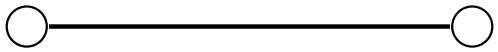


$$\mathrm{Tr}\,\rho_A^n = \mathrm{Tr}\,e^{-2\pi n K_A} = \frac{Z_{\mathcal{R}_n}}{Z_{\mathcal{R}_1}^n}$$

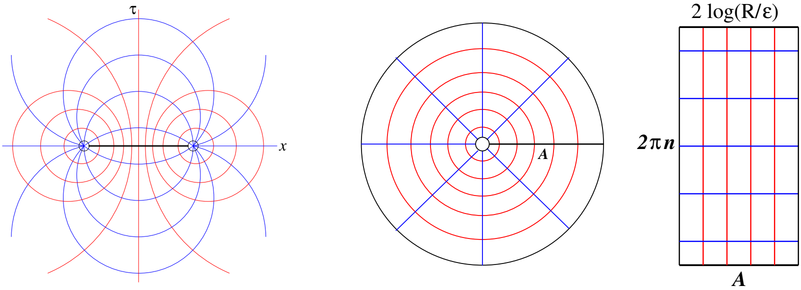
This problem was considered by Holzhey *et al.* [1994]
[Calabrese-JC, 2004].

In this approach, $Z_{\mathcal{R}_1}$ was trivial and $(\partial/\partial R)\log Z_{\mathcal{R}_n}$ came from evaluating $\langle T_{\mu\nu}\rangle_{\mathcal{R}_n}$ by conformal mapping and integrating $\int_{-\infty}^{\infty}\langle T_{xx}(0,\tau)\rangle d\tau$.

But we can regularize another way, by cutting out small discs (or slits) around the end points of the interval.



In that case, both $\mathcal{R}_n$ and $\mathcal{R}_1$ are conformally equivalent to annuli.



$$\mathrm{Tr}\,\rho_A^n = \frac{Z_{\text{annulus}}\left(\frac{2\pi n}{2\log(R/\epsilon)}\right)}{Z_{\text{annulus}}\left(\frac{2\pi}{2\log(R/\epsilon)}\right)^n}$$

So $S_A^{(n)}$ is given in terms of the free energy of a system of length $\ell = 2\log(R/\epsilon)$ at inverse temperature $\beta = 2\pi n$, *including all universal finite size effects when* $\beta \sim \ell$

Z_{annulus} is more easily expressed in terms of

$$q = e^{-\pi\,\text{height}/\text{width}} = e^{-\pi^2/\log(R/\epsilon)} \to 1$$

boundary scaling dimensions

$$Z_{\text{ann}} = q^{-c/24}\big(1 + \sum_j n_j q^{\Delta_j}\big)$$

or equivalently

$$\tilde{q} = e^{-4\pi\,\text{width}/\text{height}} \sim (\epsilon/R)^4 \to 0$$

bulk scaling dimensions

$$Z_{\text{ann}} = \tilde{q}^{-c/24}\big(b_0^2 + \sum_j b_j^2 \tilde{q}^{\Delta_j}\big)$$

Since $q \to q^n \Rightarrow \tilde{q} \to \tilde{q}^{1/n}$, the latter expansion is OK *unless* $n \gtrsim \log(R/\epsilon)$

Using only the leading term as $\tilde{q} \to 0$ then gives the standard result [Holzhey et al. (1994), Calabrese + JC (2004)]

$$\mathrm{Tr}\,\rho_A^n \sim (b_0^2)^{1-n} \frac{\left(\tilde{q}^{1/n}\right)^{-c/24}}{\left(\tilde{q}^{-c/24}\right)^n}$$

so that

$$(n-1)S_A^{(n)} \sim (c/6)(n-1/n)\log(R/\epsilon) + 2(n-1)\log b_0$$

Note however that the linear term in n now comes from the denominator $Z_{\mathcal{R}_1}^n$.

The $\log b_0$ term is the Affleck-Ludwig boundary entropy

We also get 'unusual' corrections $O(\tilde{q}^{\Delta_j/n}) \sim (\epsilon/R)^{4\Delta_j/n}$ [Calabrese-JC, 2010]

We can identify the eigenvalues of ρ_A as being of the form

$$\frac{q^{-c/24+\Delta_j+N}}{Z_{\text{ann}}(q)} \sim \frac{q^{-c/24+\Delta_j+N}}{b_0^L b_0^R \tilde{q}^{-c/24}}$$

(with degeneracies n_j) so that

$$E_0 \sim (c/6)\log(R/\epsilon) + 2\log b_0 \qquad \text{(single copy entanglement)}$$

Note this comes entirely from the denominator $Z_{\mathcal{R}_1}^n$!

The entanglement gaps are [*cf* Calabrese-Lefevre, 2008]

$$E_j - E_0 \sim \frac{\pi^2(\Delta_j + N)}{\log(R/\epsilon)}$$

More generally we see that, for this case, the Rényi n-entropy is simply related to the thermodynamic entropy of the CFT at inverse temperature $\beta = 2\pi n$ in an interval of length $2\log(R/\epsilon)$ *including universal boundary terms*, that is given by

$$Z \propto \mathrm{Tr}\,\exp\left(-\beta \int_{-\log(R/\epsilon)}^{\log(R/\epsilon)} \hat{T}_{tt}(x)dx\right)$$

From this we can undo the conformal mappings to find the result [Casini-Huerta-Myers] for the entanglement hamiltonian

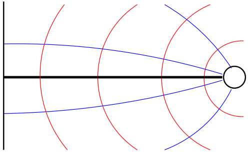
$$K_A = \int_{-R+\epsilon}^{R-\epsilon} \frac{(R^2 - x^2)}{2R}\,\hat{T}_{tt}(x)dx$$

The asymptotic density of states of K_A is just that of a CFT on an interval of length $2\log(R/\epsilon)$, i.e.

$$\mu(E) \sim e^{\mathrm{const.}\sqrt{cE\log(R/\epsilon)}}$$

(power of E simply related to the the n^{-1} behavior as $n \to 0$).

2. Single Interval $(0, R)$ at the end of a semi-infinite line



This is once again an annulus, now with

$$q = e^{-2\pi^2/\log(R/\epsilon)} \qquad \tilde{q} \sim (\epsilon/R)^2$$

and similar results apply:

$$(1-n)S_A^{(n)} \sim (c/12)(n-1/n)\log(\epsilon/R) + (n-1)\big(\log b_0^L + \log b_0^R\big)$$

The boundary terms do not have to be the same as before.

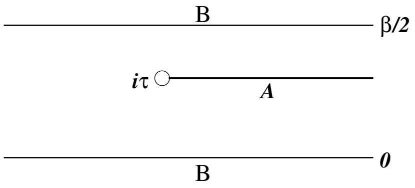
$$K_A = \int_\epsilon^{R-\epsilon} \frac{(R^2 - x^2)}{2R}\,\hat{T}_{tt}(x)dx$$

3. Entanglement growth after a Quantum Quench

In this case we consider the entanglement between $A=(0,\infty)$ and $B=(-\infty,0)$ following a quench to a CFT from a state with short-range entanglement

$$|\psi_0\rangle = e^{-(\beta/4)H_{CFT}}|\mathrm{B}\rangle$$

where $|\mathrm{B}\rangle$ is a conformal boundary state



continued to $\tau = \frac{\beta}{4} + it$. This is also an annulus with

$$q \sim \exp\left(-\frac{2\pi^2}{\log(\beta/\pi\epsilon) + \log\cosh(2\pi t/\beta)}\right) \sim e^{-\pi\beta/t}, \quad \tilde{q} \sim \left(\frac{\pi\epsilon/\beta}{\cosh(2\pi t/\beta)}\right)^2 \sim e^{-4\pi t/\beta}$$

So [Calabrese-JC 2005]

$S_A \sim (c/6)\log\cosh(2\pi t/\beta) \propto t^2$ (short times), $\quad \propto t \quad (t \to \infty)$

Entanglement gaps $E_j - E_0 \propto \Delta_j \beta/t \quad (t \to \infty)$

Entanglement hamiltonian

$$K_A = (\beta/2\pi)\int_\epsilon^\infty f(x,t)\hat{T}_{tt}(x)dx$$

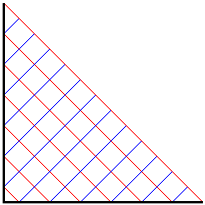
where $f(x,0) = \sinh(2\pi x/\beta)$ and

$$f(x,t) \approx 1 \quad (x < t); \quad f(x,t) \sim e^{2\pi(x-t)/\beta} \quad (x > t)$$

Thus the interval $(0,t)$ is thermalized, $x > t$ retains short-ranged entanglement.

4. Non-critical integrable lattice models

Consider the entanglement between $A = (0, \infty)$ and $B = (-\infty, 0)$ in a 1+1d *lattice* model.



$\rho_A \propto (\text{Corner Transfer Matrix})^4$.
Baxter showed that if the weights satisfy the Yang-Baxter relations (and some other assumptions), its spectrum is simple.

By Baxter magic, the Rényi entropies have *exactly* the same form as for a CFT discussed, that is they are given in terms of $Z_{\text{ann}}(q^n)/Z_{\text{ann}}(q)^n$ where now however

$$\tilde{q} \propto (\xi/a)^{-2}, \quad q \sim e^{-\text{const.}/\log(\xi/a)}$$

and a is now the lattice spacing rather than a macroscopic cut-off ϵ.

This observation is crying out for an entanglement explanation!

Higher dimensions

For a sphere in d space-time dimensions [Casini-Huerta-Myers]

$$K_A = \int_{r<R-\epsilon} \frac{(R^2 - r^2)}{2R} \hat{T}_{\tau\tau}(r) d^{d-1} r$$

For $\mathrm{Tr}\, e^{-2\pi n K_A}$ the effective local inverse temperature is

$$\beta_{\text{eff}}(r) = 2\pi n \frac{(R^2 - r^2)}{2R}$$

and as long as this is $\ll R$, i.e. $n \ll 1$, the contributions from each element $d^{d-1} r$ are additive and given by Stefan's law:

$$S_A^{(n)} \sim -\int \frac{\sigma}{\beta_{\text{eff}}(r)^{d-1}} d^{d-1} r \propto \frac{\sigma}{n^{d-1}} \left(\frac{\text{Area}}{\epsilon^{d-2}} + \cdots + \overbrace{O(\log(R/\epsilon))}^{d \text{ even}} \right)$$

$\sigma =$ Stefan-Boltzmann constant for the CFT

Asymptotic density of states of the CFT is of the form $e^{\text{const.}\sigma^{1/d} E^{1-1/d}}$

But near the boundary $\beta_{\text{eff}}(r) \sim n\epsilon$ is small unless n is very large. The contribution from this region is

$$\int^{R-\epsilon} \frac{\sigma}{(n(R-r))^{d-1}} R^{d-2} dr d\Omega \sim \frac{\sigma}{n^{d-1}} \frac{\text{Area}}{\epsilon^{d-2}}$$

So the n-dependence of the area law for the Rényi entropies is simple:

$$(n-1)S_A^{(n)} \sim \sigma \left(n - \frac{1}{n^{d-1}} \right) (R/\epsilon)^{d-2}$$

as $R/\epsilon \to \infty$ at fixed n. Note that such a simple law does not follow for the universal log or $O(1)$ corrections!

A more careful analysis shows this is valid for $n \ll (R/\epsilon)^{(d-2)/(d-1)}$, and the entanglement gaps behave like

$$E_j - E_0 \propto (\epsilon/R)^{(d-2)/(d-1)} \qquad (c.f. \ (\log(R/\epsilon))^{-1} \text{ for } d=2)$$

Summary

For several simple cases in 1+1-dimensions (single interval, semi-infinite system after a quantum quench or a non-critical integrable lattice model) with a suitable regularization the Rényi entropies and the entanglement spectrum are known exactly and are given by the finite temperature partition function of the CFT on an open interval

For entanglement of a spherical region in a $d > 2$ dimensional CFT, the $n \to 0$ behavior of the Rényi entropies is $\sim \sigma / n^{d-1}$. For the area law coefficient this extends to all n for sufficiently large R.