FBS16, KITP Santa Barbara

Three-body collisions at low energies

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funded by the NSF & Sloan

Overview

- Lesson from two-body collisions
- Wave function of 3 particles colliding
- Applications of D
- Numerical results of D
- Extensions & Open questions

Main References

- Braaten and Nieto, Eur. Phys. J. B **11**, 143 (1999)
(*Effective field theory for low-energy bosons*)
- ST, Phys. Rev. A **78**, 013636 (2008)
(*defined three-body scattering hypervolume D, related it to few-body and many-body physics, and calculated D for hard-sphere bosons, etc)*
- Shangguo Zhu and ST
(calculated D for soft-sphere bosons and for weak potentials)

Lesson from two-body collisions

Consider 2 particles colliding at low energy, $E \approx 0$

Strategy: study the collision at ZERO ENERGY first,
(and then make small corrections if necessary)

2 particles colliding in s-wave:

Outside of the range of interaction we have

$$\phi(s) = 1 - \frac{a}{s}$$

***a*: two-body scattering length**

3 particles colliding at $E \approx 0$

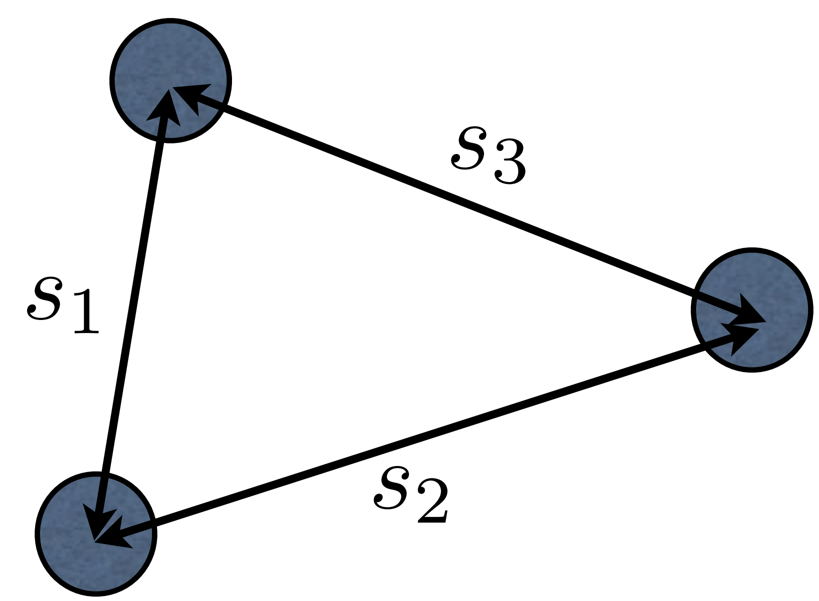
Consider 3 identical bosons with short-range interactions for simplicity.
At low energy, interaction usually dominated by 0 orbital angular momentum

To begin with, consider zero energy collision:

$$\left[-\nabla_1^2-\nabla_2^2-\nabla_3^2+V(s_1)+V(s_2)+V(s_3)+V_3(s_1,s_2,s_3)\right]\phi=0$$

can't solve it analytically

so we'll do expansions



Asymptotic expansions of the three-body wave function ϕ

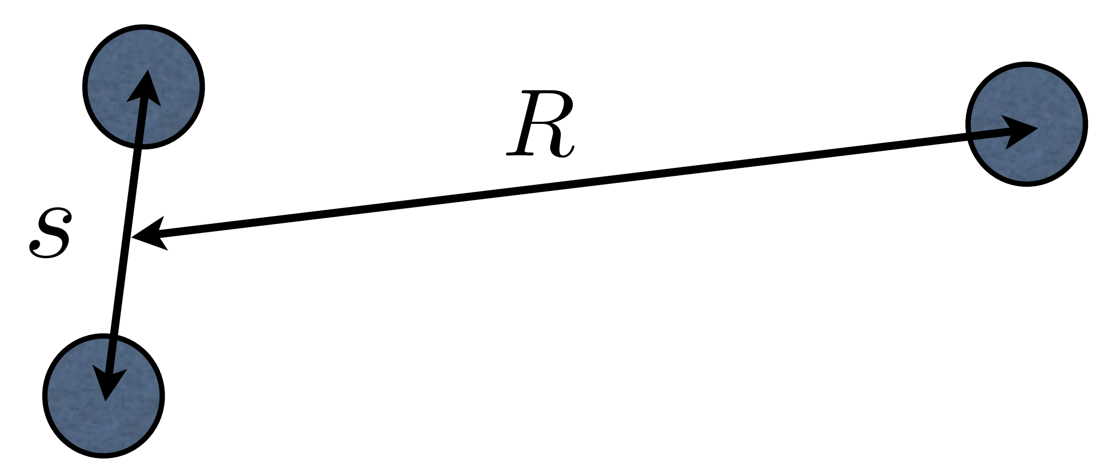
"111"-expansion:

when s1,s2,s3 go to infinity *simultaneously*, expand ϕ in powers of $1/\rho$, where

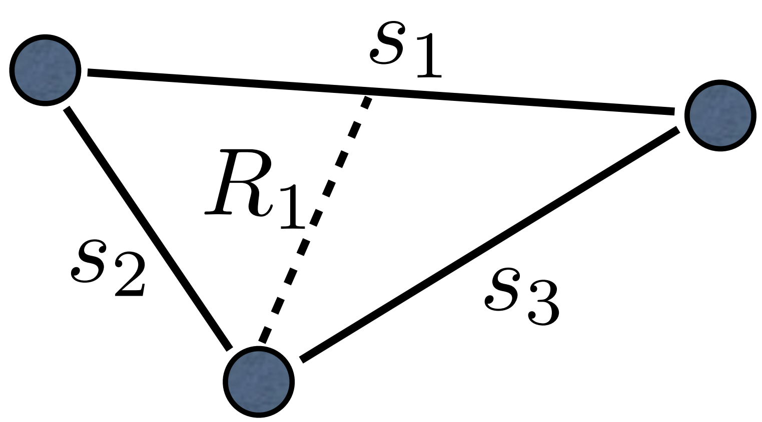
$$\rho \equiv \sqrt{\frac{s_1^2 + s_2^2 + s_3^2}{2}}$$

"21"-expansion:

when two bosons are maintained at a fixed distance *s*, but the third boson is far away, expand ϕ in powers of $1/R$.



The "111"-expansion



When *s1*, *s2*, *s3* go to infinity simultaneously:

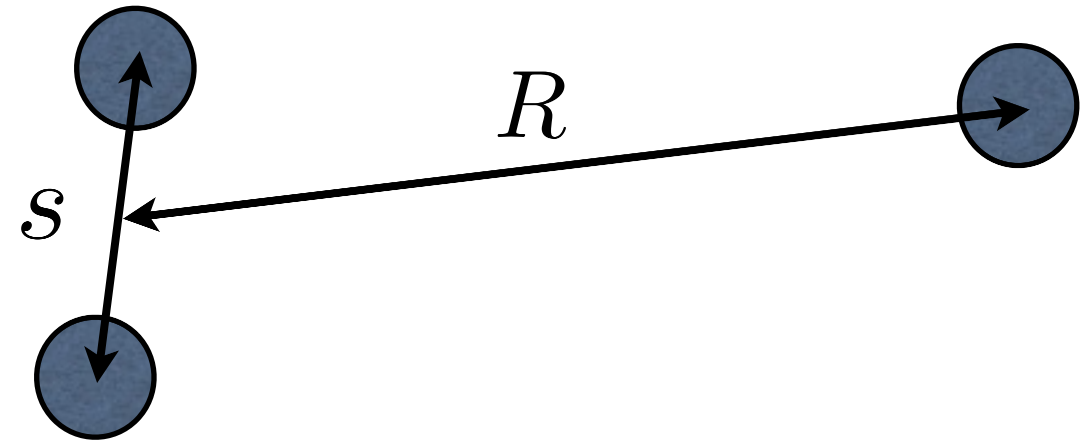
$$\phi = 1 + \left(\sum_{i=1}^{3} -\frac{a}{s_i} + \frac{4a^2\theta_i}{\pi R_i s_i} - \frac{2wa^3}{\pi \rho^2 s_i} + \frac{8\sqrt{3}\, wa^4 (\ln \frac{e^\gamma \rho}{|a|} - 1 - \theta_i \cot 2\theta_i)}{\pi^2 \rho^4} \right) - \frac{\sqrt{3}\, D}{8\pi^3 \rho^4} + \cdots + O(\rho^{-8})$$

$$\rho \equiv \sqrt{\frac{s_1^2 + s_2^2 + s_3^2}{2}}, \quad \theta_i \equiv \arctan\left(\frac{2R_i}{\sqrt{3}\, s_i}\right) \quad w \equiv \frac{4\pi}{3} - \sqrt{3} = 2.4567\cdots$$

(hyperradius)

***D*: three-body scattering hypervolume**
(dimension: length raised to the 4th power)

The “21”-expansion



When *s* is fixed, but *R* goes to infinity:

$$\phi = \left(1 - \frac{2a}{R} + \frac{2wa^2}{\pi R^2} - \frac{4wa^3}{\pi R^3} + \frac{24\sqrt{3}\,wa^4(\ln\frac{e^\gamma R}{|a|} - \frac{3}{2}) - \xi_1}{\pi^2 R^4}\right)\phi(s) + \frac{3wa^2}{\pi R^4} f(s)$$

$$+\left(-\frac{15a}{2R^3} + \frac{40(2\pi - 3\sqrt{3})a^2}{\pi R^4}\right)\phi^{(d)}_{\hat{\mathbf{R}}}(\mathbf{s}) + \cdots + O(R^{-8})$$

$$\xi_1 \equiv \frac{\sqrt{3}}{8\pi} D - 8\left(\sqrt{3} - \frac{\pi}{3}\right) wa^4 - \frac{3\pi w}{2} a^3 r_s$$

r_s: two-body effective range

$$\left[-2\nabla^2 + V(s)\right] f(s) = \phi(s), \ \text{all } s;$$

$$f(s) = -\frac{s^2}{6} + \frac{as}{2} - \frac{ar_s}{2}, \quad s > r_0.$$

Connection to the Effective-Field Theory

Lagrangian density of EFT:

$$\mathcal{L} = \cdots - \frac{1}{36} g_3 \psi^* \psi^* \psi^* \psi\psi\psi + \cdots$$

$g_3 \equiv g_3(\kappa)$: running coupling constant

Braaten and Nieto, Eur. Phys. J. B ***11****, 143 (1999)*

I found $\quad g_3(|a|^{-1}) = 6(D + 12\pi^2 a^3 r_s + 977.736695 a^4)$

Application of *D*: 3-body scattering amplitude (first investigated by Braaten and Nieto in 1999)

incoming momenta: $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ outgoing momenta: $\mathbf{q}_1', \mathbf{q}_2', \mathbf{q}_3'$

T-matrix element: $\mathrm{T}(\mathbf{q}_1'\mathbf{q}_2'\mathbf{q}_3';\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3) = \sum_{s=-2}^{\infty} \mathrm{T}^{(s)}(\mathbf{q}_1'\mathbf{q}_2'\mathbf{q}_3';\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3)$

$$\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 = \mathbf{q}_1' + \mathbf{q}_2' + \mathbf{q}_3' = 0; \quad E = \frac{q_1^2 + q_2^2 + q_3^2}{2m} = \frac{{q_1'}^2 + {q_2'}^2 + {q_3'}^2}{2m}$$

$$\mathrm{T}^{(-2)} = 64\pi^2 a^2 \sum_{i,j=1}^{3} G^E_{\mathbf{q}_j'\mathbf{q}_i} \qquad \mathrm{T}^{(-1)} = \sum_{i,j=1}^{3} \left[-512\pi^3 a^3 c_1^E(\mathbf{q}_j', \mathbf{q}_i) - \mathrm{i}\, 64\pi^2 a^3 (p_j' + p_i) G^E_{\mathbf{q}_j'\mathbf{q}_i} \right]$$

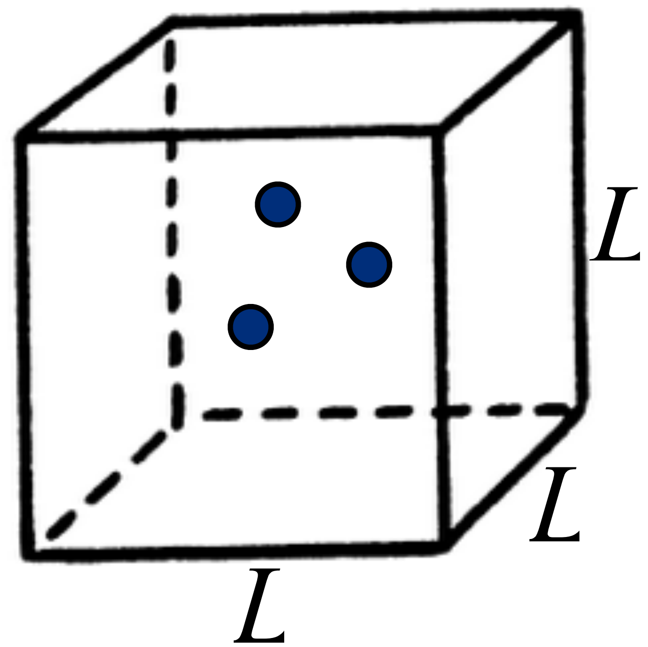
$$\mathrm{T}^{(0)} = -6\left[D - 24\pi w\left(7\pi/\sqrt{3} - 8\right)a^4 - 36\pi^2 a^3 r_s\right]$$
$$+64\pi^2 a^3 \sum_{i,j=1}^{3} \left[32\pi^2 a c_2^E(\mathbf{q}_j', \mathbf{q}_i, |a|^{-1}) + (r_s/2 - a)({p'}_j^2 + p_i^2) G^E_{\mathbf{q}_j'\mathbf{q}_i} - a p_j' p_i G^E_{\mathbf{q}_j'\mathbf{q}_i} + \mathbf{i}\, 8\pi a (p_j' + p_i) c_1^E(\mathbf{q}_j', \mathbf{q}_i)\right]$$

where $G^E_{\mathbf{q}'\mathbf{q}} \equiv ({q'}^2 + \mathbf{q}' \cdot \mathbf{q} + q^2 - E/m - \mathrm{i}\epsilon)^{-1}$

$$c_1^E(\mathbf{q}', \mathbf{q}) \equiv \int \frac{d^3k}{(2\pi)^3} G^E_{\mathbf{q}'\mathbf{k}} G^E_{\mathbf{k}\mathbf{q}}$$

$$c_2^E(\mathbf{q}', \mathbf{q}, \kappa) \equiv \lim_{K\to\infty} -\frac{w}{16\pi^3} \ln\frac{K}{\kappa} + \int_{k<K} \frac{d^3k}{(2\pi)^3} G^E_{\mathbf{q}'\mathbf{k}} \left(2c_1^E(\mathbf{k}, \mathbf{q}) - \frac{\sqrt{3k^2/4 - E/m - \mathrm{i}\epsilon}}{4\pi} G^E_{\mathbf{k}\mathbf{q}}\right)$$

Application of D: ground state energy of 3 bosons in a large volume



$$E = \frac{12\pi a}{L^3}\left[1 + 2.83729748\frac{a}{L} + 9.72533081\frac{a^2}{L^2} + \left(-39.30783036\ln\frac{L}{|a|} + 95.85272360\right)\frac{a^3}{L^3} + \frac{3\pi a^2 r_s}{L^3} + \left(-669.16804795\ln\frac{L}{|a|} + 810.05328680\right)\frac{a^4}{L^4} + 53.48179751\frac{a^3 r_s}{L^4}\right] + \frac{D}{L^6} + 17.02378488\frac{aD}{L^7} + O(L^{-8})$$

$O(L^{-6})$: Beane, Detmold, Savage, 2007; ST 2007

Application of D: ground state energy of dilute Bose-Einstein condensates

$$\frac{E}{N} = \frac{4\pi n a}{2}\left[1 + \frac{128}{15\sqrt{\pi}}(na^3)^{1/2} + 8wna^3\ln(na^3) + na^3\mathcal{E}_3'\right] + o(n^2)$$

4th term: known to depend on 3-body parameter and many-body physics by T. T. Wu in 1959.
Many-body contributions calculated by Braaten and Nieto 1999.

I did independent calculations and found

$$\mathcal{E}_3' = \frac{D}{12\pi a^4} + \frac{\pi r_s}{a} + 118.498\,920\,346\,444$$

Application of D:
ground state energy of dilute BEC with $a = 0$

$$\frac{E}{N} = \frac{Dn^2}{6} + o(n^2)$$

Gross-Pitaevskii equation becomes:

$$i\frac{\partial}{\partial t}\Psi = -\nabla^2\Psi + \frac{D}{2}|\Psi|^4\Psi + V_{\rm trap}(\mathbf{r})\Psi$$

Applications of *D*: many more things to be learned

Equation of state of Bose gases at finite temperatures;
Critical temperature;
Dynamical properties;
Few-body systems in LOOSE traps (cf somewhat related works by Blume, Johnson, Tiesinga and others);
Interferometric properties of cold atoms;
......

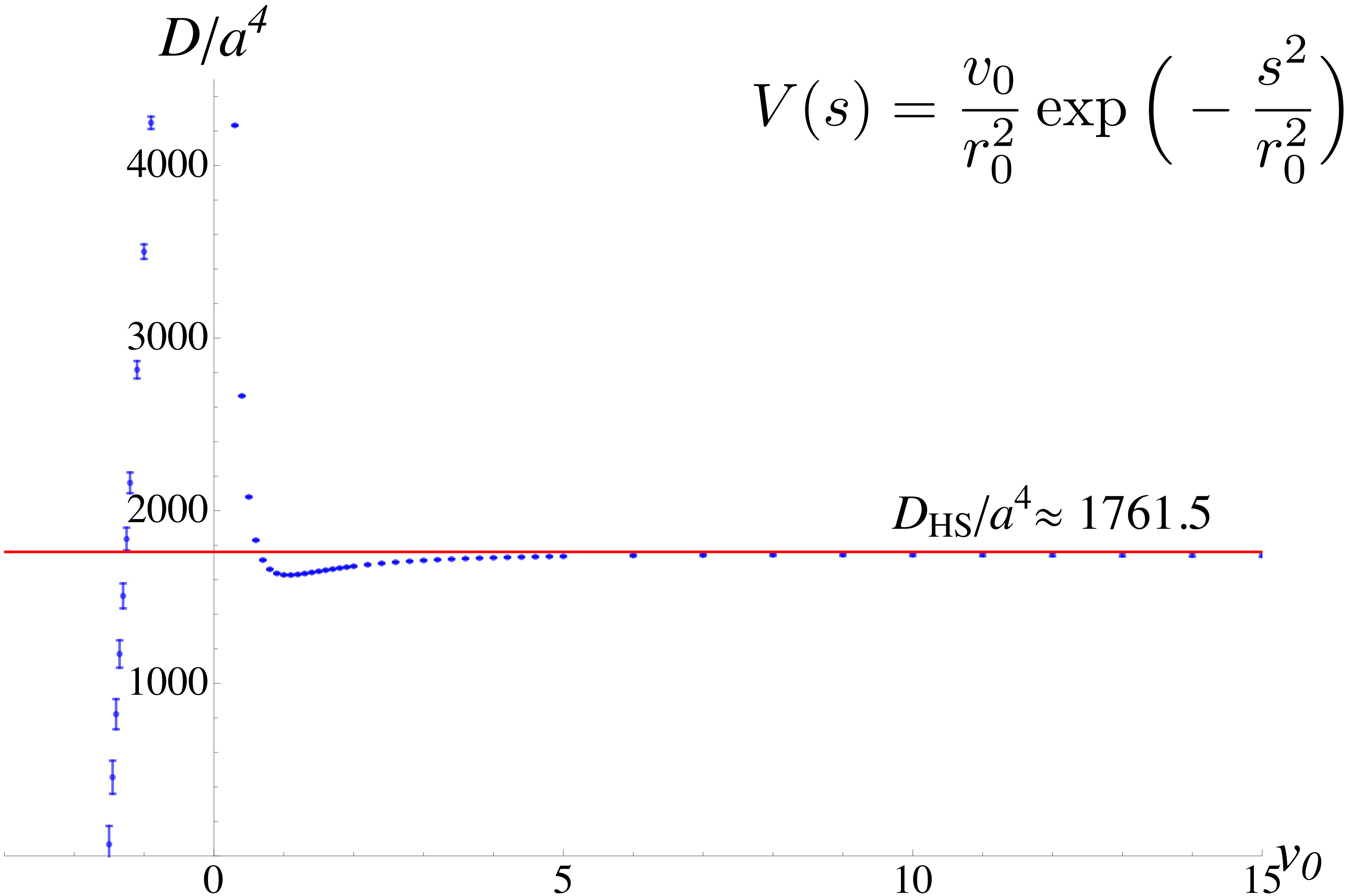
D for three HARD spheres

pairwise potential: $V(s) = \begin{cases} 0, & s > a \\ +\infty, & s < a \end{cases}$

$$\boxed{D = (1761.5430 \pm 0.0024)\, a^4 \quad \text{for hard-sphere bosons}}$$

ST, 2008

D for three SOFT spheres



Shangguo Zhu and ST, 2015

Extensions & Open questions

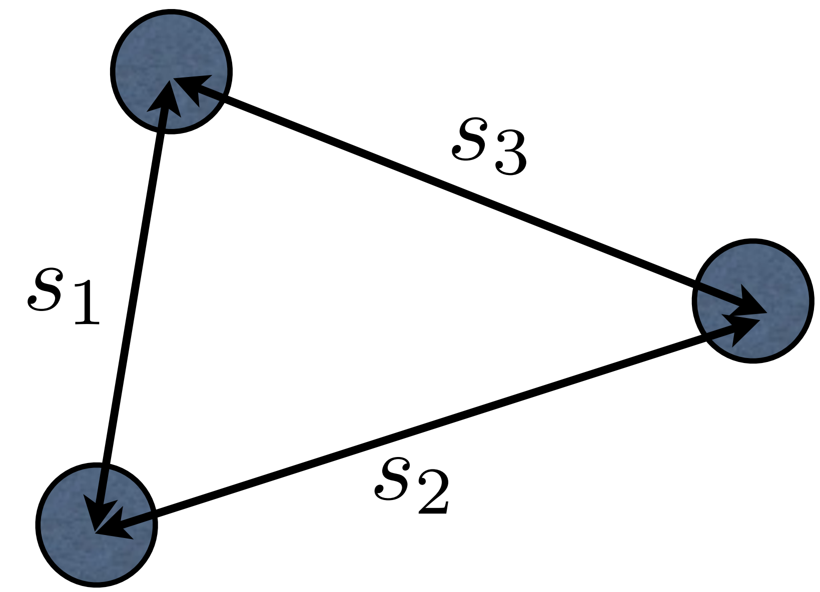
When there are two-body bound states, D is complex

$\mathrm{Im}D$: determines the three-body recombination rate

cf Braaten, Hammer, Mehen, PRL 88, 040401 (2002)

Extensions & Open questions

At large hyperradii, $\rho \gg r_0, |a|$, can we calculate ϕ to high precision for ALL shapes of the triangle?



If we can, the numerical solution of the 3-body Schrödinger eq will be greatly simplified.

$$\phi = 1 + \left(\sum_{i=1}^{3} -\frac{a}{s_i} + \frac{4a^2\theta_i}{\pi R_i s_i} - \frac{2wa^3}{\pi \rho^2 s_i} + \frac{8\sqrt{3}\, wa^4 (\ln \frac{e^\gamma \rho}{|a|} - 1 - \theta_i \cot 2\theta_i)}{\pi^2 \rho^4} \right) - \frac{\sqrt{3}\, D}{8\pi^3 \rho^4} + \cdots + O(\rho^{-8})$$

$$\phi = \left(1 - \frac{2a}{R} + \frac{2wa^2}{\pi R^2} - \frac{4wa^3}{\pi R^3} + \frac{24\sqrt{3}\, wa^4 (\ln \frac{e^\gamma R}{|a|} - \frac{3}{2}) - \xi_1}{\pi^2 R^4} \right) \phi(s) + \frac{3wa^2}{\pi R^4} f(s)$$
$$+ \left(-\frac{15a}{2R^3} + \frac{40(2\pi - 3\sqrt{3})a^2}{\pi R^4} \right) \phi^{(d)}_{\hat{\mathbf{R}}}(\mathbf{s}) + \cdots + O(R^{-8})$$

Extensions & Open questions

If the bosons interact with pairwise Van der Waals potential

$$V(s) = -\frac{r_{vdw}^4}{ms^6}$$

which supports infinitely many two-body bound states, do we have a universal function of the following form?

$$D = D(r_{vdw}, a)$$

cf universality of a_- in the Efimov physics.

Do we need to introduce 3-body phase(s) at $\rho \ll r_{vdw}$?

Three-body wave function highly oscillatory at small distances, so Schrödinger eq hard to solve numerically

Extensions & Open questions

Heteronuclear systems
(eg, two heavy and one light atoms)

Three-body collisions in lower spatial dimensions