Geometrical interpretation of the Casimir effect

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- An object or objects immersed in vacuum modify quantum fluctuations of the EM field: there is measurable fluctuation-induced force (Casimir, 1948).
- Also true for any ground state and other fields – analog Casimir effects in condensed matter.
- There also could be topological Casimir/Aharonov-Bohm effects.
- ✓ How can we calculate it efficiently?
- ✓ How to make sense of the (sometime) infinite results?

Only $T=0$ limit, sharp boundaries and pure vacuum will be considered.

- The EM field has zero-point energy

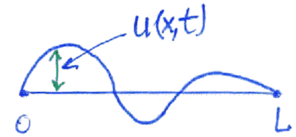
$$E = \sum_{\nu} \frac{1}{2}\hbar\omega_\nu$$ which is infinite

(ν: modes)

- An object  modifies the spectrum. This gives a self-energy relative to the vacuum. This is also infinite.

-  Objects that are close to each other have overlapping influence. This gives a finite change in the self-energy. This is the Casimir interaction (forces and torques).

Example: a scalar field $u(x,t)$ on a one-dimensional Dirichlet interval:



$\omega_n = c k_n = \pi c n / L$ (c: speed of light/sound)

$$E = \frac{\pi\hbar c}{2L}\sum_{n=1}^{\infty} n$$ diverges

But real materials become transparent at short wavelengths. So we can cut off the sum at $n = N$. Then

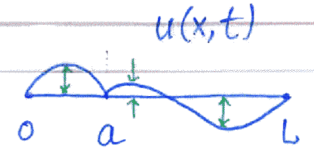
$$E = \frac{\pi\hbar c}{4L}\left(N^2 + N + 0\right)$$

(0: sharp; N: cutoff frequency)

The upper limit is of order $N = \frac{\omega_0 L}{c}$:

$$E \simeq \underbrace{\frac{\hbar\omega_0^2}{c}L}_{\text{bulk}} + \underbrace{\hbar\omega_0}_{\text{ends}} + \underbrace{(\#)}_{\text{smooth}}\underbrace{\frac{\hbar c}{L}}_{\text{finite-size}}$$

Insert another Dirichlet partition at $x = a$.



$$E(L) = \frac{\hbar\omega_0^2}{c}L + \hbar\omega_0 + (\#)\frac{\hbar c}{L}$$

$$\mathcal{E} = E(a) + E(L-a) =$$

$$= \underbrace{\#\hbar c\left(\frac{1}{a} + \frac{1}{L-a}\right)} + \underbrace{\frac{\hbar\omega_0^2}{c}(\cancel{a} + L - \cancel{a})} + \underbrace{\hbar\omega_0(1+1)}$$

- universal
- size determined by $\hbar, c$ and macroscopic length scales.

- a-independent
- $\omega_0 \to \infty$ limit – infinities are subtracted

a, L – independent

non-universal

- if $L \to \infty$, then $\mathcal{E} \approx \hbar c/a$ is uniquely determined by dimensional analysis.
- although it is electromagnetic in origin, the charge quantum e does not appear.
- determination of # requires smooth cutoff; the sign determines if it is attractive or repulsive.

Determining the numerical prefactor.

- Assume a smooth cutoff function $F\left(\frac{n}{N}\right)$

$$E = \frac{\pi\hbar c}{2L}\sum_{n=1}^{\infty} n e^{-n/N} = -\frac{\pi\hbar c}{2L}\frac{\partial}{\partial(1/N)}\left(\sum_{n=1}^{\infty} e^{-n/N}\right) =$$

$$= \frac{\pi\hbar c}{2L}\frac{e^{-\frac{1}{N}}}{(1-e^{-1/N})^2} \xrightarrow{N\gg1} \frac{\pi\hbar c}{2L}\left(N^2 - \frac{1}{12}\right)$$

$\leftarrow$ any $F(y)$

So $\mathcal{E} = -\frac{\pi\hbar c}{24}\left(\frac{1}{a} + \frac{1}{L-a}\right)$ – attractive

- Another route: the Riemann ζ-function: $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ – defined for $s > 1$. There is a singularity at $s=1$, but it is possible to continue analytically past the singularity. Then

$E \overset{R}{=} \frac{\pi\hbar c}{2L}\zeta(-1)$. It turns out $\zeta(-1) = -\frac{1}{12}$ which gives same result as above.

This "works" most of the time but there are exceptions when "infinities remain".

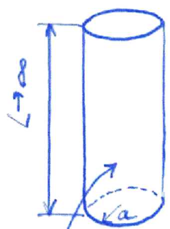
Exceptions

Always for curved solitary boundaries:

- For a spherical shell in d spatial dimensions the Casimir pressure is infinite for even d (Bender+Milton, 1994)

Does it mean that conductive ring in two dimensions is unstable?

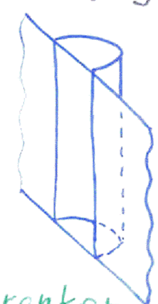
- The Casimir energy is finite for a cylinder, but infinite for half-cylinder!



$$\frac{\mathcal{E}}{L} = \# \frac{\hbar c}{a^2}$$

DeRaad+ Milton, 1981

hollow



Nesterenko+ Lambiase+ Scarpetta, 2001

Formalism

We have found an efficient way to calculate the Casimir energy that does not require an explicit knowledge of the spectrum.

- It is based on the path-integral representation of the zero-point energy
- It works for a scalar field in the presence of arbitrary number of Dirichlet and/or Neumann boundaries.
- It relates two ideas that the Casimir effect can be viewed either as due to zero-point motion of the vacuum or due to fluctuations at the boundaries.

Only the case of single Dirichlet boundary is considered.

Imaginary time action for a scalar field is

$$S[w] = \frac{1}{2}\int_0^{\hbar/T} d\tau\, d^d x \left(\frac{1}{c^2}\left(\frac{\partial w}{\partial \tau}\right)^2 + (\nabla w)^2\right)$$

imag. time

$w(\vec{r},0) = w(\vec{r}, \frac{\hbar}{T})$ - periodicity on the Matsubara circle.

"partition function"

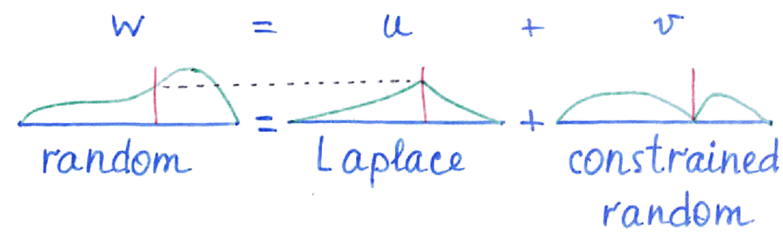
Calculate $Z[w] = \int Dw(\vec{r},\tau)\exp(-S[w]/\hbar)$

over all possible forms for $w(\vec{r},\tau)$ satisfying various boundary conditions. Functions slowly varying in space and time contribute most.

The zero-point energy is the "free energy" per unit "length" in the imaginary time direction.

$$E = -\hbar \ln Z[w]/(\hbar/T) = -T\ln Z[w]$$

Introduce a new Dirichlet boundary. This will constrain the field:



There is a unique way to associate the unconstrained field w with a constrained field v (satisfying new boundary condition):

$$w(\vec{r},\tau) = v(\vec{r},\tau) + u(\vec{r},\tau)$$

Solution to $\left(\frac{\partial^2}{c^2\partial\tau^2} + \Delta\right)u = 0$ agreeng with w on the boundary.

Then $S[w] = S[v] + S[u] \rightarrow Z[w] = Z[v]Z[u]$.

$$\mathcal{E}_{casimir} = E_{constrained} - E_{unconstrained} = T\ln Z[u]$$

The rule

In words: the Casimir energy due to a Dirichlet boundary is negative of the zero-point energy of the modes suppressed by this boundary.

In symbols: need to solve the boundary value Laplace problem:

$$\left(\frac{\partial^2}{c^2\partial\tau^2}+\Delta\right)u=0,\quad u\Big|_{\text{boundary}}=f(\vec r,\tau)$$

dynamical field

After Fourier expansion

$$u(\vec r,\tau)=\sum_\omega u_\omega(\vec r)\exp i\omega\tau \rightarrow \text{Helmholtz}$$

$$\left(\Delta-\frac{\omega^2}{c^2}\right)u_\omega=0,\quad u_\omega\Big|_{\text{boundary}}=f_\omega(\vec r)$$ – put into $S[u]$:

$$S\{u[f]\}=\frac{1}{2}\int_0^{\hbar/T}d\tau\int\underbrace{[u\nabla u]}_{\text{discontinuity}}d\vec\sigma=\frac{\hbar}{2T}\sum_\omega\int f_\omega[\nabla u_{-\omega}]\,d\vec\sigma=$$

discontinuous

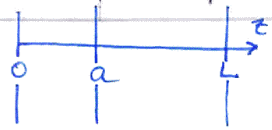
$u_\omega\propto f_\omega$

$$=\frac{\hbar}{2T}\sum_{\omega,\nu}\frac{|f_{\omega\nu}|^2}{\lambda_\nu(|\omega|/c)}$$ – Gaussian

modes geometry

$$\mathcal{E}=\frac{T}{2}\sum_{\omega,\nu}\ln\left(2\pi T\lambda_\nu(|\omega|/c)/\hbar\right)\xrightarrow{T\to0}\frac{\hbar c}{2\pi}\sum_\nu\int_0^\infty d\kappa\ln\lambda_\nu(\kappa)$$

An example: three planes at $z=0,a,L$



in d dimensions

What is the Casimir pressure on the plane at $z=a$?

$$u_\omega(\vec r)=\sum_{\vec q}u_{\omega\vec q}(z)\exp i\vec q\vec r_\perp$$

to z

$$\left(\frac{d^2}{dz^2}-q^2-\frac{\omega^2}{c^2}\right)u_{\omega\vec q}=0,\quad u_{\omega\vec q}\Big|_{0,L}=0,\quad u_{\omega\vec q}\Big|_a=f_{\omega\vec q}$$

The particular solution

$$\begin{cases}u_{\omega\vec q}(z)=f_{\omega\vec q}\dfrac{\sinh(|\varkappa|z)}{\sinh(|\varkappa|a)}, & 0\le z\le a,\quad \varkappa^2=q^2+\dfrac{\omega^2}{c^2}\\ u_{\omega\vec q}(z)=f_{\omega\vec q}\dfrac{\sinh(|\varkappa|(L-z))}{\sinh(|\varkappa|(L-a))}, & a<z\le L\end{cases}$$

becomes localized for large $|\varkappa|$

$$S=\frac{\hbar \overset{\text{area}}{A}}{2T}\sum_{\omega,\vec q}|\varkappa|\underbrace{\left(\coth|\varkappa|a+\coth|\varkappa|(L-a)\right)}_{}|f_{\omega\vec q}|^2$$

Gaussian

$$\frac{\mathcal{E}}{A}\to-\frac{\hbar}{2}\int\frac{d\omega\, d^{d-1}q}{(2\pi)^d}\ln(\ldots)$$ – divergent

$$P=-\frac{\partial(\mathcal{E}/A)}{\partial a}=\#\,\hbar c\left[(L-a)^{-d-1}-a^{-d-1}\right]$$ – finite

known d-dependence, Ambjørn + Wolfram, 1983

An example: the circle of radius a

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}-\frac{\omega^2}{c^2}\right)u_\omega=0,\quad u_\omega\Big|_a=f_\omega(\varphi)$$

The solution is

$$\begin{cases} u_\omega=\displaystyle\sum_{n=-\infty}^{\infty}\frac{I_n(|\omega|\rho/c)}{I_n(|\omega|a/c)}f_{\omega n}e^{in\varphi}, & \rho\le a\\ u_\omega=\displaystyle\sum_{n=-\infty}^{\infty}\frac{K_n(|\omega|\rho/c)}{K_n(|\omega|a/c)}f_{\omega n}e^{in\varphi}, & \rho>a\end{cases}$$

where $I_n(z)$ and $K_n(z)$ are modified Bessel functions

$$S=\frac{\pi\hbar}{T}\sum_{\omega,n}\frac{|f_{\omega n}|^2}{I_n(|\omega|a/c)K_n(|\omega|a/c)}$$ – Gaussian

$$\mathcal{E}=\frac{\hbar c}{2\pi}\sum_{n=-\infty}^{\infty}\int_0^\infty d\kappa\,\ln\big(I_n(\kappa a)K_n(\kappa a)\big)$$

divergent

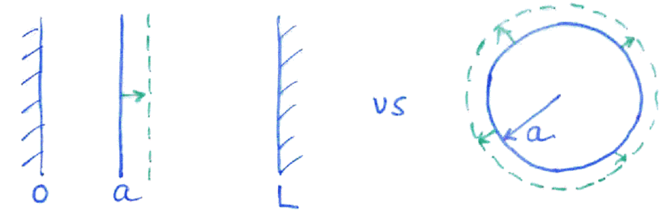
$$P=-\frac{\partial\mathcal{E}}{2\pi a\,\partial a}=-\frac{\hbar c}{4\pi^2a^3}\sum_{n=-\infty}^{\infty}\int_0^\infty x\,dx\,\frac{d}{dx}\ln\big(I_n(x)K_n(x)\big)$$

Sen, 1981

Divergences exist in any curved geometry!

Why? Largely (but not entirely) due to infinite self-energy (per unit area) of a boundary (Deutsch+Candelas, 1979; Jaffe et. al., 2002+, Barton, 2004). How come?

The force is the energy change per infinitesimal displacement:



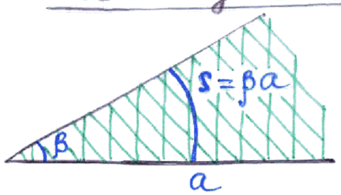
planar: the area does not change, the force is finite

curved: the area changes, the force is infinite

Our contention: geometry-dependent "infinities", sometime "removed" and sometime not, are finite non-universal pieces of the effect having geometrical origin and measurable consequences.

Test: need a curved geometry where various geometrical and universal pieces of the effect can be unambiguosly separated; ideally a system characterized by more than one length scale so that the area of the boundary is independent of its curvature.

The wedge-arc geometry in 2d



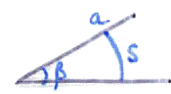
Circular Dirichlet arc with Dirichlet $u(\rho,0)=u(\rho,\beta)=0$, or periodic, $u(\rho,\varphi)=u(\rho,\varphi+\beta)$, edges.

- close relative of the circle problem; periodic version includes the circle as a special case $\beta=2\pi$
- if $s=\beta a$ is kept fixed, the answer is expected to be universal (almost true)
- Both the force and torque exist here.

Solution employs:

- general cutoff function reflecting material's transparency to high-energy modes
- extremely useful and accurate Debye expansion (Milton+DeRaad+Schwinger, 1978).

$\beta = s/a \ll 1$



line tension universal log-accuracy

Dirichlet: $\mathcal{E} = \alpha\cdot s - \frac{\hbar c s}{256\pi a^2}\ln\frac{\omega_0 s}{c} + \frac{\pi\hbar c}{48 s}$

non-universal nearly-universal universal

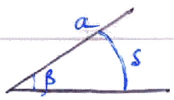
Periodic: $\mathcal{E} = \alpha\cdot s - \frac{\hbar c s}{256\pi a^2}\ln\frac{\omega_0 s}{c} + \frac{\pi\hbar c}{12 s} - \frac{0.08808\,\hbar c}{2\pi a}$

$\alpha \simeq \frac{\hbar\omega_0}{c/\omega_0} = \frac{\hbar\omega_0^2}{c}$

can be given exactly

for $s=2\pi a$ and exponential cutoff function this was first given by Sen, 1981

Solution again:



$$\mathcal{D}: \quad \mathcal{E} = \alpha\cdot s - \frac{\hbar c s}{256\pi a^2}\ln\frac{\omega_0 s}{c} + \frac{\pi\hbar c}{48 s}$$

$$\frac{\hbar\omega_0^2}{c}$$

$$P: \quad \mathcal{E} = \alpha\, s - \frac{\hbar c s}{256\pi a^2}\ln\frac{\omega_0 s}{c} + \frac{\pi\hbar c}{12 s} - \frac{0.08808\,\hbar c}{2\pi a}$$

Summary of properties

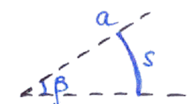
Contribution \ Property	sensitive to topology	additive
non-universal	no	yes
nearly-universal	no	nearly
universal	yes	no

Why are the non-universal terms insensitive to topology and (largerly) additive?

Because they are dominated by high energy localized modes – they are (almost) local properties of the boundary.

Geometrical interpretation of non-universal contributions

$$\mathcal{E}(\omega_0) = \underbrace{\overset{\hbar\omega_0^2/c}{\alpha}\cdot s}_{\text{line energy}} - \underbrace{\frac{\hbar c s}{256\pi a^2}\ln\frac{\omega_0 s}{c}}_{\text{How to understand this?}}$$



has to be attributed to arc itself.

- Finite-size correction to line tension

$$\alpha \to \alpha - \frac{\hbar c}{256\pi a^2}\ln\frac{\omega_0 s}{c} \quad \text{or}$$

- Marginally non-local contribution to curvature (bending) rigidity

$K = \frac{1}{a}$ – local curvature

$$\left(-\frac{\hbar c}{256\pi a^2}\ln\frac{\omega_0 s}{c}\right)s = \left(-\frac{\hbar c}{256\pi}K^2\ln\frac{\omega_0 s}{c}\right)s$$

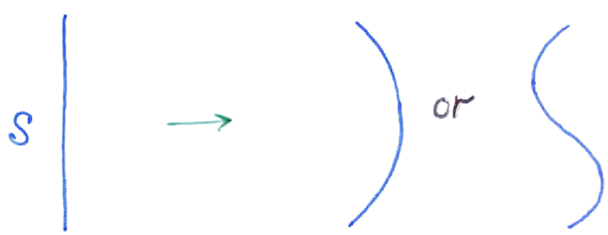
Arbitrary smooth boundary

$$\mathcal{E}(\omega_0) = \alpha\cdot s - \frac{\hbar c s}{256\pi} K^2 \ln\frac{\omega_0 s}{c}$$

For infinitesimally small s this looks almost like a differential of $\mathcal{E}(\omega_0)$. Integrate over the boundary and fix the log by hand:

$\int ds$

$$\mathcal{E}(\omega_0) = \alpha\cdot S - \underbrace{\left(\frac{\hbar c}{256\pi}\int K^2(s)\,ds\right)\ln\frac{\omega_0 S}{c}}_{\substack{\text{Casimir wrinkling of}\\ \text{sufficiently long straight}\\ \text{fixed length boundary}}} \quad \text{(Sen, 1981)}$$



Conclusions

- Casimir energy of a smooth boundary is generally non-universal.
- "Infinities" removed by formal methods are either irrelevant geometry-independent constants or measurable local contributions into elastic properties of the boundary.
- "Infinities" that cannot be removed by formal techniques represent non-local (logarithmic in ω_0) contributions into boundary's elasticity. This happens for a spherical shell in even dimensions (Sen almost mentions this as a possibility in 2d).
- These issues do not exist when the Casimir force is not due to shape change.

How to see that even dimensions are special?

For a sphere of radius a the non-universal part of the effect must have the form

even function of curvature

$$\mathcal{E}(\omega_0) = \int \epsilon(K)\, d\sigma = \int \sum_{n=0}^{N} \gamma_n K^{2n} d\sigma \simeq$$

$$\simeq \sum_{n=0}^{N} \gamma_n a^{-2n} a^{d-1} = \sum_{n=0}^{N} \gamma_n a^{d-1-2n}$$

The rigidity constants γ_n have dimension of energy/(length)$^{d-1-2n}$ and they are determined by microscopic physics, i.e. by $\hbar\omega_0$ and c/ω_0 (microscopic energy + length scales)

$$\gamma_n \simeq \frac{\hbar\omega_0}{\left(\frac{c}{\omega_0}\right)^{d-1-2n}} = \hbar c \left(\frac{\omega_0}{c}\right)^{d-2n}$$

$$\mathcal{E}(\omega_0) \simeq \hbar c \sum_{n=0}^{N} \left(\frac{\omega_0}{c}\right)^{d-2n} a^{d-1-2n} -$$

more restrictive than dimensional analysis implies

$d-2N \geq 0$ fixes the # of terms.

For d even there are $\frac{d}{2}+1$ terms and the last one is ω_0-independent → marginal case → deviation from locality.