Regularities in Frequency/Period Spacings: How Asteroseismology Works

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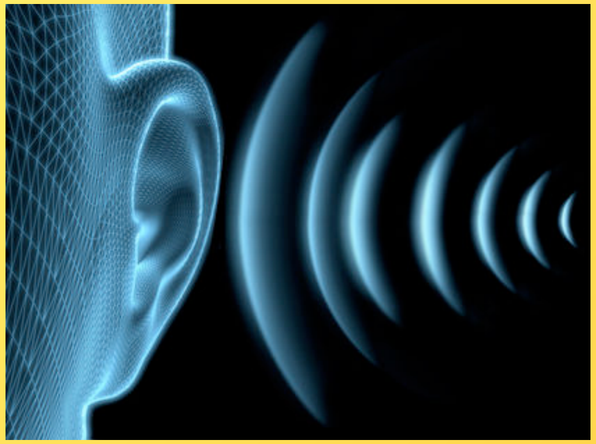
Waves in Stars

Stars are gravitationally stratified $\longrightarrow$ two types of waves

Acoustic (Pressure) Waves



Dispersion Relation

$$\omega^2 = c^2(k_\mathrm{r}^2 + k_\mathrm{h}^2)$$

Sound Speed

$$c^2 = \frac{\gamma P}{\rho}$$

Gravity (Buoyancy) Waves

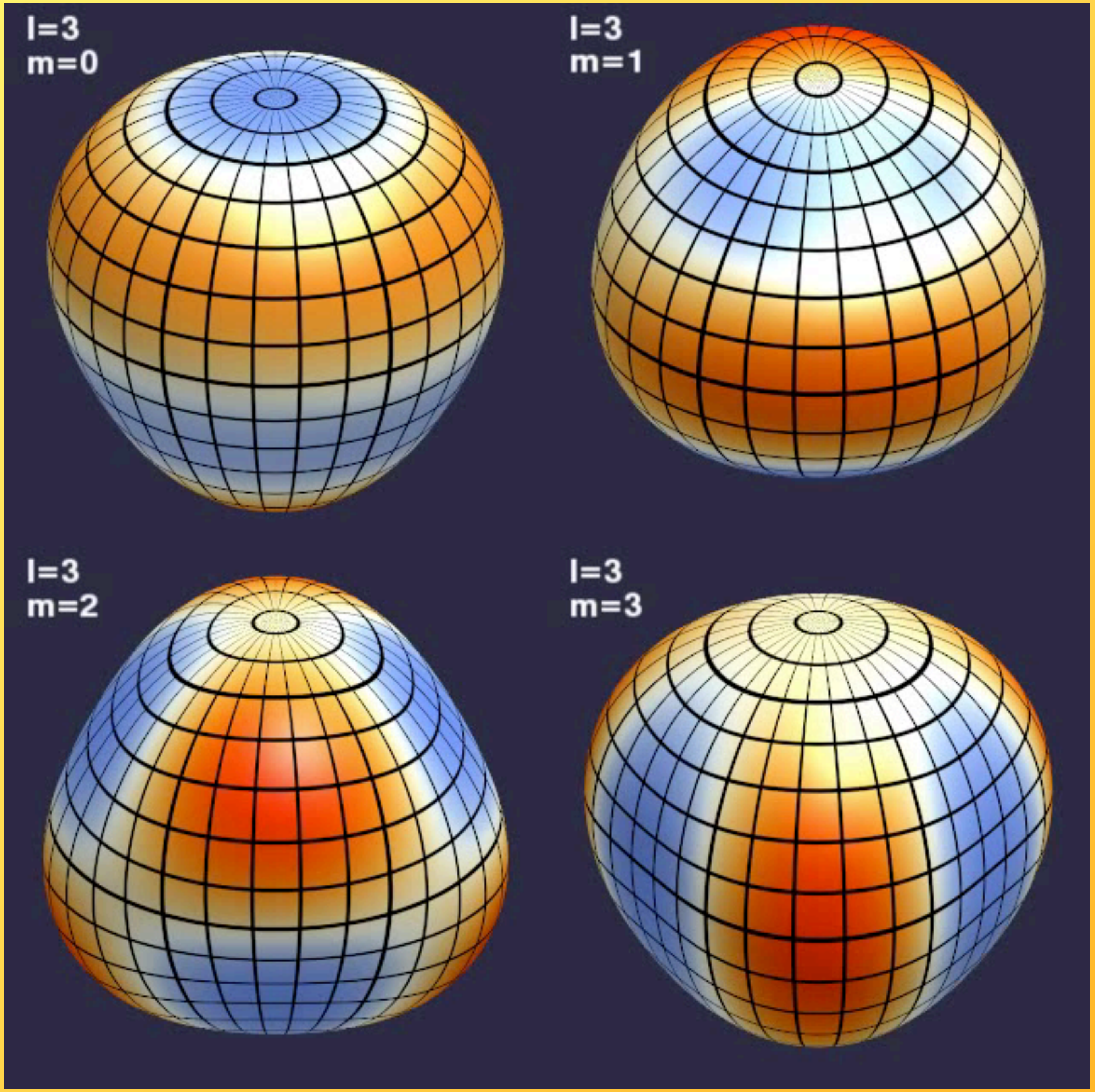
Dispersion Relation

$$\omega^2 = \frac{k_\mathrm{h}^2}{k_\mathrm{r}^2 + k_\mathrm{h}^2} N^2$$

Brunt-Väisälä Frequency

$$N^2 = \frac{g}{r}\left(\frac{1}{\gamma}\frac{\mathrm{d}\ln P}{\mathrm{d}\ln r} - \frac{\mathrm{d}\ln \rho}{\mathrm{d}\ln r}\right)$$

Applying Periodic Boundary Conditions



Angular dependence of waves: spherical harmonics

$$Y_\ell^m \propto P_\ell^m(cos\theta)\ \exp(\mathrm{i}m\phi)$$

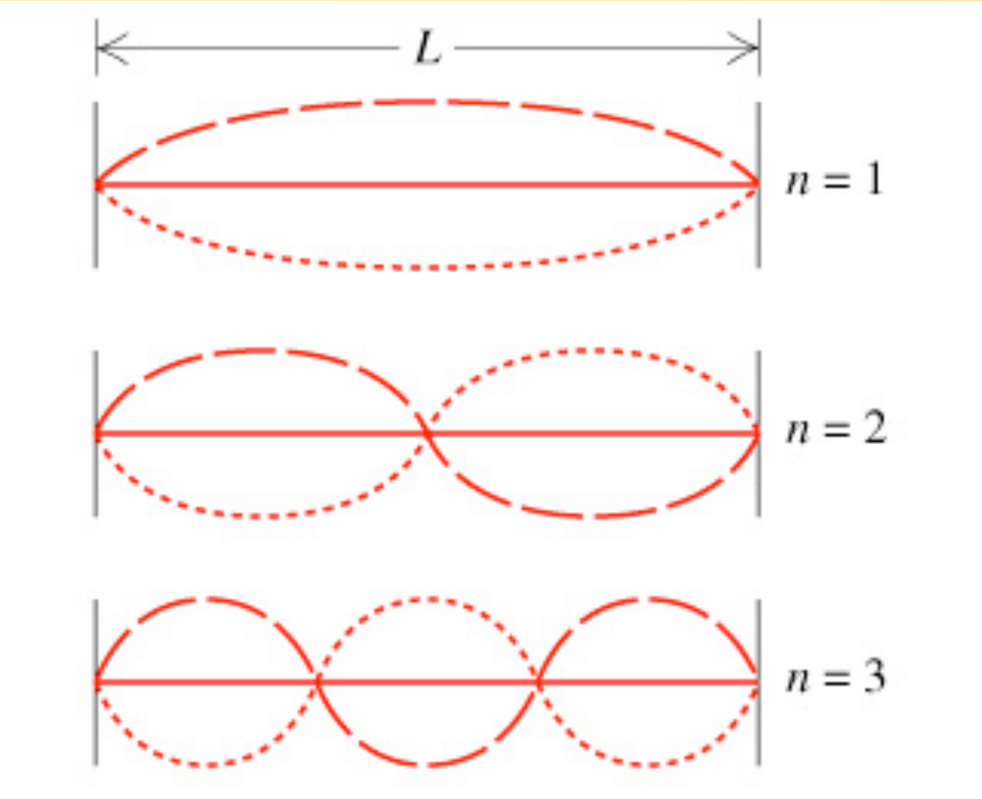
Spherical degree ℓ sets horizontal wavenumber k_h

$$k_\mathrm{h}^2 = \frac{\ell(\ell+1)}{r^2}$$

Applying Radial Boundary Conditions

<u>*On a stretched string…*</u>

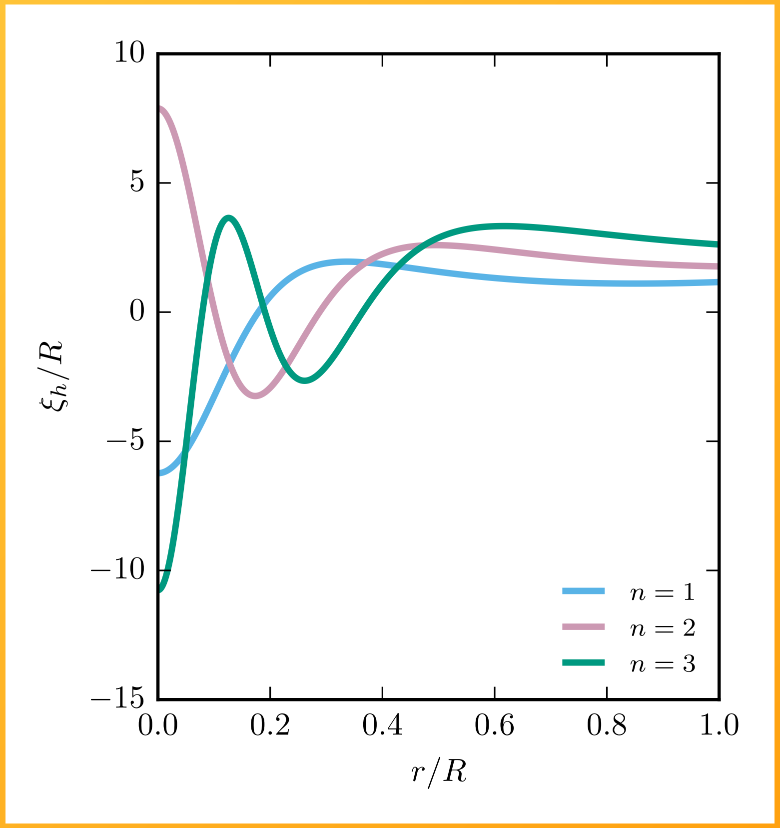
$$kL = n\pi$$



<u>*In a star…*</u>

boundary phase term

$$\int_0^R k_r \, \mathrm{d}r = (n + \alpha)\pi$$



Polytrope model

Asymptotic Frequencies of Pressure (p) Modes

Radial boundary condition

$$\int_0^R k_\mathrm{r}\,\mathrm{d}r = (n+\alpha)\pi$$

Asymptotic dispersion relation

$$\omega^2 = c^2(k_\mathrm{r}^2 + k_\mathrm{h}^2) \xrightarrow{k_r \gg k_h} k_r \approx \frac{\omega}{c}$$

Characteristic equation

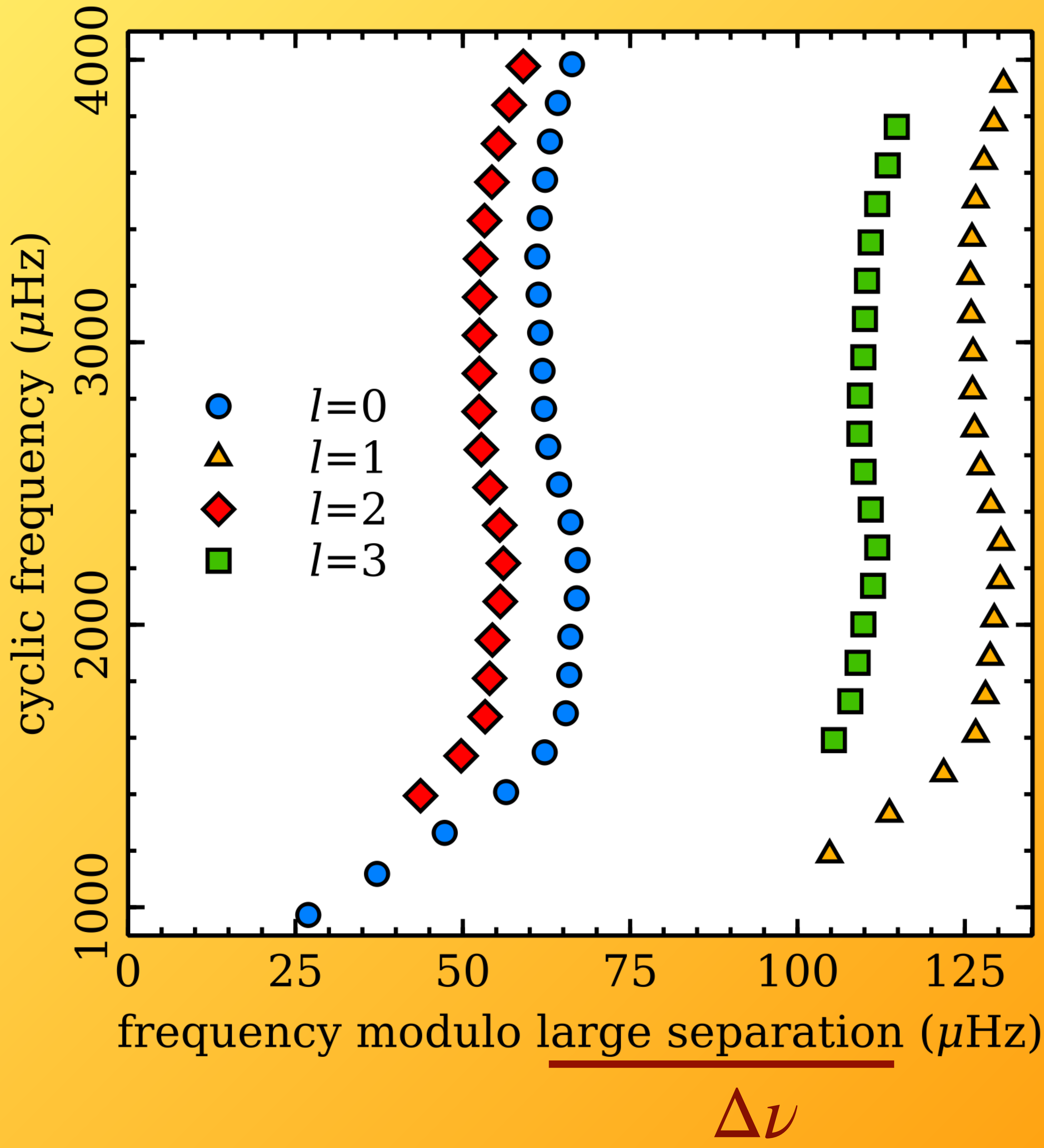
$$\omega \int_0^R \frac{\mathrm{d}r}{c} \approx (n+\alpha)\pi$$

Solve for frequencies...

$$\nu_n \equiv \frac{\omega}{2\pi} \approx (n+\alpha)\underbrace{\left[2\int\frac{\mathrm{d}r}{c}\right]^{-1}}_{\Delta\nu}$$

...p-modes with consecutive n will show a uniform spacing in frequency Δν

Echelle Diagram for the Sun



Modes with consecutive n stack vertically in echelle diagram

Warrick Ball (BiSON data)

p-Mode Frequencies Revisited

Asymptotic dispersion relation $$\omega^2 = c^2 k^2 \xrightarrow{k_r \gg k_h} k_r \approx \frac{\omega}{c}\sqrt{1 - \frac{k_\mathrm{h}^2 c^2}{\omega^2}}$$

Angular boundary condition $$k_\mathrm{h}^2 = \frac{\ell(\ell+1)}{r^2}$$

Characteristic Equation $$\omega \int_{r_1}^{R} \sqrt{1 - \frac{\ell(\ell+1)c^2}{\omega^2 r^2}} \frac{\mathrm{d}r}{c} \approx (n+\alpha)\pi$$

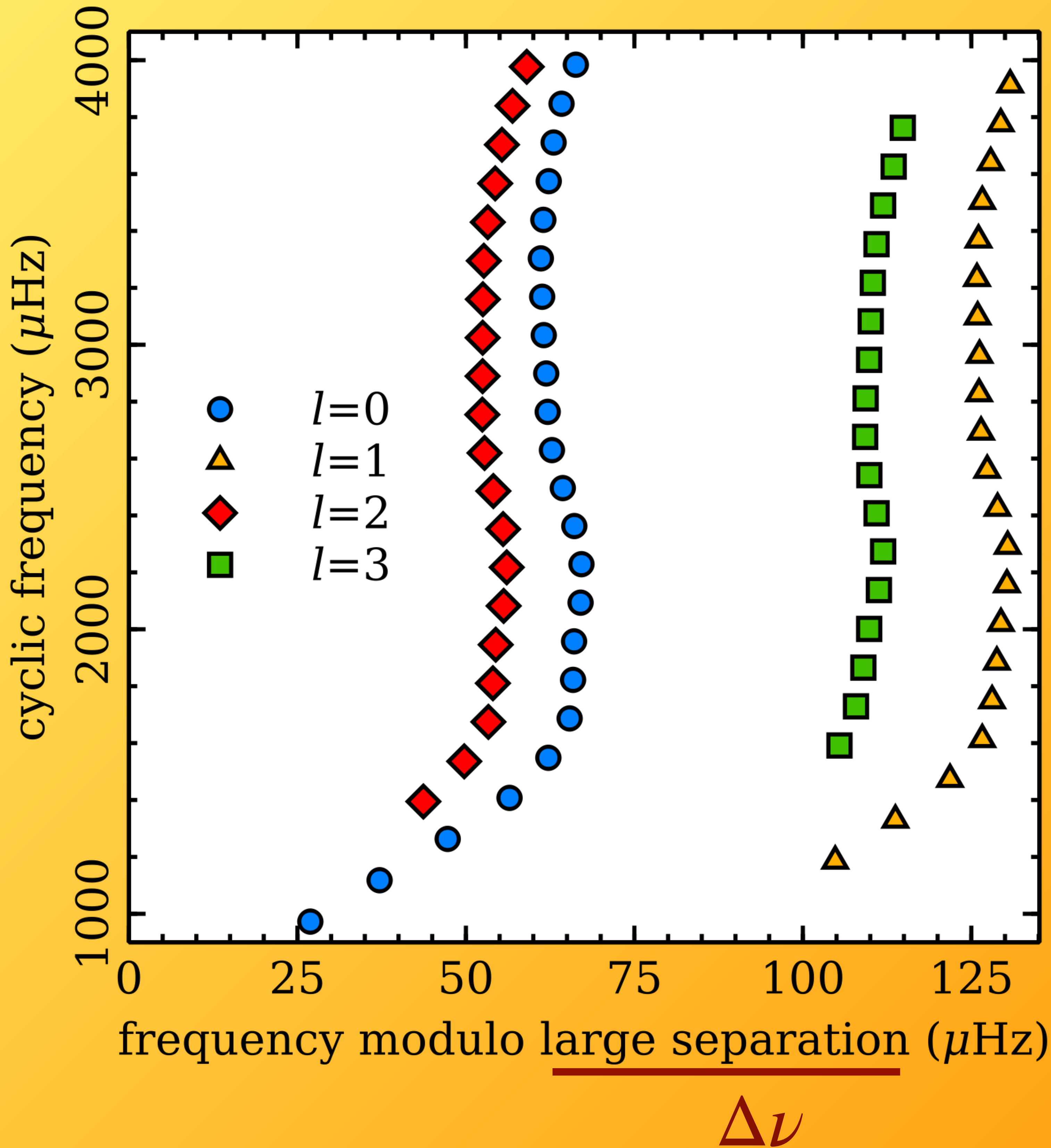
turning point ($k_r \to 0$)

this step involves math!

Solve for frequencies... $$\nu_{nl} \equiv \frac{\omega}{2\pi} \approx \left(n + \frac{\ell}{2} + \frac{1}{4} + \alpha\right) \left[2 \int \frac{\mathrm{d}r}{c}\right]^{-1}$$

...p-modes with $(n',\ell') = (n-1,\ell+2)$ will have (almost) the same frequencies

Echelle Diagram for the Sun



Odd-ℓ modes are offset by Δν/2 from even-ℓ modes

Warrick Ball (BiSON data)

Asymptotic Periods of Gravity (g) Modes

Asymptotic dispersion relation

$$\omega^2 = \frac{k_\mathrm{h}^2}{k^2} N^2 \quad \xrightarrow{k_r \gg k_h} \quad k_r \approx \frac{k_h N}{\omega}$$

Angular boundary condition

$$k_\mathrm{h}^2 = \frac{\ell(\ell+1)}{r^2}$$

Characteristic Equation

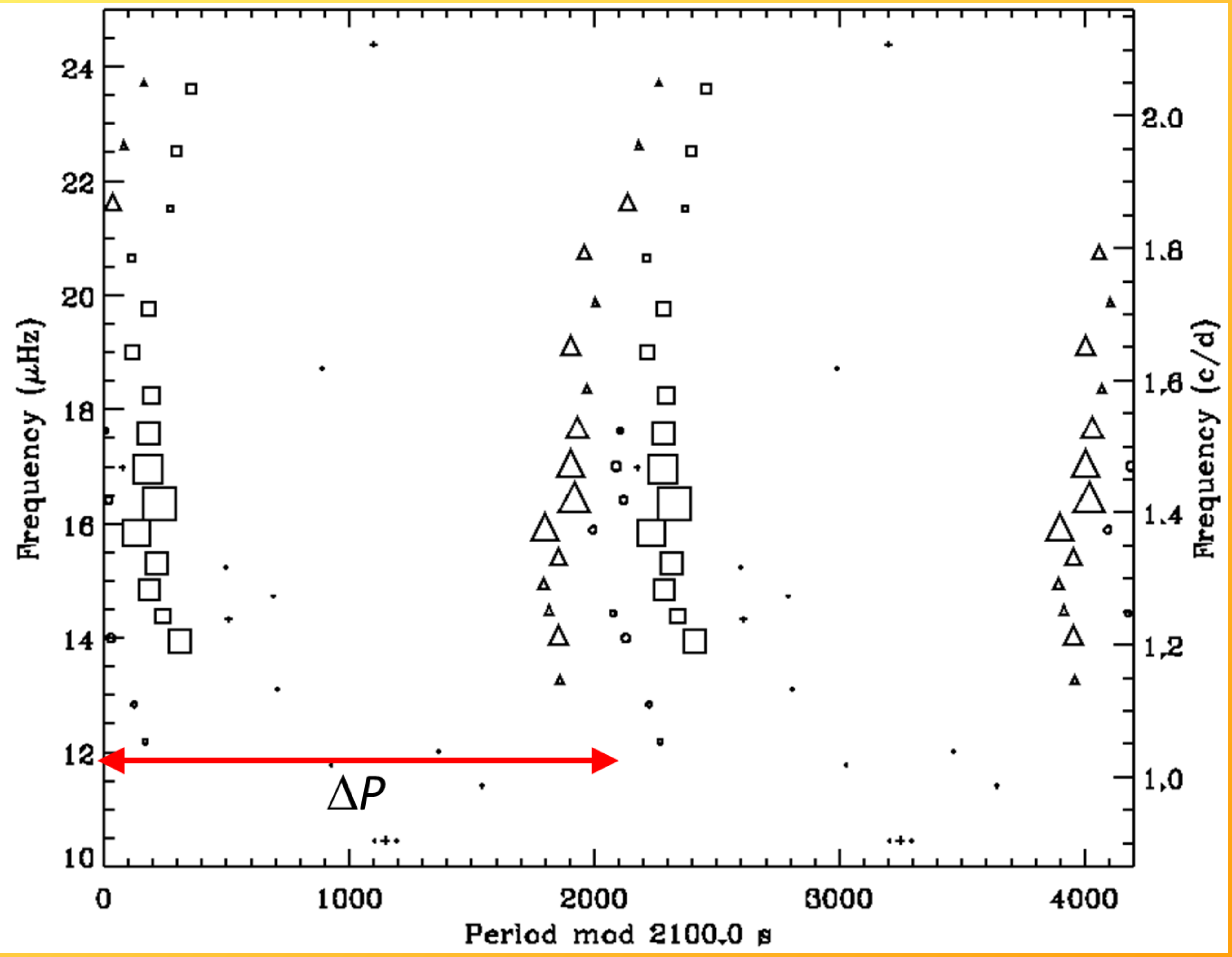
$$\frac{\sqrt{\ell(\ell+1)}}{\omega} \int_{r_1}^{r_2} \frac{N}{r}\,\mathrm{d}r \approx (n+\alpha)\pi$$

Solve for periods...

$$P_{n\ell} \equiv \frac{2\pi}{\omega} \approx (n+\alpha) \overbrace{\left[\frac{\sqrt{\ell(\ell+1)}}{2\pi^2} \int_{r_1}^{r_2} \frac{N}{r}\,\mathrm{d}r \right]^{-1}}^{\Delta P}$$

...g-modes with same ℓ and consecutive n will show a uniform period spacing ΔP

Period Echelle Diagram



Bedding et al. (2015);
KIC 11145123

Mixed Modes: Dispersion Relation

Gravito-acoustic dispersion relation

$$k_r^2 = \frac{1}{\omega^2 c^2} \left(\omega^2 - N^2\right) \left(\omega^2 - S_\ell^2\right)$$

$$S_\ell^2 = \frac{\ell(\ell+1)c^2}{r^2}$$

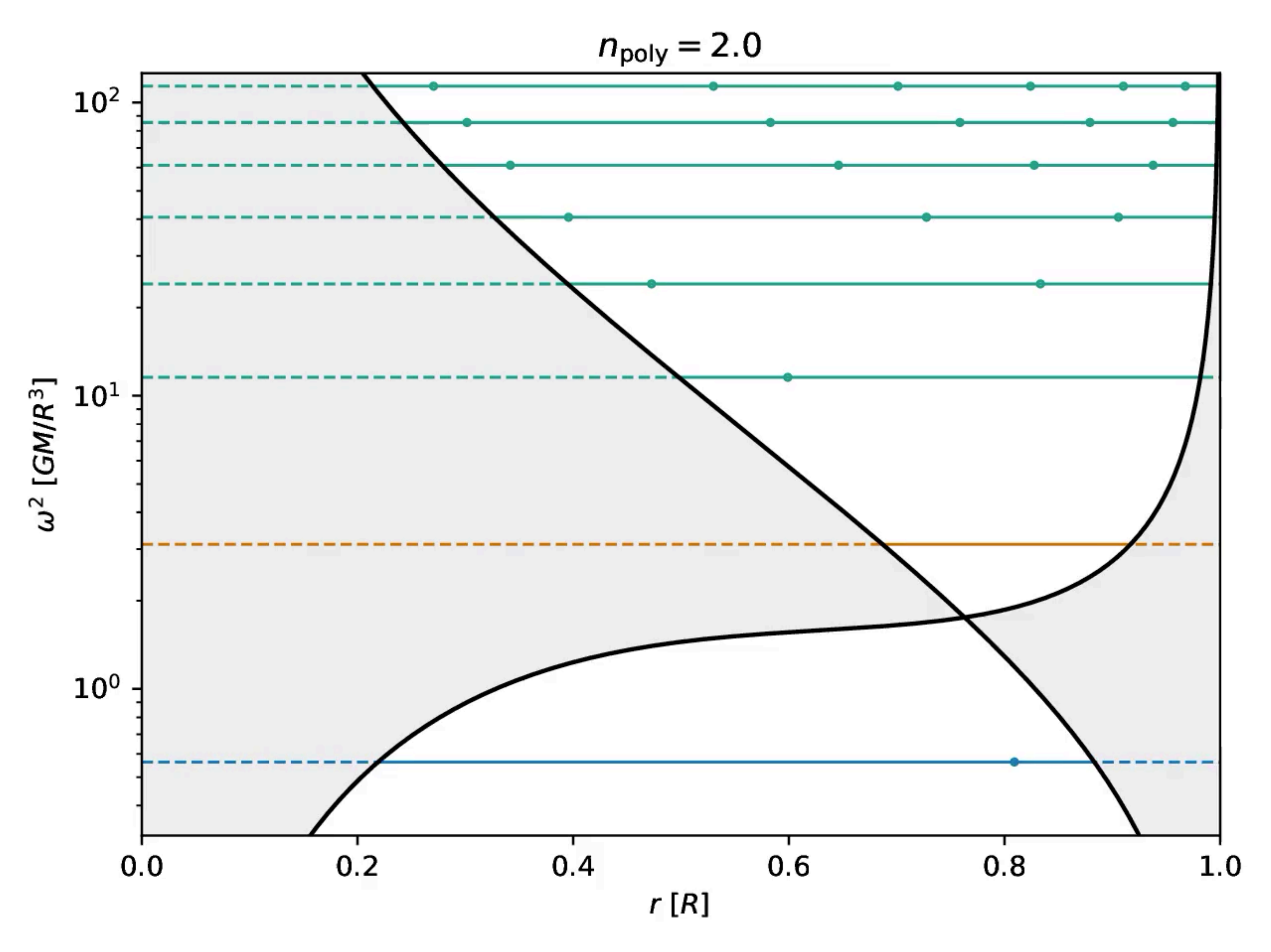
Lamb Frequency

$\omega^2 < N^2, S_\ell^2 \longrightarrow$ mode behaves locally as a gravity wave

$\omega^2 > N^2, S_\ell^2 \longrightarrow$ mode behaves locally as an acoustic wave

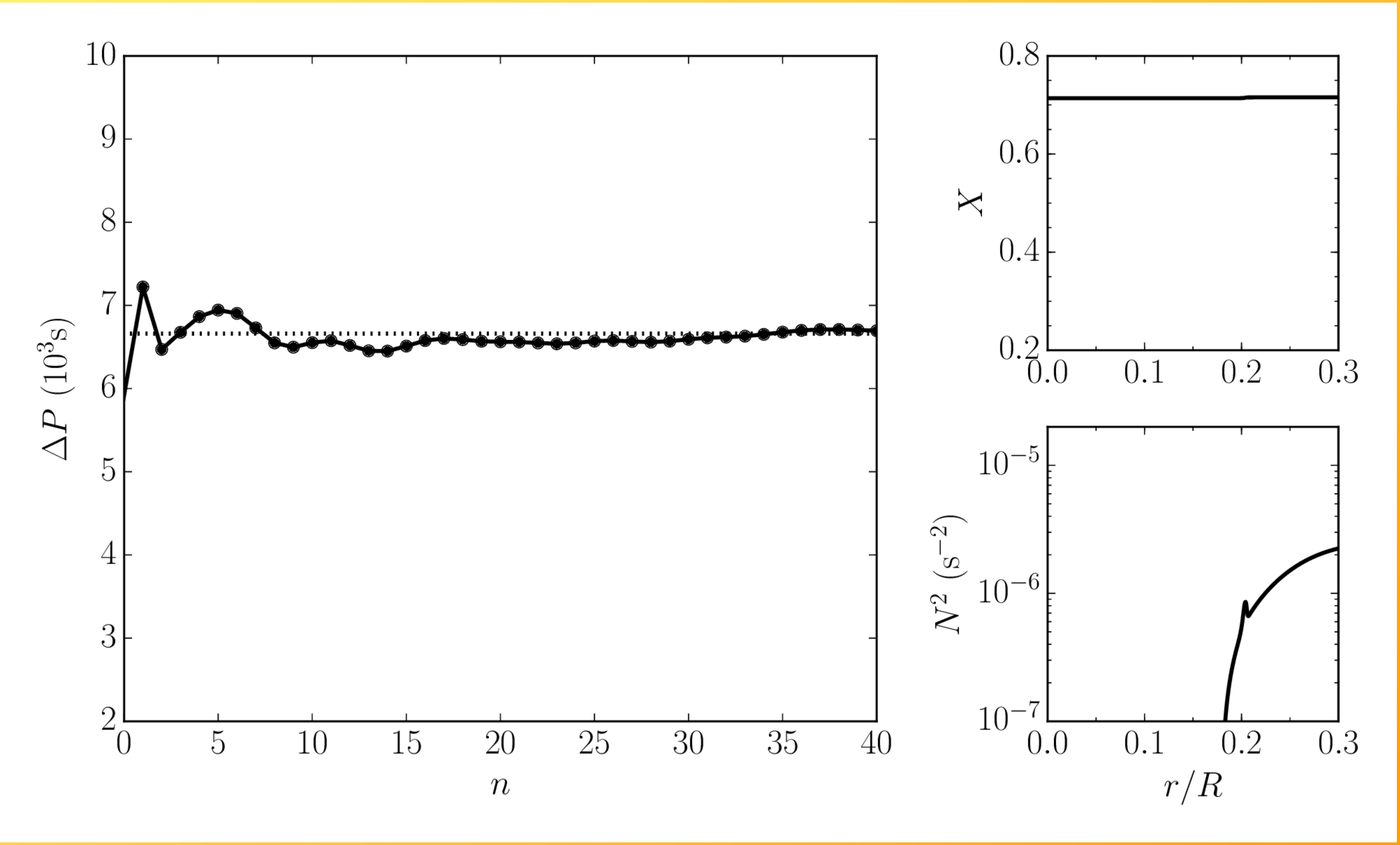
$$\left.\begin{array}{l} N^2 < \omega^2 < S_\ell^2 \\ S_\ell^2 < \omega^2 < N^2 \end{array}\right\} \longrightarrow \text{mode is locally evanescent}$$

Mixed Modes: Propagation Diagram



Calculations done with GYRE — what real stellar seismologists use!

Period Spacings in a 4.21M⊙ SPB Model



...departures from uniform ΔP allow us to probe the core boundary — "real" asteroseismology