Confining sheets, shells, and membranes

0. Elastic energy & Confinement of solid sheets and shells
   Question #0: response to compression in the absence of 2D shear modulus.

I. Isometry, the elastica & Geometrically Incompatible confinement.
   Question #1: response of unsupported solid sheets to biaxial/isotropic confinement

II. Substrate-induced & Tension-induced confinement.
    Question #2: nearly-isometric non-wrinkly deformations

III. Boundary-induced confinement.
   Question #3: why Pogorelov’s ridge is unstable?
The elastic energy of solid “surfaces”
Elastic energy of solid sheets

thickness: $t$

- stretching modulus: $Y \sim E \cdot t$
- bending modulus: $B \sim E \cdot t^3$

For solids:
- two types of strain
  - area change
    - (also in fluid membranes)
  - no area change
    - (shear)

- (compress)
- (stretch)

$U_{\text{strain}} = Y \cdot (\text{strain})^2$

$U_{\text{bend}} = B \cdot (\text{curvature})^2$

$\frac{\lambda}{2} (Tr(\varepsilon^2))^2 + \mu Tr(\varepsilon^2)$
Elastic energy of solid sheets

\[ t \]

- **stretching modulus:** \( Y \sim E t \)
- **bending modulus:** \( B \sim E t^3 \)

**FvK Formalism**
- displacement \((u_i, u_j, \zeta) \rightarrow \text{strain, curvature}\)
- small slope (Monge) approx.
- Hookean: energy \( \sim (\text{strain})^2 \)
- geometrically nonlinear

\[
U_{\text{strain}} = Y \cdot (\text{strain})^2 \quad \text{and} \quad U_{\text{bend}} = B \cdot (\text{curvature})^2
\]
Elastic energy of **solid sheets**

- **stretches modulus:** \(Y \sim E \, t\)
- **bending modulus:** \(B \sim E \, t^3\)

**FvK Formalism**

- displacement \((u_i, u_j, \zeta) \rightarrow \text{strain, curvature}\)
- small slope (Monge) approx.
- Hookean: energy \(\sim (\text{strain})^2\)
- geometrically nonlinear

\[
\varepsilon_{ij} \approx \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_j} \right)
\]

\[
\kappa_{ij} \approx \frac{\partial^2 \zeta}{\partial x_i \partial x_j}
\]
Some generalizations of FvK formalism
(not discussed in this talk)

\[
\begin{align*}
\text{thickness: } t \quad & \begin{cases}
\text{stretching modulus: } Y \sim E t \\
\text{bending modulus: } B \sim E t^3
\end{cases}
\end{align*}
\]

**FvK Formalism**
- displacement \((u_i, u_j, \zeta) \rightarrow \text{strain, curvature}\)
- small slope (Monge) approx.
- Hookean: energy \(\sim (\text{strain})^2\)
- geometrically nonlinear

**Shallow shell theory**
(Donnel-Mushtari-Vlasov / Koiter-Hutchinson)

\textit{displacement expressed w.r.t (parabolic) shell}

**Incompatible (non-Euclidean) elasticity**

\textit{deviation from “target” metric & curvature} \(\rightarrow U_{\text{strain}}, U_{\text{bend}}\)
In this presentation

**NO list**

**NO** dissipation

**NO** plasticity

**NO** dynamics
\[ t \text{ is for “thickness”} \]

**NO** thermodynamics
\[ T \text{ is for “tension”} \]

**NO** material-dependent response

**YES list**

Energy minimization

Hookean response

Geometric nonlinearity
Confinement of solid sheets & shells
Uniaxial confinement: buckling

thickness: $t$

- stretching modulus: $Y \sim E t$
- bending modulus: $B \sim E t^3$

This is a continuous (pitchfork) bifurcation

$\zeta \sim \sqrt{\delta - \delta_c}$

deflection ($\zeta$)

confinement ($\delta$)
buckling instability

\[ U_{\text{strain}} \sim Y \cdot \delta^2 \]

\[ U_{\text{bend}} \sim B \cdot \delta/W \]

\[ \delta_{c}/W \sim \sqrt{\sigma/Y} \sim t^2 \]
Uniaxial confinement of supported sheet: wrinkling

\[ U_{\text{strain}} \sim Y \cdot (\text{strain})^2 \]
\[ U_{\text{bend}} \sim B \cdot (\text{curvature})^2 \]
\[ U_{\text{subst}} \sim K \cdot (\text{deflection})^2 \]

hard “skin” on compliant substrate
Bowden et al. (1998) Yu et al. (2015)

PS sheet on liquid bath
Huang et al. (2010), Pocivavsek et al. (2008)

effective “stiffness”

edge-clamped ribbon under uniaxial tension \( T \)
Cerda & Mahadevan (2003)

\[ K \sim \frac{E_{\text{subst}}}{\lambda} \]
\[ K \sim \rho_{\text{liq}} g \]
\[ K \sim \frac{T}{L^2} \]
wrinkling instability

\[ \sigma_{\text{plane}} \sim Y \cdot \delta \rightarrow U_{\text{strain}} \sim \sigma_{\text{plane}} \cdot \delta \]

\[ \sigma \sim B/\lambda^2 \rightarrow U_{\text{bend}} \approx U_{\text{subst}} \sim \sigma \cdot \delta \]

\[ \lambda \sim (B/K)^{1/4} \sim t^{3/4} \]
naïve question #0
uniaxial buckling/wrinkling of a liquid phase?

solid sheet (finite shear mod.)
planar stress may be anisotropic

\[ \sigma_{yy} = \sigma_0 < 0 \ ; \ \sigma_{xx} = \gamma > 0 \]

periodic (uniaxial) wrinkles energetically favorable

liquid film (shear mod. = 0)
planar stress is isotropic

\[ \sigma_{ij} = \sigma_0 \cdot \delta_{ij} \]

What is the energetic landscape of surface deformations when \( \sigma_0 < 0 \)?

- are periodic patterns favorable? “protected”?

Buckling of Langmuir monolayers
Milner-Joanny-Pincus (1989)

Huang et al. (2010), Pocivavsek et al. (2008)
Isometry and the *elastica*
Euler’s approach to buckling/wrinkling

thickness: $t$

- stretching modulus: $Y \sim E \ t$
- bending modulus: $B \sim E \ t^3$

no “equipartition”!

owing to Euler, it is common/useful to consider the sheet as “inextensible” (i.e. accommodates no strain)

minimize $U_{\text{bend}} + U_{\text{subst}}$
subject to: $U_{\text{strain}} = 0$
buckling/wrinkling & the elastica

\[ U_{str} \sim Y \cdot \delta^2 \]

\[ U_{bend} \sim B \cdot \delta/W \]

\[ \text{strain} = \frac{\partial u}{\partial x} = -\frac{\delta}{W} \]

\[ \text{strain} \approx \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial \zeta}{\partial x} \right)^2 \approx 0 \]

geometrical nonlinearity!

near threshold (weakly nonlinear)
\[ \left| \frac{\partial \zeta}{\partial x} \right|^2 \ll \left| \frac{\partial u}{\partial x} \right| \]

far from threshold (strongly nonlinear)
\[ \left| \frac{\partial \zeta}{\partial x} \right|^2 \sim \left| \frac{\partial u}{\partial x} \right| \]

\[ \delta_c \sim B/YW \]
From instability to “isometry”

\[ U_{\text{strain}} \sim Y \cdot \delta^2 \]

\[ U_{\text{bend}} \sim B \cdot \delta/W \]

far from threshold

\[ \frac{U_{\text{strain}}}{U_{\text{bend}}} \sim \frac{\delta_c}{\delta} \ll 1 \]

planar

\[ U_{\text{bend}} = 0 \]

buckled/wrinkled (no strain)

\[ \delta_c \sim B/YW \]

planar (strain)

\[ W \]

\[ \delta \]
**question #1a: “equi-biaxial” buckling**

Consider:
- a disk-like sheet: radius $W$, thickness $t$
- Its edge ($r=W$) is displaced inward, $u_r(W) = -\delta$, and clamped, $\zeta(W) = 0$.
- Assume $t^2/W \ll \delta \ll W$ (small slope, but **far from buckling threshold**).

- **Is** $U_{\text{bend}} \gg U_{\text{strain}}$ ?
- How does the confining force $F = \partial U/\partial \delta$ scales with $\delta$ & $t$ ?
question #1b: Isotropic confinement

A.D. Cambou and N. Menon: *Three-dimensional structure of a sheet crumpled into a ball* (2011, 2015)

Consider:
• Sheet of radius $W$ and thickness $t$, confined in a ball of radius $t \ll R \ll W$
• Assume pattern minimizes elastic energy (self-avoiding constraint)

• What is the compression force $F = \frac{\partial U}{\partial R}$?
• What an effective “order parameter” describes the structure?
Geometrically Incompatible Confinement (GIC)
Gauss Theorem & geometrical incompatibility

Gauss theorema egregium

Gaussian curvature "mismatch" → strain
Gauss Theorem & geometrical incompatibility

\[ G_{\text{tar}} = \frac{1}{R^2} \]

“target” (spherical surface)

\[ G_{\text{sub}} = 0 \]

“substrate” (planar map)

\[ G_{\text{tar}} \neq G_{\text{sub}} \implies \text{strain} \]
What is “geometrical incompatible confinement”

generically compatible (developable)

\[ G_{\text{tar}} = 0 \]
\[ G_{\text{sub}} = 0 \]

Euler elastica OK

minimize \( U_{\text{bend}} \)
subject to: \( U_{\text{strain}} = 0 \)

geometrically incompatible (non-developable)

\[ G_{\text{tar}} = 0 \]
\[ G_{\text{sub}} \neq 0 \]

Euler elastica NOT OK

minimize \( U_{\text{strain}} + U_{\text{bend}} \)
Geometrical incompatibility: a daunting problem

Confined sheet \textbf{must} have \\
$G \neq 0$

"... These equations are very complicated and cannot be solved exactly, even in very simple cases ..." \\
(Landau & Lifshitz, Theory of Elasticity)

\begin{itemize}
\item \textbf{Euler elastica NOT OK}
\item FvK
\end{itemize}

\begin{align*}
\text{minimize} \quad & U_{\text{strain}} + U_{\text{bend}} \\
\end{align*}
Substrate-mediated & Tension-mediated confinement

generalizing the *elastica*

Motivated by:

- Beautiful experiments & applications (supported polymers, elasto-capillary phenomena,...)
- Seeking theoretical/conceptual simplicity
Substrate-mediated gross ("envelope") shape is **given**

- "spherical stamping"  
  Hure et al. 2012  
  BD, Sun, Grason 2019

- Flattening spherical shell  
  Aharoni et al. 2017, Tobasco et al. 2020

\[ G_{\text{tar}} = 0 \]
\[ G_{\text{sub}} = 1/R^2 \]

\[ G_{\text{tar}} = 1/R^2 \]
\[ G_{\text{sub}} = 0 \]

Tension-mediated gross ("envelope") shape is **emergent**

- "liquid wrapper"  
  Py et al. 2007  
  Paulsen et al. 2015  
  King et al. 2012

- pulled-twisted ribbon  
  Chopin et al. 2013-16

- indenting floating sheet  
  Holmes-Crosby 2010,  
  Vella et al. 2015, Paulsen et al. 2016,  
  Ripp et al. 2020

\[ \zeta(r) \approx \zeta_0 \cdot \text{Ai} \left( \frac{r}{\ell_c^{2/3} R^{1/3}} \right) \]

\[ G = 1 - \frac{R}{R_0} \]

\[ U_{\text{strain}} \to 0 \]

**Gauss-Euler elastica** ("asymptotic isometry")

\[
\text{minimize} \quad U \approx U_{\text{bend}} + U_{\text{sub}} \quad \text{subject to: } U_{\text{strain}} \to 0
\]

\[
\text{minimize} \quad U \approx U_{\text{bend}} - \text{Work} \quad \text{subject to: } U_{\text{strain}} \to 0
\]
**Substrate-mediated**

Gross ("envelope") shape is **given**

- "spherical stamping"
  - Hure *et al.* 2012

- Flattening spherical shell
  - Aharoni *et al.* 2017, Tobasco *et al.* 2020

\[
G_{tar} = 0
\]
\[
G_{sub} = 1/R^2
\]

\[
G_{tar} = 1/R^2
\]
\[
G_{sub} = 0
\]

**Tension-mediated**

Gross ("envelope") shape is **emergent**

- "liquid wrapper"
  - Py *et al.* 2007
  - Paulsen *et al.* 2015
  - King *et al.* 2012

- Stretched-twised ribbon
  - Chopin *et al.* 2013-16

- Indenting floating sheet
  - Holmes-Crosby 2010,
  - Vella *et al.* 2015, Paulsen *et al.* 2016,
  - Ripp *et al.* 2020

\[
\zeta(r) \approx \zeta_0 \cdot \text{Ai} \left( \frac{r}{\ell_c^{2/3} R^{1/3}} \right)
\]

**minimize**

\[
U \approx U_{bend} + U_{sub}
\]

**subject to:**

\[
U_{strain} \to 0
\]

**minimize**

\[
U \approx - \text{Work}
\]

**subject to:**

\[
U_{strain} \to 0
\]
Substrate-mediated gross ("envelope") shape is given

"spherical stamping"  
Hure et al. 2012  
Flattening spherical shell  
Aharoni et al. 2017, Tobasco et al. 2020

\[
\begin{align*}
G_{\text{tar}} &= 0 \\
G_{\text{sub}} &= 1/R^2
\end{align*}
\]

\[
\begin{align*}
G_{\text{tar}} &= 1/R^2 \\
G_{\text{sub}} &= 0
\end{align*}
\]

Gauss-Euler elastica ("asymptotic isometry")

\[
\text{minimize} \quad U \approx U_{\text{bend}} + U_{\text{sub}}
\]

subject to: \( U_{\text{strain}} \to 0 \)
The “Winkler ball” model

- Stretching modulus: \( Y \sim E \cdot t \)
- Bending modulus: \( B \sim E \cdot t^3 \)
- Spring constant: \( K \)
- Rest length: \( R \)

What is the characteristic stress?

\[ \sigma \sim Y \cdot (W/R)^2 \]
\[ \sigma \sim \sqrt{B \cdot K} \]
Stress in sheet on a “Winkler ball”
Stress in sheet on a “Winkler ball” elastica recovered despite Gaussian curvature mismatch

\[ u_r \approx -\frac{W^3}{6R^2} \]

\[ U_{strain} \gg U_{bend}, U_{subst} \]

FvK

\[ U_{bend}, U_{subst} \gg U_{strain} \]
question #2: nearly-isometric deformation of supported shell?

Success to find asymptotically-isometric deformations relies on the presence of designated “tension lines”, dictated by actual tensile load or by substrate topography

(see Vella’s presentation and Tobasco et al. 2021)
Consider:
- thin shell of radius $R$, surrounding compliant sphere (stiffness $K$)
- pressure is exerted: $\Delta P = P_e - P_i > 0$
- Assume pattern minimizes $U_{strain} + U_{bend} + U_{subst}$

- Is the response “asymptotically isometric” ($U_{strain} \ll U_{bend}; U_{subst}$)?
- What is the mechanical response $\Delta P = \partial U/\partial V$?

**question #2: nearly-isometric deformation of supported shell?**
Consider:
- thin shell of radius $R$, surrounding compliant sphere (stiffness $K$)
- pressure is exerted: $\Delta P = P_e - P_i > 0$
- Assume pattern minimizes $U_{\text{strain}} + U_{\text{bend}} + U_{\text{subst}}$

- Is the response “asymptotically isometric” $(U_{\text{strain}} \ll U_{\text{bend}}, U_{\text{subst}})$?
- What is the mechanical response $\Delta P = \partial U / \partial V$?

Stoop et al. (2015)
(near threshold analysis)

$U_{\text{strain}} \gg U_{\text{bend}}$

$U_{\text{bend}}, U_{\text{subst}} \gg U_{\text{strain}}$

$\Delta P$
Boundary-mediated confinement
(no substrate, no tensile loads)
daunting problem
sheet has “too much” freedom

\[ G \neq 0 \]

\[ G_{\text{tar}} = 0 \]

(hopefully) less daunting ...

almost-developable cone

contractional inclusion
(Courtesy: M. Xin)

indenting spherical shell
question #3: better than Pogorelov?

Indenting hemi-spherical shell
Vaziri & Mahadevan 2008

Pogorelov’s ridge

courtesy: D. Vella, J. Kierfeld
question #3: better than Pogorelov?  
A possible clue: localized wrinkling instability of Pogorelov’s ridge  
(Knoche & Kierfeld EPJE 2014)

• A new type of **isometric shell deformation** emerges from **localized** buckling?
Thanks

V. Demery  E. Hohlfeld  Y. Sun  G. Grason  N. Menon  J. Paulsen  D. Vella