# THE FOURTH VIRIAL COEFFICIENT OF THE UNITARY FERMI GAS

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## The system:

- 3D spin-1/2 Fermi gas at thermal equilibrium in the grand canonical ensemble in an isotropic harmonic potential $U(\mathrm{r}) = \frac{1}{2}m\omega^2 r^2$
- $s$-wave opposite-spin interactions in the unitary limit: (i) negligible interaction range $b$, (ii) infinite scattering length $a$
- interactions then replaced by contact conditions: $\forall i \in \uparrow$, $\forall j \in \downarrow$, $\exists A_{ij}(\mathrm{R}_{ij}, (\mathrm{r}_k)_{k \neq i,j})$ such that

$$\psi(\mathrm{r}_1, \ldots, \mathrm{r}_N) \underset{r_{ij} \to 0}{\overset{\mathrm{R}_{ij}\text{fixed}}{=}} \frac{A_{ij}}{r_{ij}} + O(r_{ij})$$

- These contact conditions are scale invariant (no Efimov effect)

## The problem:

- low density limit $\mu_\sigma \to -\infty$ at fixed $T$: low-fugacity expansion of the grand potential = cluster expansion

$$\Omega = -k_B T Z_1 \sum_{N_\uparrow, N_\downarrow} B_{N_\uparrow, N_\downarrow}(\omega) z_\uparrow^{N_\uparrow} z_\downarrow^{N_\downarrow}$$

- cumulants known up to order 3 numerically (Liu, Hu, Drummond, 2009) and analytically (Castin, Werner, 2013)
- getting 4th-order cumulants is an experimental and theoretical challenge. A case where four-body quantum correlations play an essential role.

## First step of the solution: reduce to a few-body problem

- Grand pot. in terms of canonical partition functions:

$$\Omega = -k_B T \ln \left( \sum_{N_\uparrow, N_\downarrow} Z_{N_\uparrow, N_\downarrow} z_\uparrow^{N_\uparrow} z_\downarrow^{N_\downarrow} \right)$$

- Tricks: (i) separate out the center-of-mass (only relative motion left), (ii) consider deviation $\Delta B$ from ideal gas value to eliminate interaction-insensitive states.
- $n^{\text{th}}$order cumulants from solution of $k$-body pbs, $k \leq n$:

$$\Delta B_{1,1} = \Delta Z_{1,1}^{\text{rel}}$$

$$\Delta B_{2,1} = \Delta Z_{2,1}^{\text{rel}} - Z_1 \Delta B_{1,1}$$

$$\Delta B_{3,1} = \Delta Z_{3,1}^{\text{rel}} - Z_1 Z_{2,0}^{\text{rel}} \Delta B_{1,1} - Z_1 \Delta B_{2,1}$$

$$\Delta B_{2,2} = \Delta Z_{2,2}^{\text{rel}} - Z_1^2 \Delta B_{1,1} - Z_1 \left[ \frac{1}{2} \Delta B_{1,1}^2 + \Delta B_{2,1} + \Delta B_{1,2} \right]$$

## Second step: reduce to free space $E = 0$

- Consider $E = 0$ scale invariant free space solutions with center-of-mass at rest, with $s > 0$ (no Efimov effect):

$$\psi_0(\mathrm{r}_1, \ldots, \mathrm{r}_N) = R^{s-(3N-5)/2}\phi(\Omega)$$

- In trap: still separability in hyperspherical coordinates

$$\epsilon_q^{\mathrm{rel}} = (s + 1 + 2q)\hbar\omega, \quad \forall q \in \mathbb{N}$$

## Third step: equation for the scale exponents $s$:

- Faddeev Ansatz depends linearly on $A_{ij}$. Must obey contact conditions $\Rightarrow$ linear equation on $A$: $M[A] = 0$
- Scale invar. $A = R_A^{s+1-(3N-5)/2}\phi_A(\Omega_A)$: $\mathcal{M}_s[\phi_A] = 0$
- Allowed $s$ = roots $u_n$ of Efimov's function:

$$\Lambda(s) \equiv \det \mathcal{M}_s = 0$$

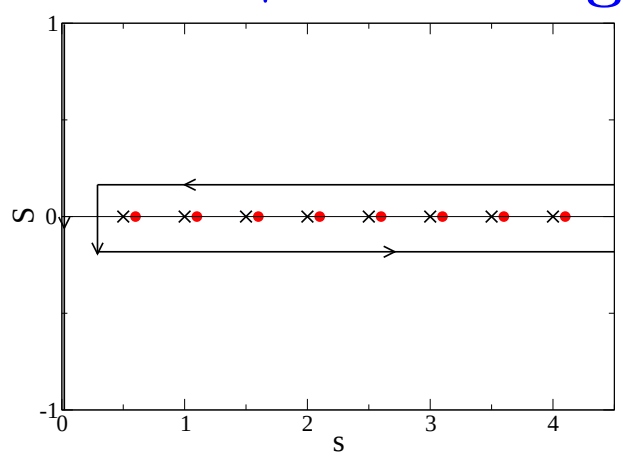
- Ideal-gas scale exponents are 1+poles $v_n$ of $\Lambda$

## Fourth step: summing over roots and poles by inverse residue formula

- The roots $u_n$ and poles $v_n$ of $\Lambda$ indeed give the unitary-gas and ideal-gas spectra.
- Can be collected as ($\bar{\omega} \equiv \beta\hbar\omega$)

$$S_{N_\uparrow,N_\downarrow} = \sum_{n,q} e^{-(u_n+2q+1)\bar{\omega}} - e^{-(v_n+2q+1)\bar{\omega}}$$

- A sum of residues of $z \mapsto \frac{\Lambda'(z)}{\Lambda(z)} e^{-z\bar{\omega}}$ (Castin, Werner, 2013). Cauchy theorem + unfolding on imaginary axis



- Defining in full generality

$$I_{N_\uparrow,N_\downarrow} \equiv \frac{1}{2\,\mathrm{sh}\,\bar{\omega}} \int_{\mathbb{R}} \frac{\mathrm{d}S}{2\pi} \sin(\bar{\omega}S) \frac{\mathrm{d}}{\mathrm{d}S} \left[\ln \Lambda_{N_\uparrow,N_\downarrow}(\mathrm{i}S)\right]$$

one thus expects

$$S_{N_\uparrow,N_\downarrow} \overset{\text{Cauchy}}{=} I_{N_\uparrow,N_\downarrow}$$

- True for third-order cumulants:

$$\Delta B_{2,1} \overset{\text{algebra}}{=} S_{2,1} \overset{\text{Cauchy}}{=} I_{2,1}$$

- Wrong for fourth-order cumulants ($I_{3,1}$ has a finite limit when $\omega \to 0^+$):

$$\Delta B_{3,1} \overset{\text{algebra}}{=} S_{3,1} - Z_1 \Delta B_{2,1} \neq I_{3,1} - Z_1 \Delta B_{2,1}$$

- Failure of Cauchy may be due to (i) branch cuts, (ii) nonzero contribution of big quarter-circles, (iii) extraneous poles.

## Conjecture for 4th-order cumulants from 3rd-order ones:

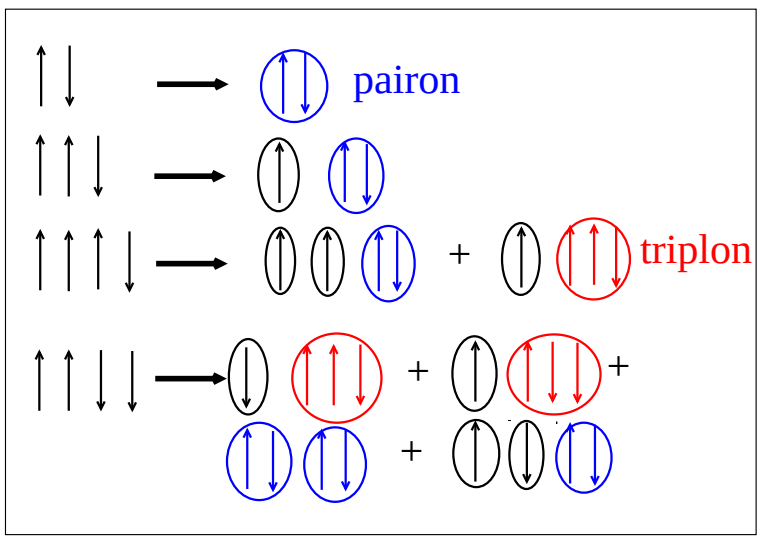
$$I_{2,1} = \Delta Z^{\rm rel}_{2,1} - \boxed{Z_1 \Delta B_{1,1}} \longrightarrow \text{partition function of DAO}$$

$$I_{3,1} \overset{\rm conj}{=} \Delta Z^{\rm rel}_{3,1} - Z_{2,0}\Delta B_{1,1} - Z_1 \Delta B_{2,1}$$

$$I_{2,2} \overset{\rm conj}{=} \Delta Z^{\rm rel}_{2,2} - Z_1 \Delta B_{2,1} - Z_1 \Delta B_{1,2}$$

$$-\Delta Z^{\rm rel}_{2\,\rm pairons} - Z_1 (Z_1 - Z^{{\rm rel},a=0}_{1\,\rm pairon}) \Delta B_{1,1}$$

Decoupled Asymptotic Objects
(at large quantum numbers)




**The triplon partition function involves subtraction of the one atom+one pairon partition function to avoid double counting.**

**Final result:**

$$\Delta B_{3,1} \overset{\text{conj}}{=} I_{3,1}$$
$$\Delta B_{2,2} \overset{\text{conj}}{=} I_{2,2} + \frac{1}{32\,\mathrm{ch}^3\frac{\bar\omega}{2}\,\mathrm{ch}\,\bar\omega}$$

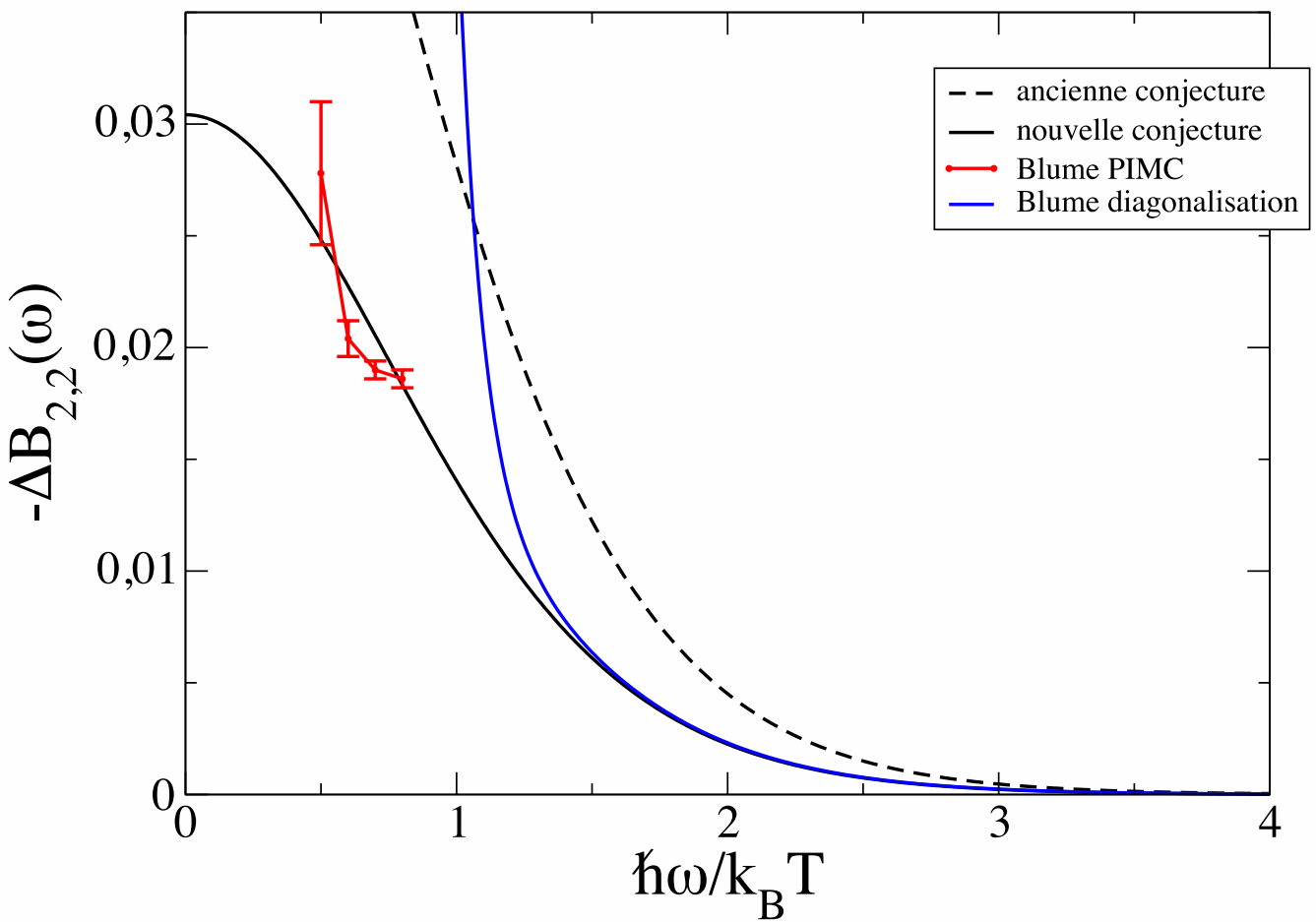
**Interpretation of the corrective term: pairons are bosons and are (statistically) correlated even if they do not interact (are decoupled)**

$$\frac{1}{32\,\mathrm{ch}^3\frac{\bar\omega}{2}\,\mathrm{ch}\,\bar\omega} = Z_1^{-1}\left[\Delta Z_{2\ \text{pairons}} - \frac{1}{2}\Delta(Z_{1\ \text{pairon}}^2)\right]$$

**Comparison to other results in the literature:**

- **Good agreement of $\Delta B_{3,1}(0^+)$ and $\Delta B_{2,2}(0^+)$ with an approximate result of Parish and Levinsen, 2015, based on incomplete summation of Feynman diagrams.**

# Comparison to Blume's Path Integral Monte Carlo



# Vs experiments and Werner's diagrammatic Monte Carlo