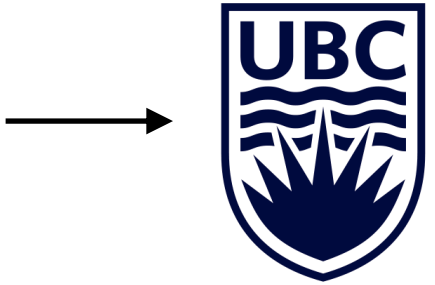
Phase transitions and universality in circuit sampling

Max McGinley

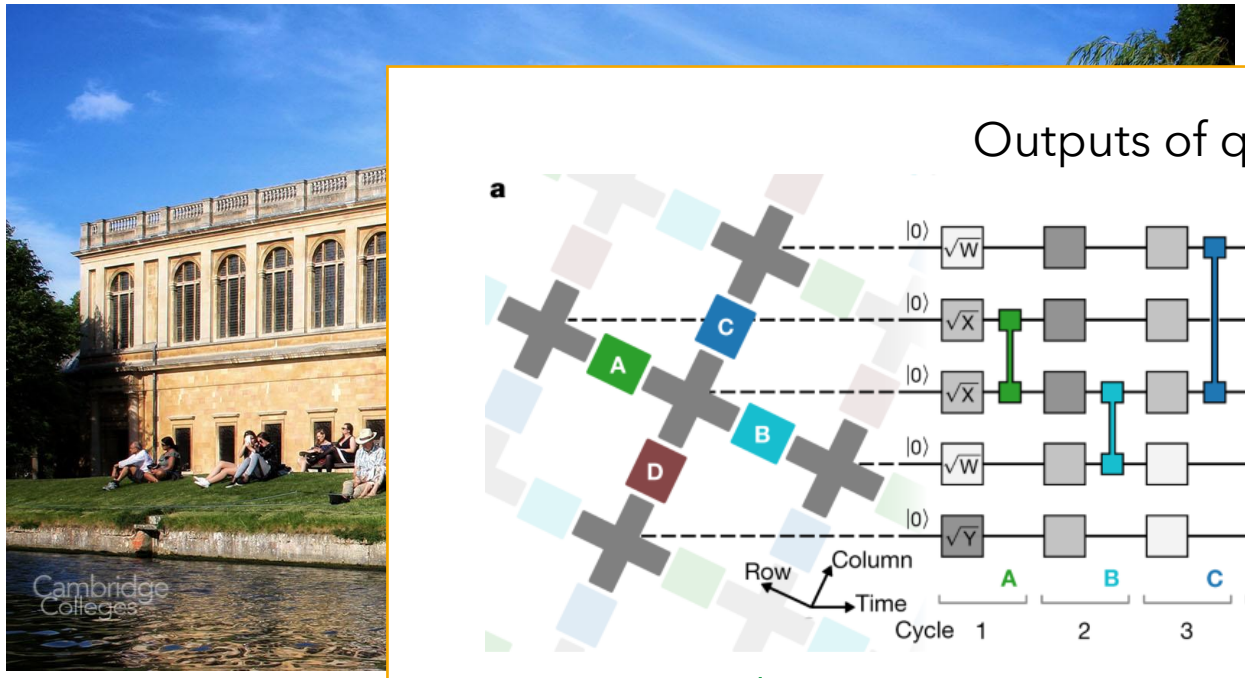
September 29th, 2025

KITP

UNIVERSITY OF CAMBRIDGE



Some very fine structures



Outputs of quantum circuits

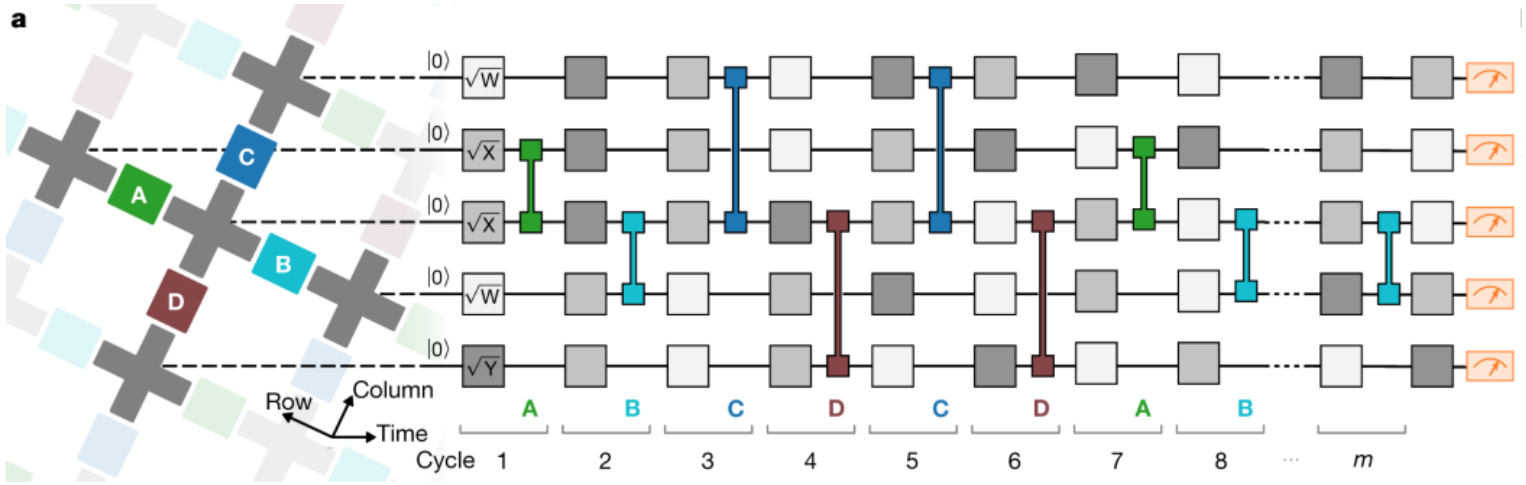
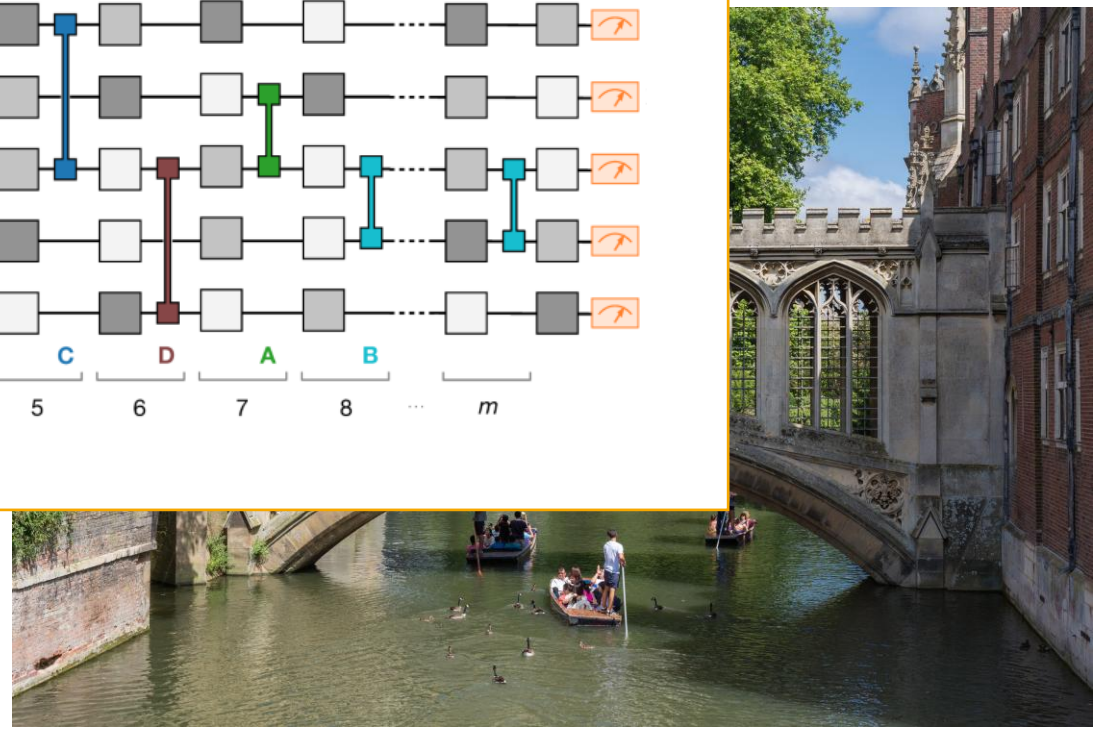


Image: Arute *et al.* (2019)



What does a quantum computer output?

Wish list:

- **Hope 1**: Something a classical computer couldn't output
- **Hope 2**: Something that helps us solve a useful computational problem

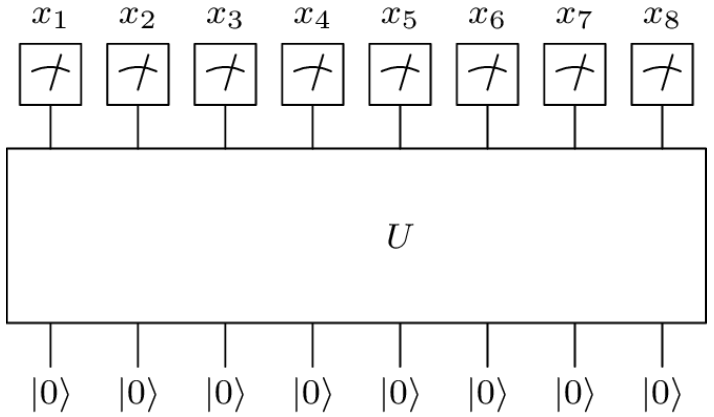
Randomness

- Fundamentally: Q. Computers output *random variables*
  - Distribution determined by Born's rule:

    $x \sim p(x) = |\langle x|U|0\rangle|^2$

- Samples x can be processed:
  - Decision problems (BQP)
  - Expectation values $\langle O \rangle$
  - … depends on what computational problem we want to solve

$x = (x_1, x_2, \ldots, x_n)$



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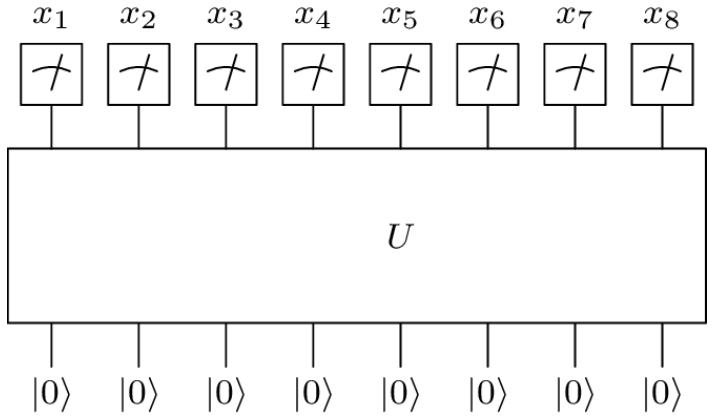
$$p(x) = |\langle x|U|0\rangle|^2$$



What does a quantum computer output?

- **Ho** …t
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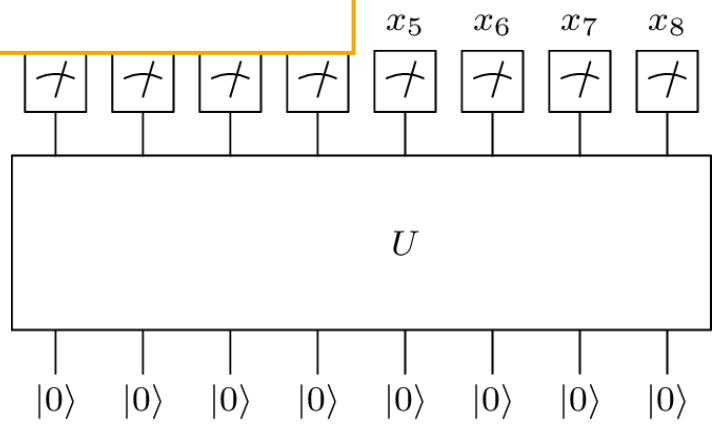
Is there a classical algorithm that takes U as input and generates samples from $p(x)$?

This talk: U = Random Circuit

$$p(x) = |\langle x|U|0\rangle|^2$$



'Weak simulation' (Terhal & DiVincenzo 2004)

Different perspectives on quantum advantage

'Computer science'

- Computational complexity theory (SampBQP)
  - Relationships between different computational tasks
- Show that classical sim. would allow solution of problems believed to be hard

Bremner *et al.* (2010)
Aaronson & Arkhipov (2011)
⋮

'Physics'

- System of qubits undergoing a dynamical process
- *Universality*: phases of matter
- *Typicality*: generic behaviour over specific instances
  - (here: random circuits)

NISQ

- A Complementary motivation:
  - Generating samples from random circuits is typically less space + time intensive than full quantum algorithms (n**IS**q)
  - Verification helped by mathematical understanding of RCS
  - Pre-FTU: Gates + measurements are noisy (**N**isq). Does q. advantage survive?
- Sheds light on computational power of NISQ devices
  - Many well-publicised experiments!

Boixo *et al.* (2018)
Bouland *et al.* (2019)
⋮

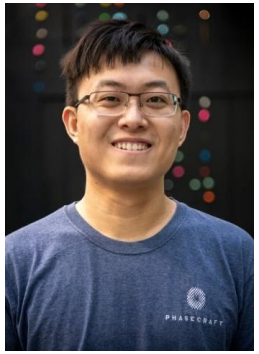
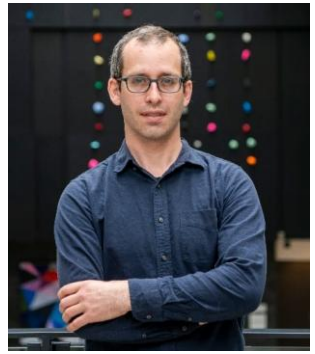
Summary of this talk

1) A phase transition in RCS tuned by circuit depth d
   - Argue that $d < d_c$ 'easy', $d > d_c$ 'hard'

MM, W.W. Ho, D. Malz Phys Rev X (2025)

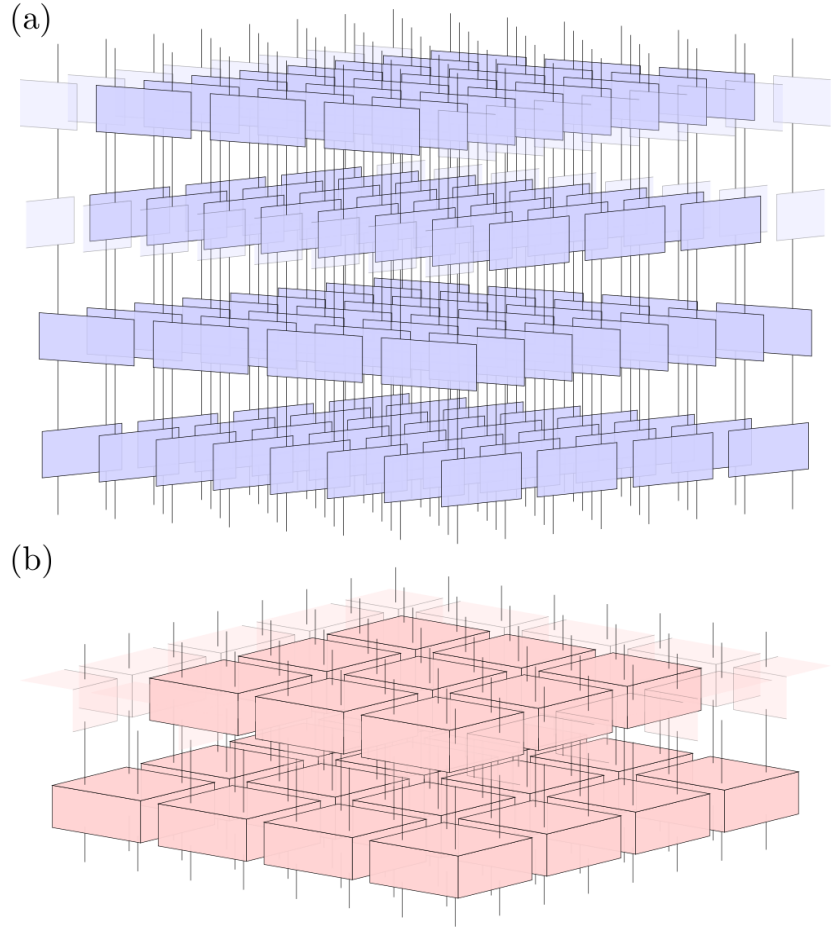
2) Universal behaviour for $d > d_c$
   - Implications for noise-robustness

work with T. Schuster, Y. Liu, D. Gosset

Setup

- Ensemble of random unitaries $U \sim \mathcal{U}$
  - Gate architecture fixed (dimensionality, depth)
  - Gates drawn i.i.d. from chosen dist. (e.g. Haar)
- Collection of gates defines output distribution $p(x) = |\langle x|\psi\rangle|^2 \quad |\psi\rangle = U|0\rangle$
  - Statistical properties of $p(x)$? Implications for classical simulation strategies?



Review: Deep circuits

- Theory + experiment usually focuses on extensive depth $d = \Omega(n)$
- Typically generate (ϵ-approximate) k-designs

$$\mathbb{E}_{\psi} f(\psi) \approx_{\epsilon} \mathbb{E}_{\phi \sim Haar} f(\phi)$$

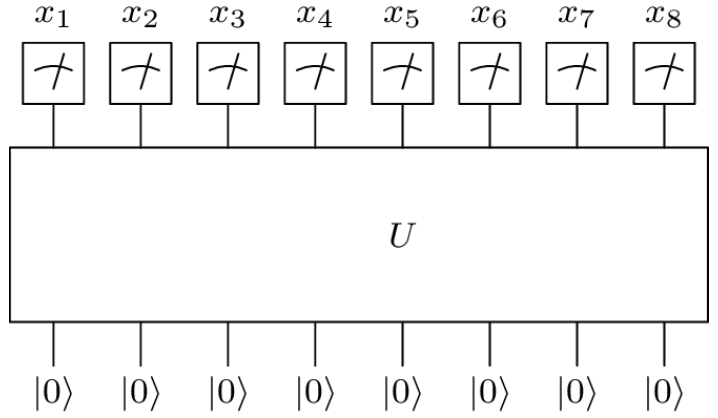
If f is a kth order polynomial of $|\psi\rangle\langle\psi|$

- ψ "looks like" a globally random state. Spatiotemporal structure of circuit is approximately forgotten.
- What can we say about output distribution? ...

Review: Deep circuits

- Conditional states look Haar-random
- Tracing out $O(\log 1/\epsilon)$ qubits gives a flat distribution (*anticoncentration*)
- Sampling most of the bits is easy but the last few bits are hard
- Fragile to noise: Flat distribution for readout error $\gamma > O(n^{-1} \log 1/\epsilon)$
- Fine structure!

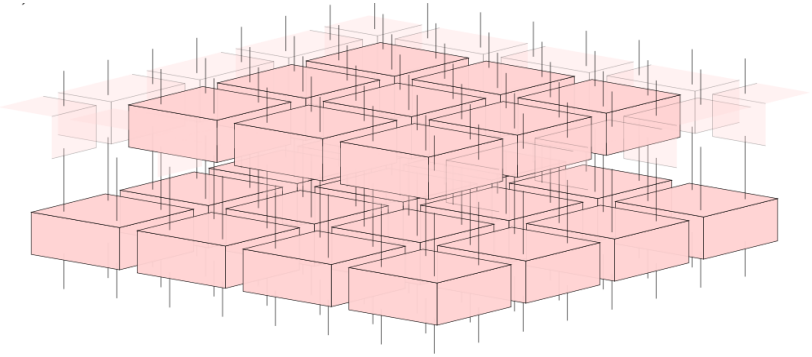
$$p(x_1) = f(\psi^{1\ldots n})$$
$$p(x_2|x_1) = f(\psi_{x_1}^{2\ldots n})$$
$$p(x_3|x_1x_2) = f(\psi_{x_1x_2}^{3\ldots n})$$
$$\vdots$$
$$p(x_n|x_1 \ldots x_{n-1}) = f(\psi_{x_1\ldots x_{n-1}}^{n})$$



Shallow circuits

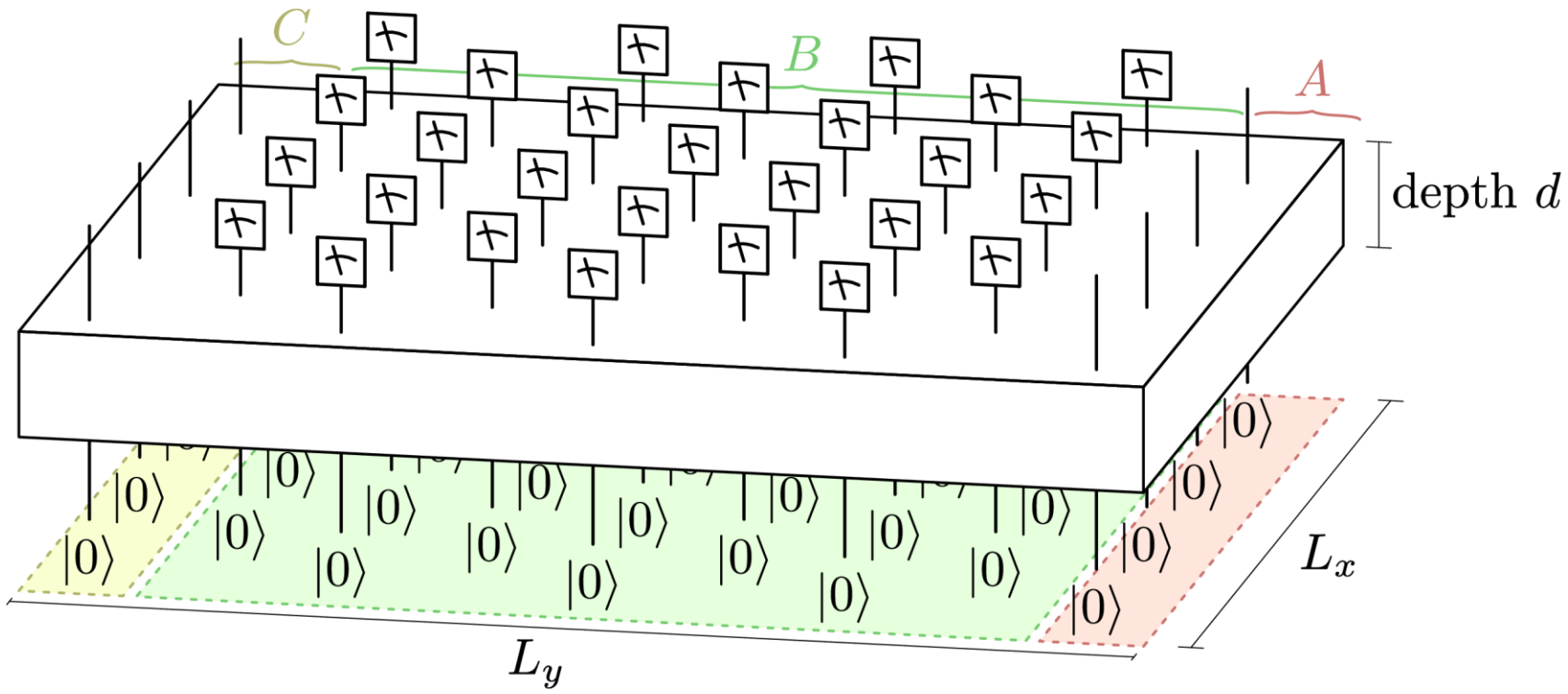
- 'Shallow' will mean constant-depth: $d = O(1)$ as $n \to \infty$
  - We focus on 2D: Minimal dimension for worst-case hardness
- Do **not** form k-designs for $k \geq 2$:
  - Marginals of small number of qubits $p(x_1 \dots x_r)$ not flat
  - Circuit architecture reflected in entanglement structure: area law
- Succinct TNS representation of ψ. Basis of efficient classical algorithm?
  - What do the conditional states look like?

Terhal & DiVincenzo (2004)
Bermejo-Vega *et al.* (2018)



Conditional states

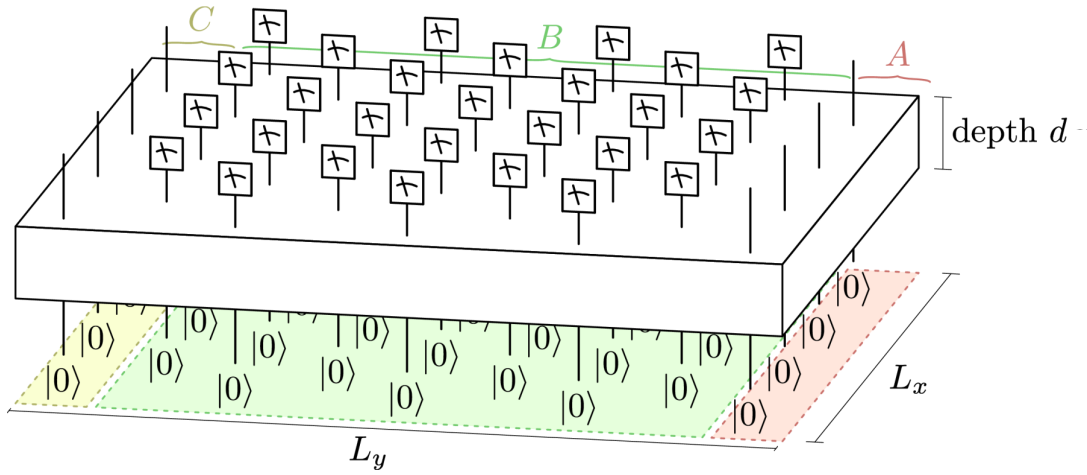
$$\psi_{x_B}^{AC} =$$



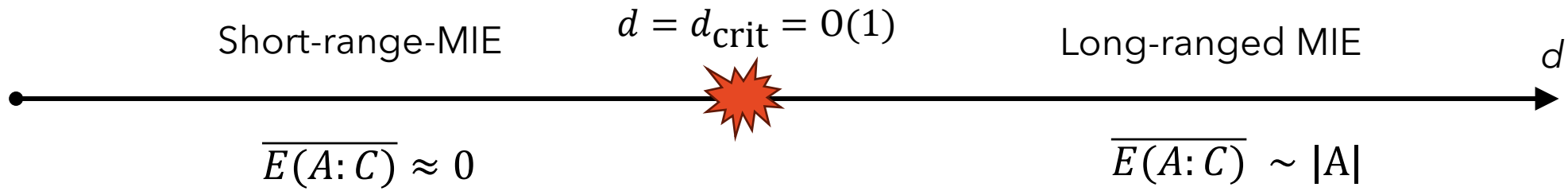
Conditional states

- Before measurement: A and C are uncorrelated (lightcone-separated)
- After measurement: A and C can be entangled
  - *Measurement-induced entanglement* (MIE)
  - E.g. Entanglement swapping …
- **Claim**: A Phase transition in MIE

$$\overline{E(A:C)} = \mathbb{E}_{U,x_B} S\left(\rho^A_{x_B}\right)$$



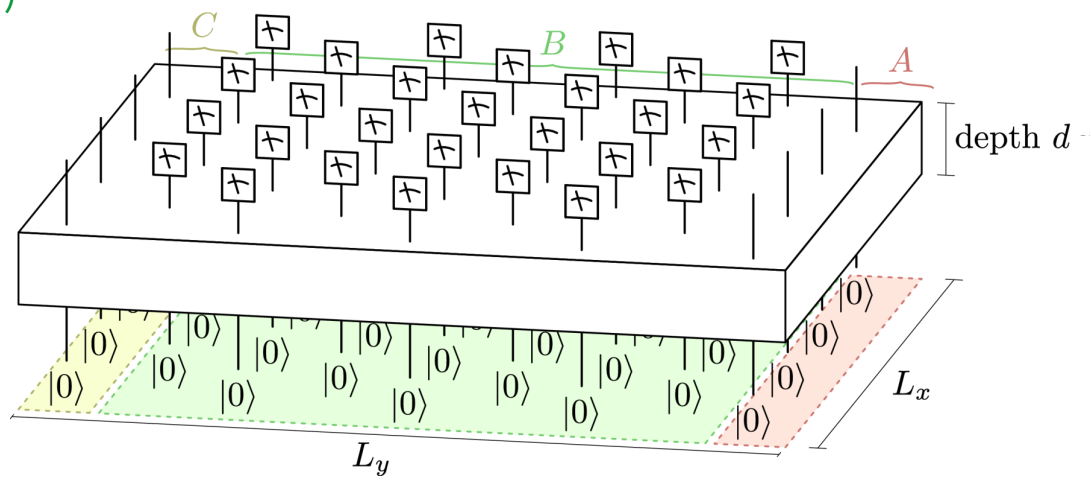
MIE Phase transition



- Proposed MIE / 'teleportation' transition conjectured before

Napp *et al. PRX* (2022)
Bao *et al. PRL* (2024)

- Here: *Rigorous proof* of long-ranged MIE for $d > d_{\mathrm{crit}}$
  - Alternative view: proof of volume law phase in (1+1)d MIPTs
- N.B.: No transition in 1D!



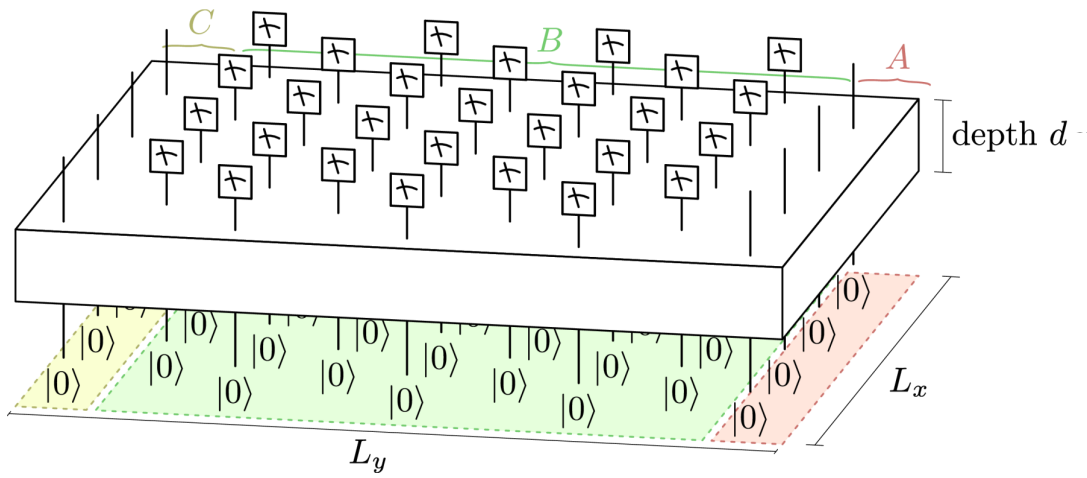
MM, W.W. Ho, D. Malz Phys Rev X (2025)

Long-range MIE phase

Theorem: In several natural families of random quantum circuits of depth d acting on qudits of dimension q, there exist values $(d, q) = O(1)$ for which

$$\overline{E(A{:}C)} \geq \alpha L_x$$

- Previously: only known for $d = \Omega(\log n)$ or $q = \mathrm{poly}(n)$, which are both effectively log-depth circuits (back to k-designs)
  Hayden *et al* JHEP (2016); Schuster *et al*. Science (2024)
- New theoretical techniques based on multi-user information theory Colomer & Winter, CMP (2024)



Classical algorithms

- In the short-ranged MIE phase: $\psi_{x_B}^{AC} \approx \psi_{x_B}^{A} \otimes \psi_{x_B}^{C}$

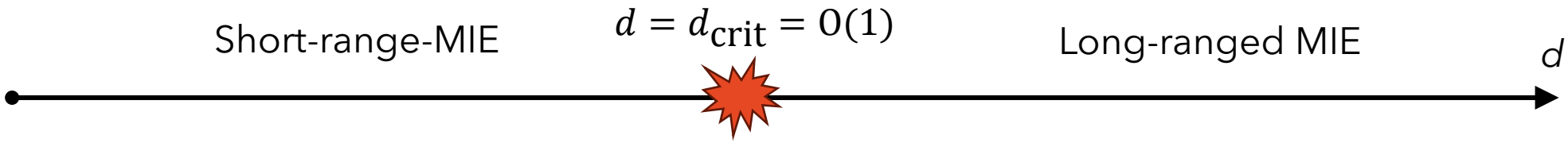
$$p(x_A|x_B x_C) \approx p(x_A|x_B)$$

  - Can ignore conditional correlations between distant qubits → efficient classical simulation

$$p(x_1) = f(\psi^{1\dots n})$$
$$p(x_2|x_1) = f(\psi_{x_1}^{2\dots n})$$
$$p(x_3|x_1 x_2) = f(\psi_{x_1 x_2}^{3\dots n})$$
$$\vdots$$
$$p(x_n|x_1 \dots x_{n-1}) = f(\psi_{x_1 \dots x_{n-1}}^{n})$$

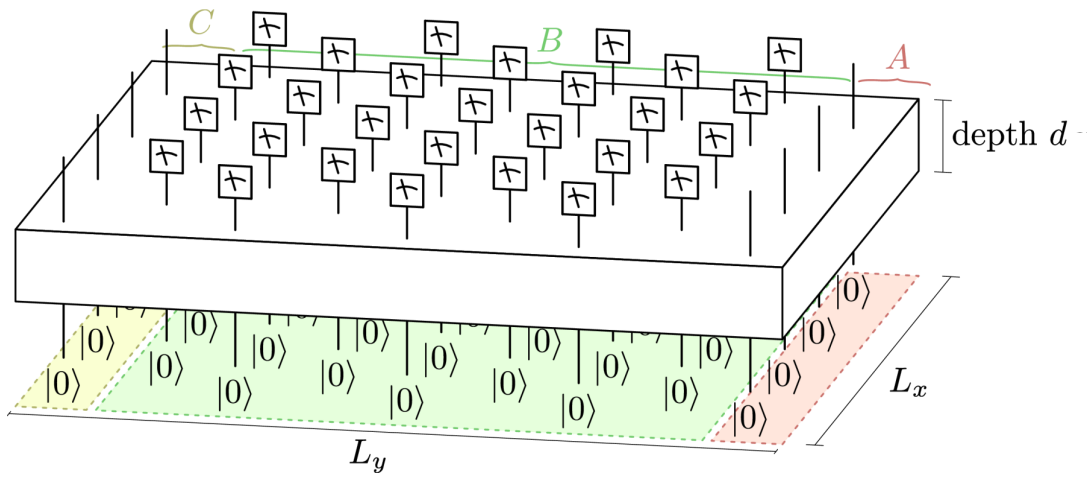
- In the long-ranged MIE phase:
  - Conditional state $\psi_{x_B}^{AC}$ is highly entangled
  - No efficient TNS description (similar to deep circuits)
  - Conditional probabilities depend nontrivially on *all* previous outcomes

Complexity Phase transition?



Classically simulable

Quantum advantage (!?)



Note: known worst-case instances with complexity-theoretic hardness proofs all have long-range MIE (based on MBQC)

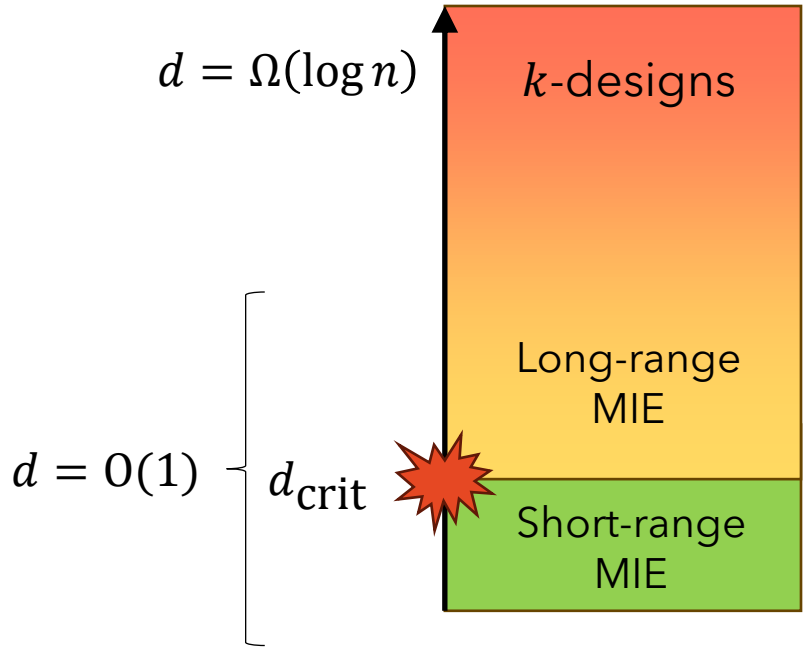
Terhal & DiVincenzo (2004)
Bermejo-Vega *et al.* (2018)

Part 2: Universality above the MIE phase transition

with T. Schuster, Y. Liu, D. Gosset

From shallow to deep

- If we keep increasing depth to $d = \Omega(\log n)$, should recover properties of the 'deep' k-design regime
  - Marginals of $[n - O(\log 1/\epsilon)]$ qubits are easy
  - Distribution collapses for readout noise rate $\gamma \sim n^{-1}$
- Do these properties also hold in the constant-depth regime?
  - Need more information about conditional states…



The ensemble of conditional states

- Recall: For deep circuits, conditional states look Haar-random
  - Not possible at constant depth: nontrivial local structure
- We identify a statistical ensemble that we conjecture describes the conditional states in constant depth circuits
  - Support with numerical evidence for *Clifford circuits*
  - Derive implications for classical simulability of marginals + noisy distributions

The Scrooge ensemble

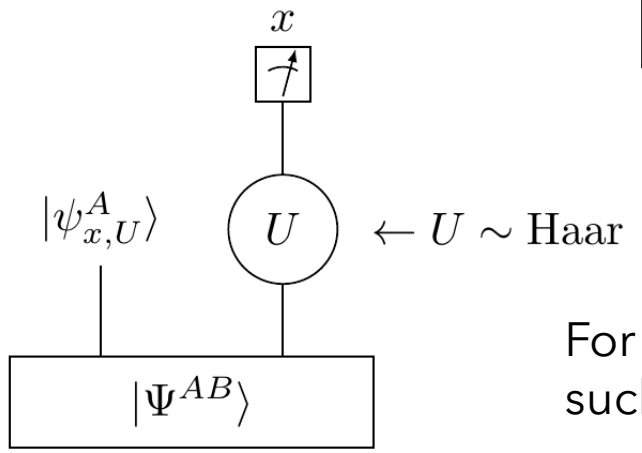
Jozsa, Robb, Wootters, PRA (1994)

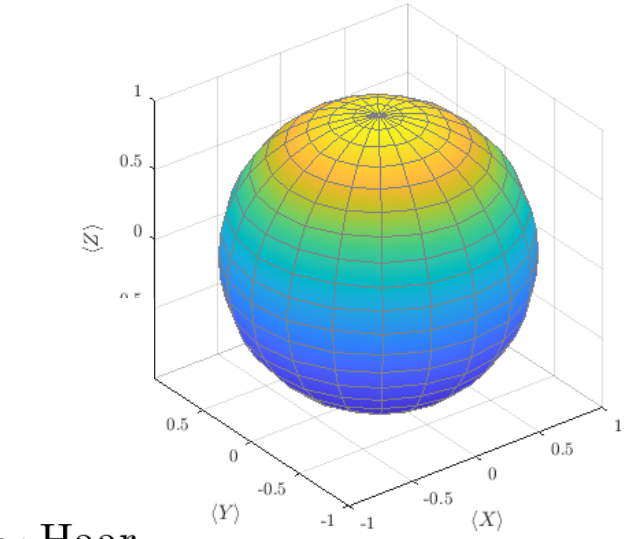
- A random ensemble of pure states with nontrivial mean

$$|\psi\rangle \sim \text{Scrooge}(\rho) \qquad \rightarrow \mathbb{E}_{\psi}[|\psi\rangle\langle\psi|] = \rho$$

- $\text{Scrooge}(\rho)$ is 'as random as possible' given this constraint
- Explicit construction: ...
- Argued to arise in various many-body settings

Cotler *et al.* (2023), Ho & Ippoliti (2023), Mark *et al* (2024)

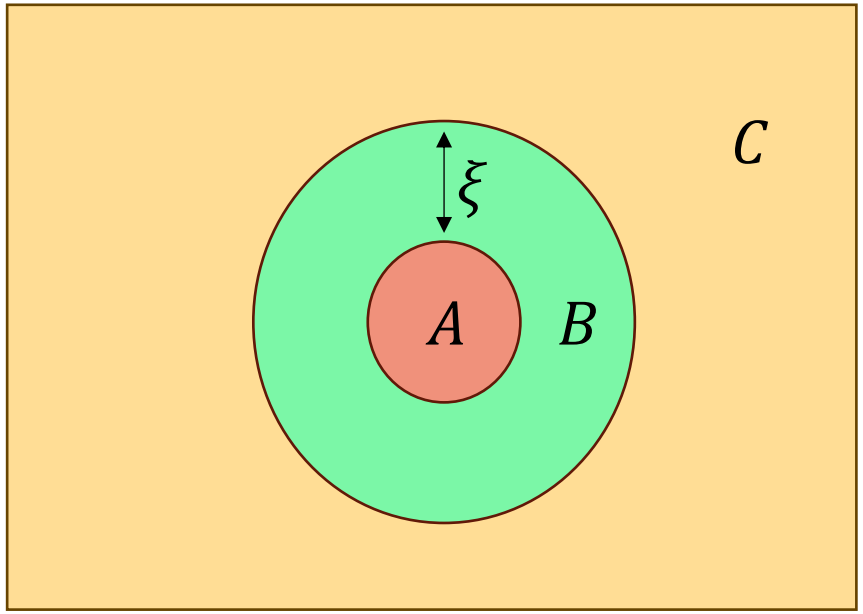




For any purification $|\Psi^{AB}\rangle$ such that $\text{Tr}_B \Psi^{AB} = \rho^A$

Our conjecture

- Take a O(1)-sized region A and 'pad' with a shielding region B



Conjecture

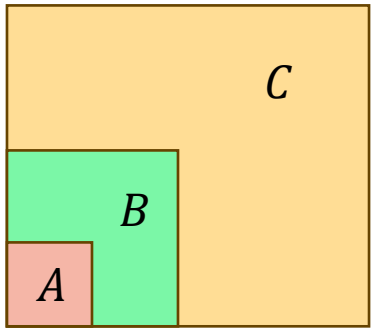
Over the random gates and measurement outcomes in C, the ensemble of conditional states on A matches the Scrooge ensemble

$$\left|\psi^{A}_{x_B x_C}\right\rangle \sim \text{Scrooge}\left(\rho^{A}_{x_B}\right)$$

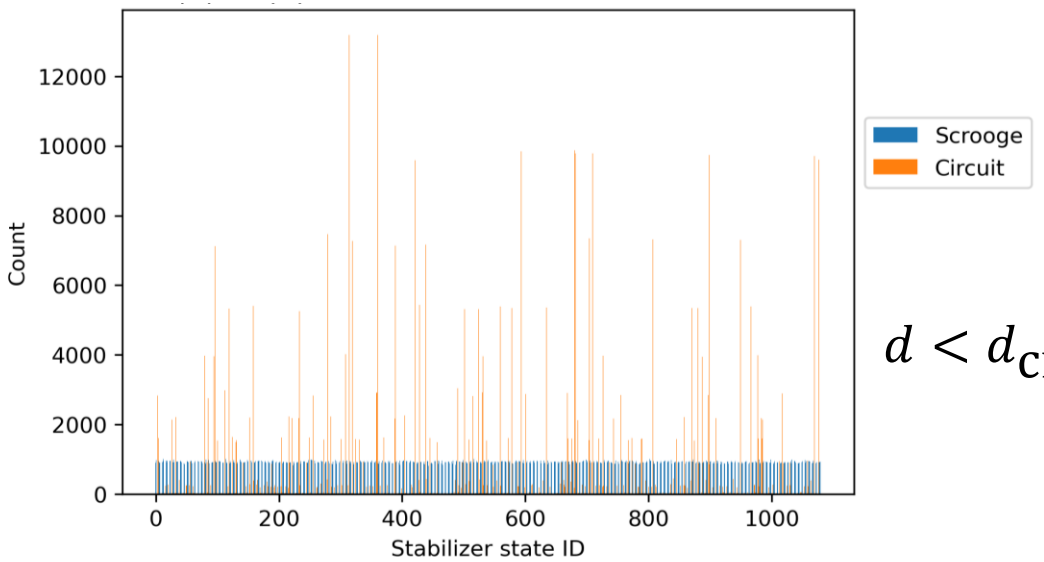
provided ξ is sufficiently large [but O(1)]

This is equivalent to replacing the gates in C with a Haar-random unitary

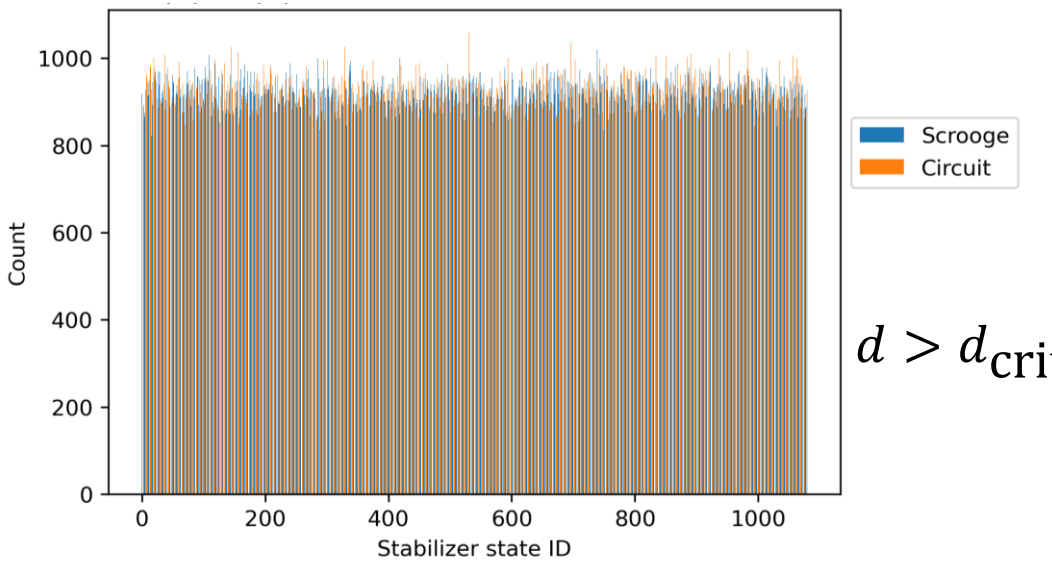
Numerical evidence



- Numerics on Clifford circuits indicates that a "stabilizer version" of our conjecture holds (wrt. a natural stabilizer generalization of Scrooge)



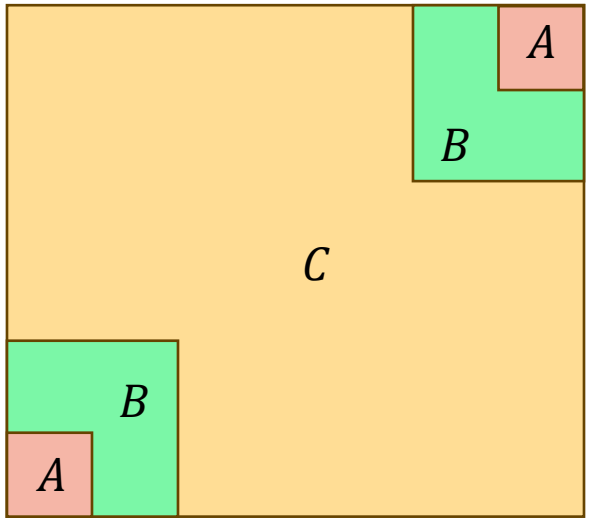
$d < d_{\text{crit}}$

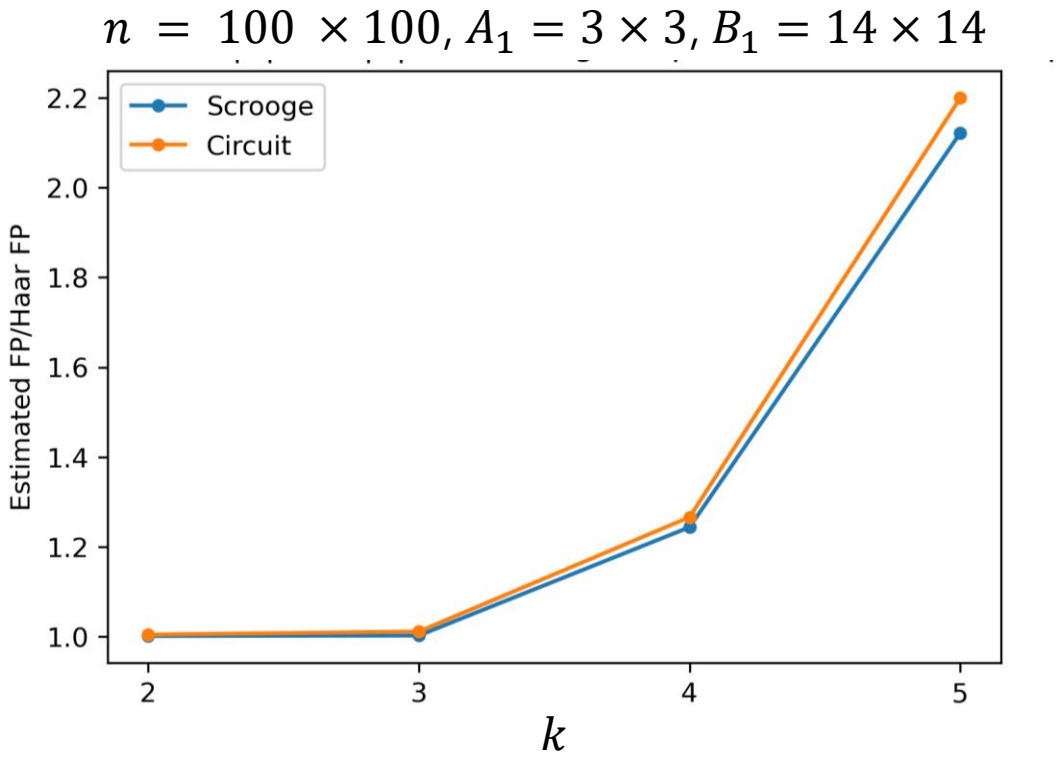


$d > d_{\text{crit}}$

Numerical evidence

- For larger sizes of A, use frame potential





Implications

- The marginals of Scrooge-random states have similar structure to Haar-random states

MM, T. Schuster (*forthcoming*)

$$\psi \sim \text{Haar} \qquad \psi \sim \text{Scrooge}(\rho)$$

$\text{Tr}_B\, \psi^{AB} \approx^* I_A/D_A$ up to error $e^{-\Omega(|B|)}$ $\quad$ $\text{Tr}_B\, \psi^{AB} \approx^* \rho^A$ up to error $e^{-\Omega(|B|)}$

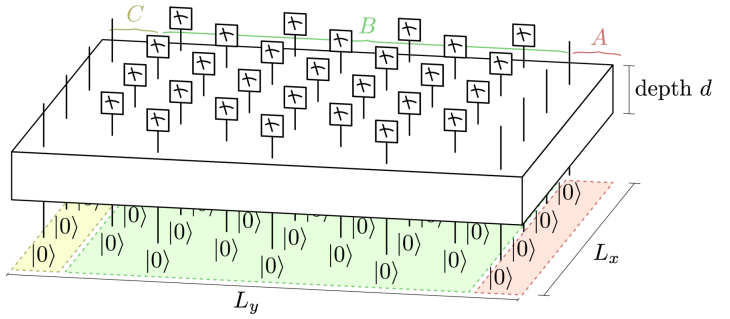
- If conjecture holds: An efficient classical algorithm for simulating marginals and noisy distributions [with $\gamma = O(n^{-1})$] based on sequential sampling

*at the level of kth moments

Conclusions

Random circuit sampling in constant-depth 2D circuits

- A phase transition in MIE tuned by depth
- Marks the boundary between classically simulable regime and (possible) quantum advantage
- Universal behaviour above the transition captured by Scrooge ensemble
  - Marginals and noisy distributions are easy



Outlook

- Beyond (Haar-) random circuits
  - Conserved charges, Hamiltonian dynamics
- Other consequences of transition:
  - Verification in sampling experiments? (XEB etc.)
  - Training landscape for variational algorithms?
  - Learning / certifying states? Constant-depth classical shadows?
- From sampling to expectation values…
  - Hybrid quantum-classical algorithms?