Fermionic and Scalar Casimir Effect within the Scattering Approach

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Together with Aurel Bulgac (Seattle) and Piotr Magierski (Warsaw):

A. Bulgac & A.W., *Phys. Rev. Lett.* **87** (2001) 120404;
A. Bulgac, P. Magierski & A.W., *Europhys. Lett.* **72** (2005) 327;
A. Bulgac, P. Magierski & A.W., *Phys. Rev. D* **73** (2006) 025007;
A.W., A. Bulgac & P. Magierski, *J. Phys. A* **39** (2006) 6815;
A.W., *J. Phys. A* **41** (2008) 164003.

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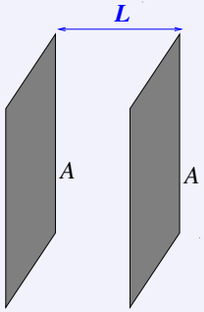

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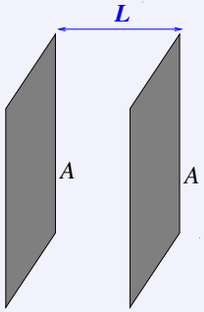

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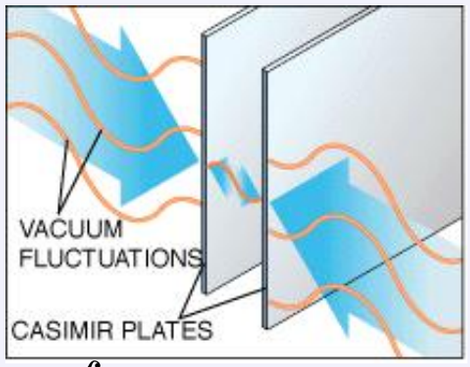

- Reason: zero-point fluctuations of the e.m. field modified by the presence of the plates relative to the free case

$\Rightarrow$ *change* in the energy of the vacuum: $\sum\!\!\!\!\!\!\int \frac{1}{2}\hbar\omega_k|_{\mathrm{plates}(L)} - \sum\!\!\!\!\!\!\int \frac{1}{2}\hbar\omega_k|_{\mathrm{free}(L\to\infty\,!)}$

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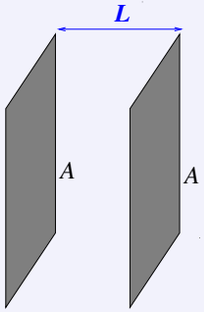

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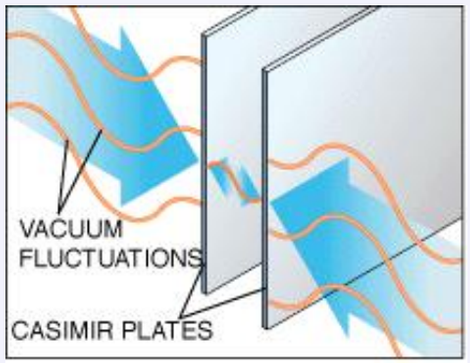

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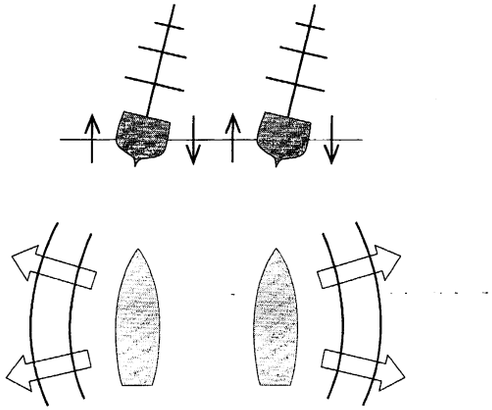

Figure 7.8 *Two nearby ships rolling in a swell of ocean waves. Some waves are excluded from the region between the ships and the ships are forced together by the higher wave pressure on their outer sides.*

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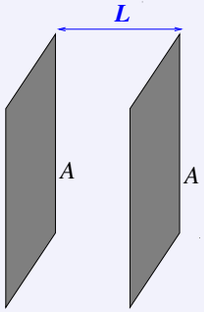

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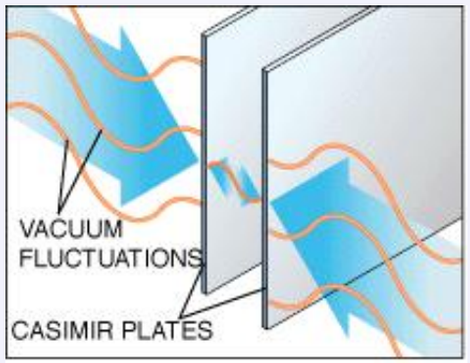

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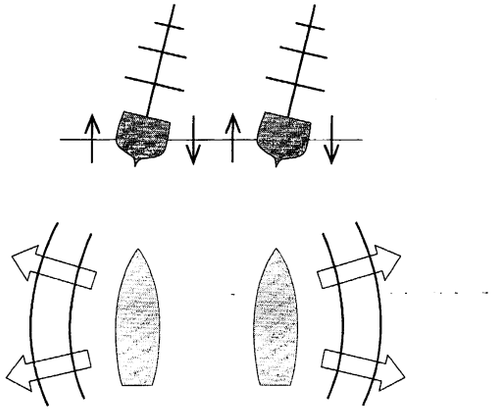

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- Distinctive Feature: Casimir calculations depend on the geometry in a non-intuitive way.

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- geom.-dep. b.c.'s (2 plates, 2 half-spheres, ...)

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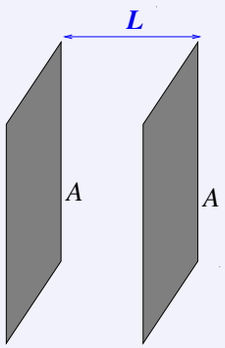

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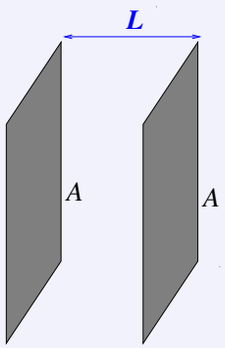

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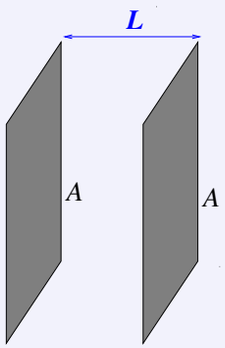

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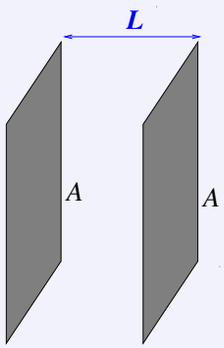

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$$= \lim_{\Lambda\to\infty} \hbar c A L \left[\underbrace{\frac{3(\Lambda/\hbar c)^4}{\pi^2}}_{\text{background}} + \underbrace{(-1+1)\frac{(\Lambda/\hbar c)^3}{4\pi L}}_{\text{bulk}} - \underbrace{\frac{\pi^2}{720 L^4}}_{\epsilon_C^{||{\rm EM}}(L)} + \mathcal{O}\left((\hbar c)^2/\Lambda^2 L^6\right) \right]$$

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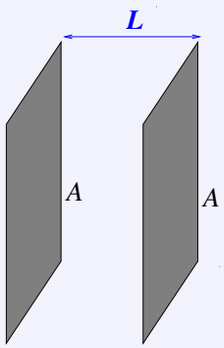

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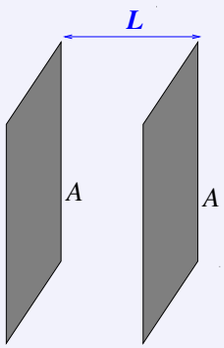

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- UV-cutoff Λ corresponds to plasma frequency.
- When the (infinitely) far separated plates are brought into a finite distance L, the background terms as well as the bulk terms of the single plates are unaffected and therefore removable $\Rightarrow$ the Casimir energy is finite, even in the limit $\Lambda \to \infty$.

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where $$\mathcal{N}_C(E) = \int_0^E dE'\, \delta\rho_C(E')$$ (*geom-dep.* part of the integrated d.o.s)

“Generalization” of concept of Casimir energy: *2) matter fields*

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- *Here*: space not "filled" with *fluctuating* EM modes, but with a gas of *non-interacting* (non-relativistic) fermions.

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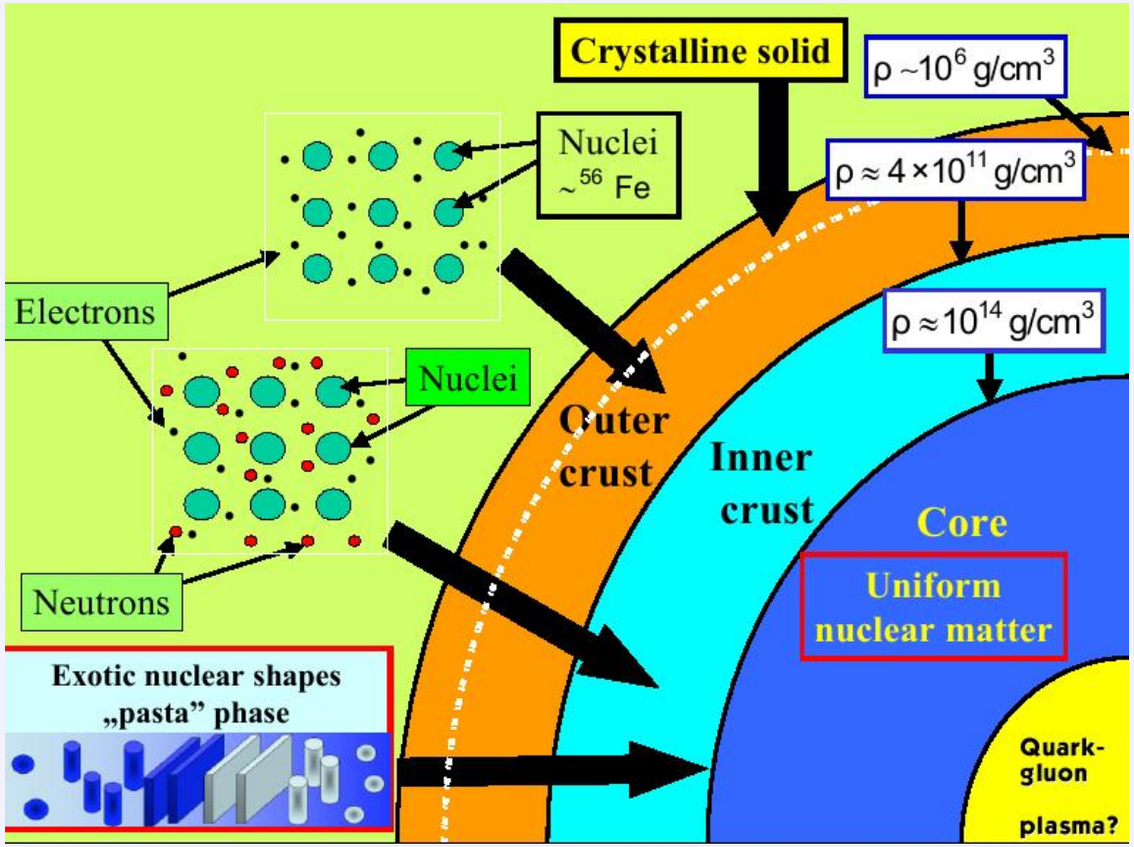
$\therefore$ effective *Casimir-type interaction* between empty regions of space in the background of *non-interacting* fermionic fields

Applications In the inner crust of a neutron star:



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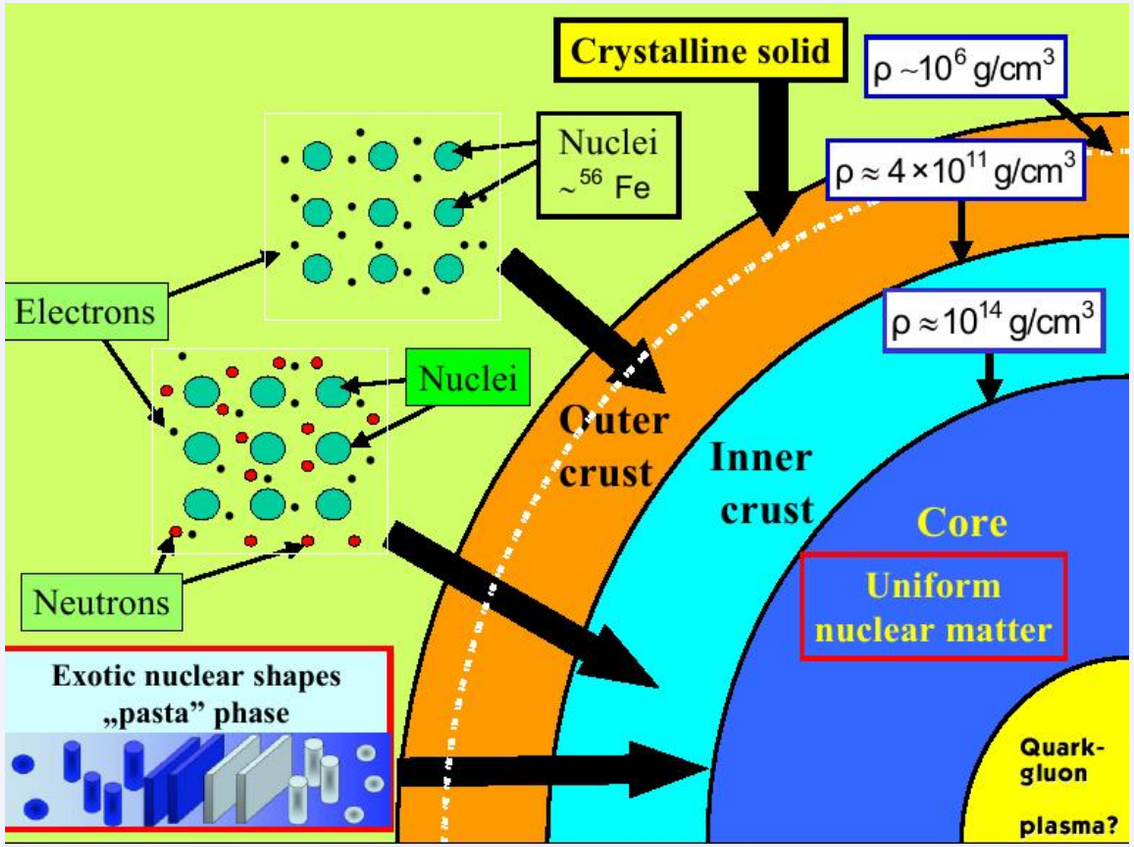

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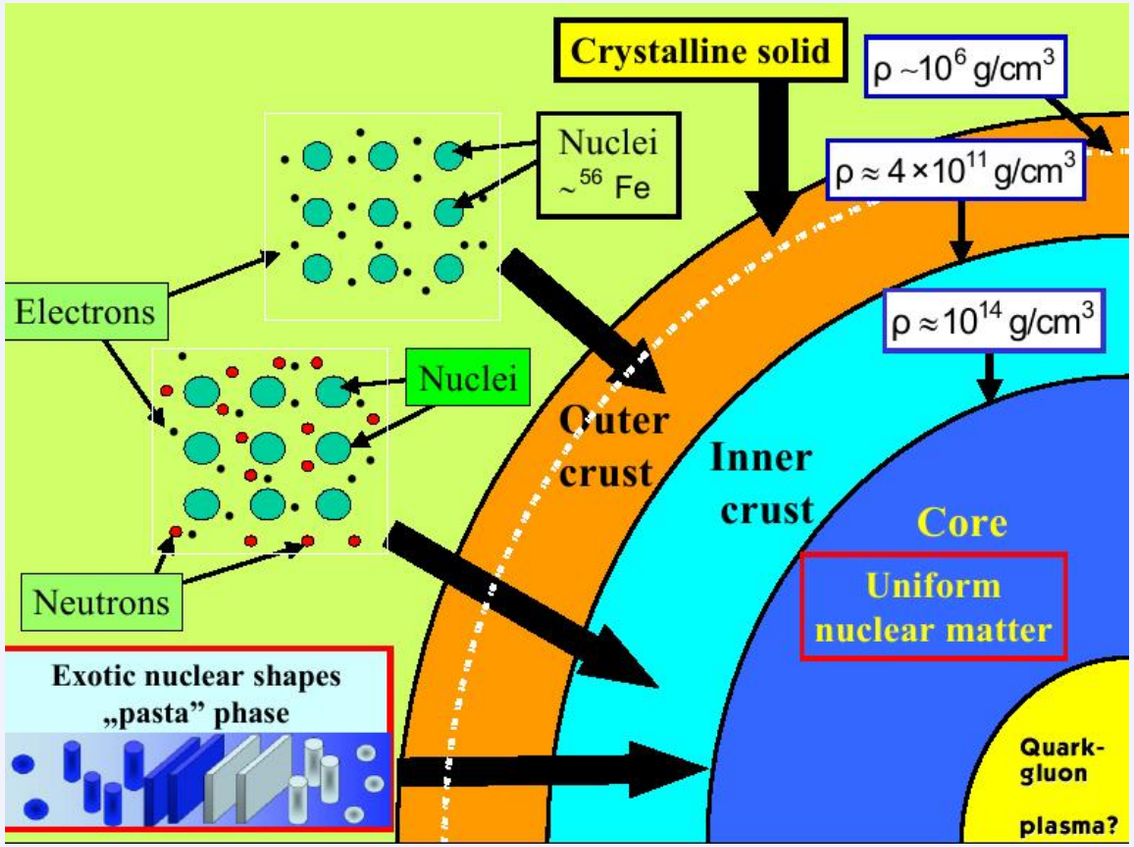

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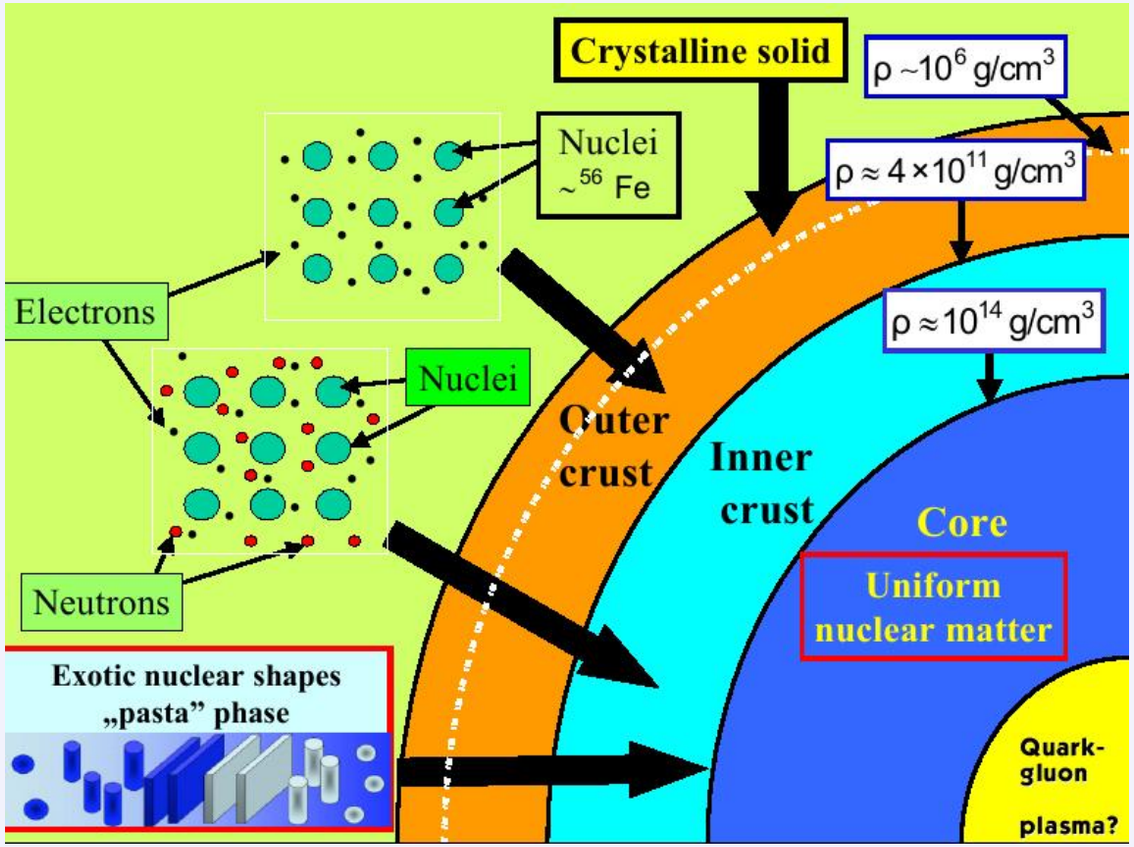

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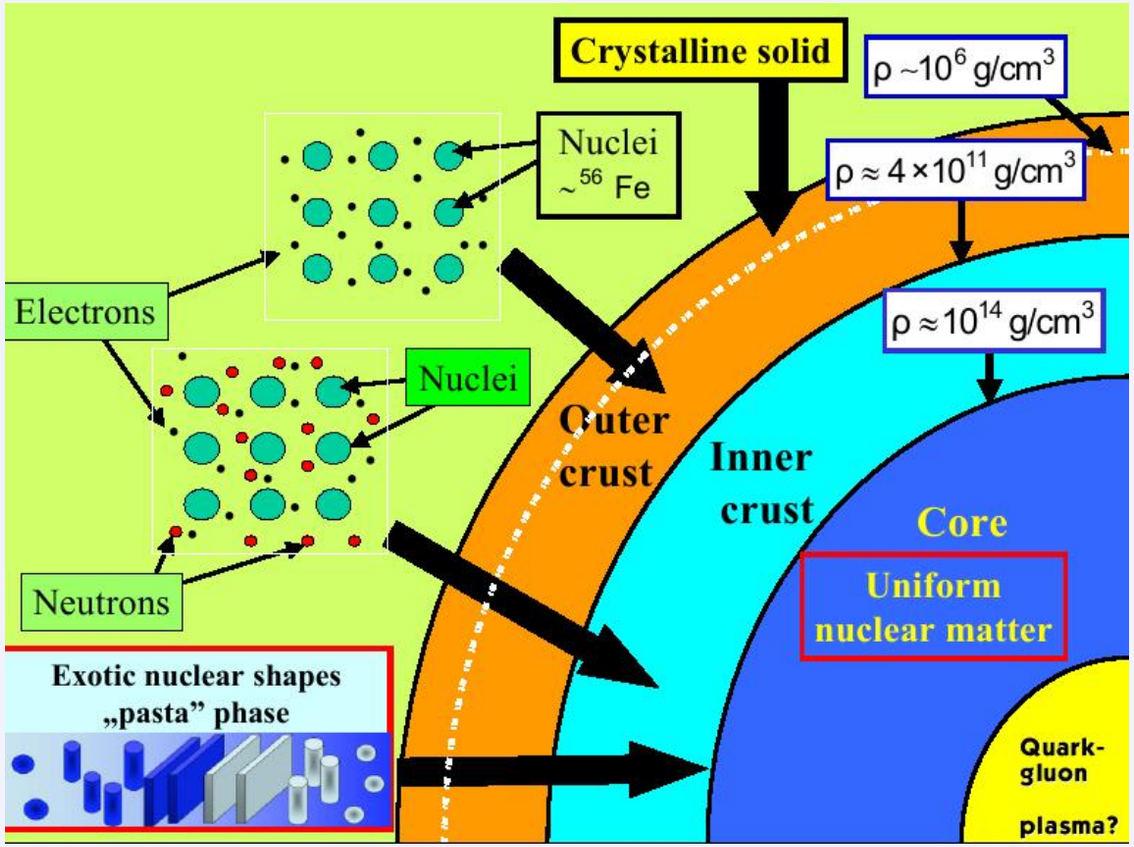

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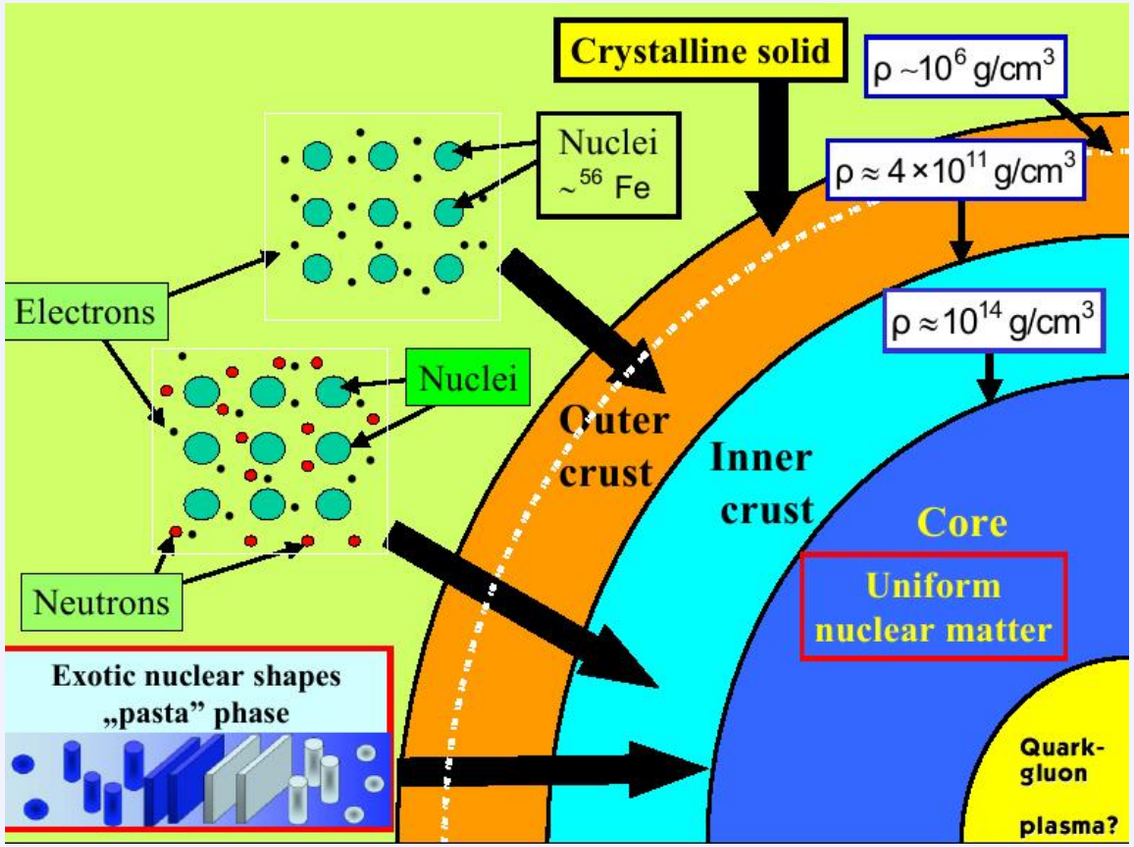

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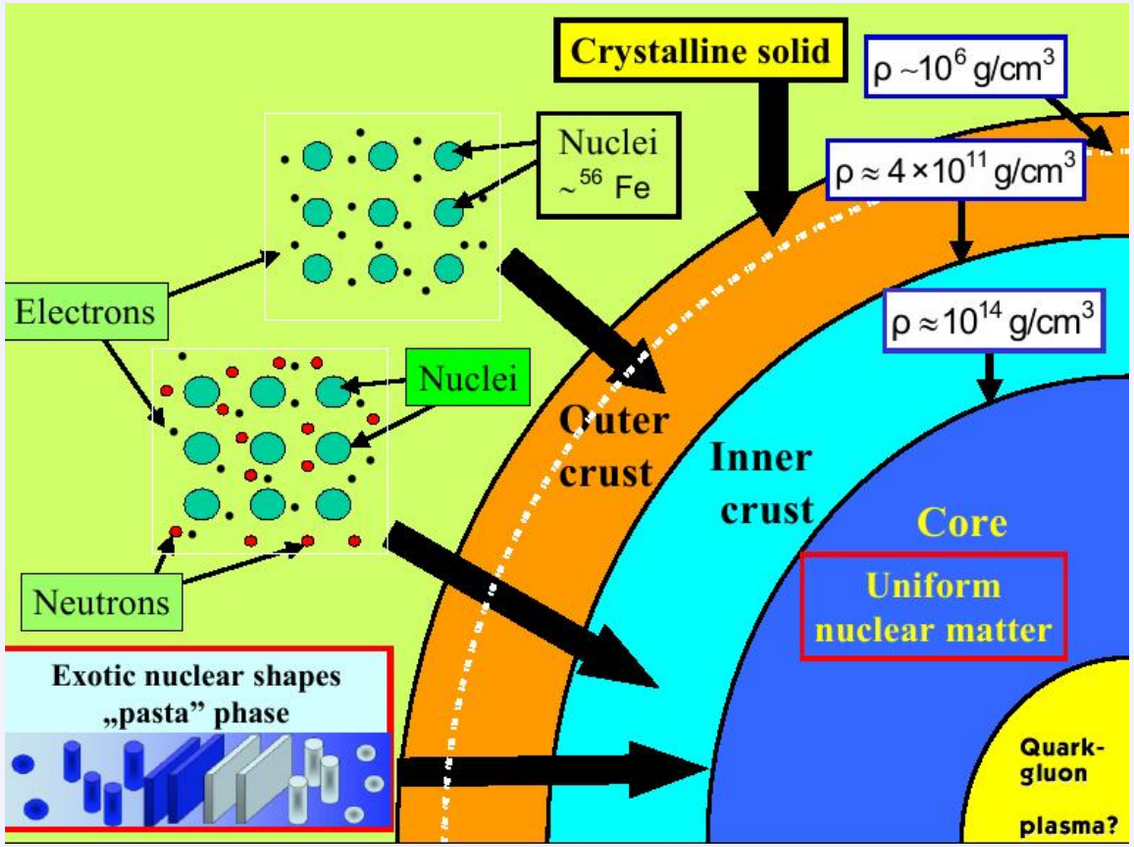

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Calculation

- Casimir energy for fermions between two impenetrable (parallel) planes at distance L:

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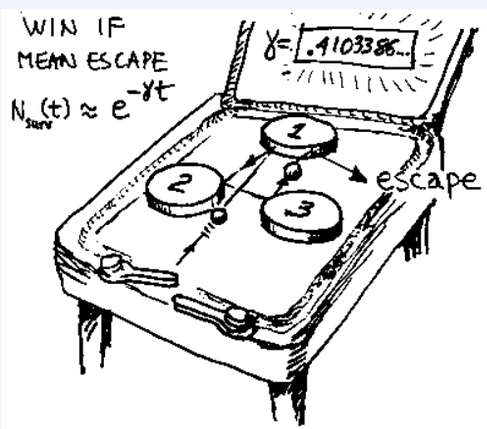
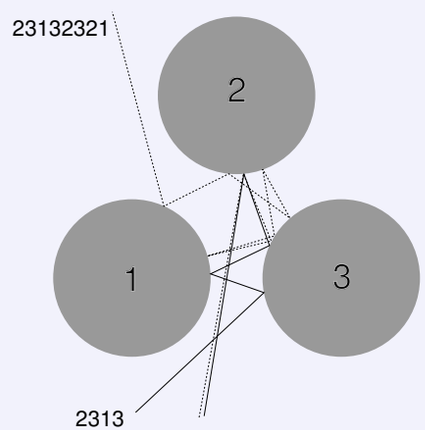
    $\Rightarrow$ Calculation mapped onto a quantum mechanical *billiard* problem:
    hyperbolic point-particle scattering off *n* spheres or *n* disks





References:
B. Eckhardt, *J. Phys.* **A20** (1987);
P. Gaspard & S. Rice, *J. Chem. Phys.* **90** (1989);
P. Cvitanović & B. Eckhardt, *Phys. Rev. Lett.* **63** (1989);
M. Henseler, A. Wirzba & T. Guhr, *Ann. Phys.* **258** (1997).

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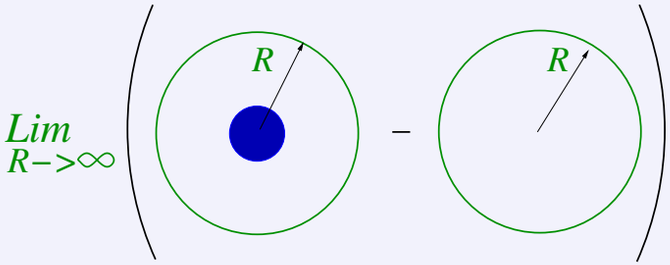

Asymptotically:

$$u_{k\ell}(r) \sim \sin\left(kr - \tfrac{1}{2}\,\ell\pi + \eta_\ell(k)\right)$$
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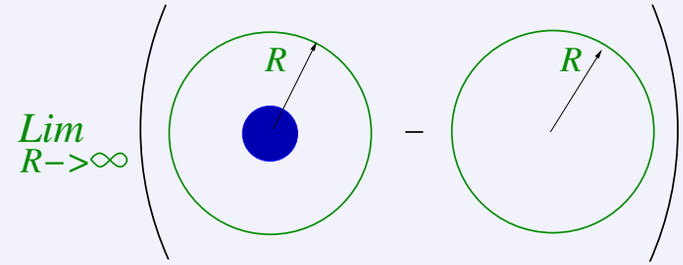

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and Dirichlet b.c.'s:
$u_{k\ell}(R) = u^{(0)}_{k\ell}(R) = 0.$

$\Rightarrow$ EV conditions ((2ℓ+1)-fold degenerate) :

$$k_{\ell,n}R - \tfrac{1}{2}\ell\pi + \eta_\ell(k_{\ell,n}) = \pi n, \quad n=0,1,2,\cdots \quad \text{(with potential)}$$
$$k^{(0)}_{\ell,n}R - \tfrac{1}{2}\ell\pi = \pi n, \quad n=0,1,2,\cdots \quad \text{(without potential)}$$

A change of the radial quantum number n by one unit, for fixed angular momentum ℓ, implies

$$\Delta k_\ell \left(R + \frac{\partial}{\partial k}\eta_\ell(k)\right) = \pi = \Delta k^{(0)}_\ell\, R\,,$$

such that the conditions $\bar\rho_\ell(k)\Delta k_\ell = \bar\rho^{(0)}_\ell(k)\Delta k^{(0)}_\ell = 2\ell+1$ (note the averaging !)

induce the formula $\bar\rho_\ell(k) - \bar\rho^{(0)}_\ell(k) = \frac{2\ell+1}{\pi}\frac{\partial}{\partial k}\eta_\ell(k)$ (independently of R)

Digression 2: *Semiclassical interpretation of Krein formula*

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Determinant of scattering matrix is semiclassically a sum over periodic orbits (+Weyl terms)

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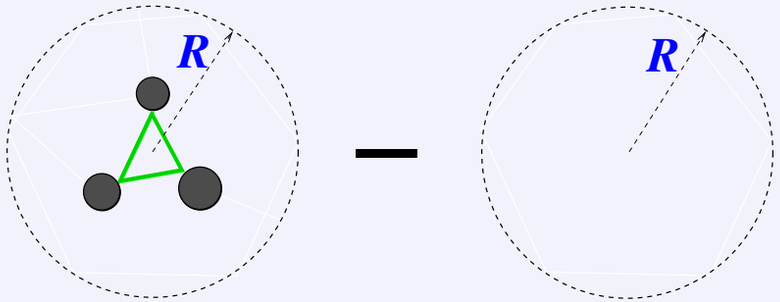

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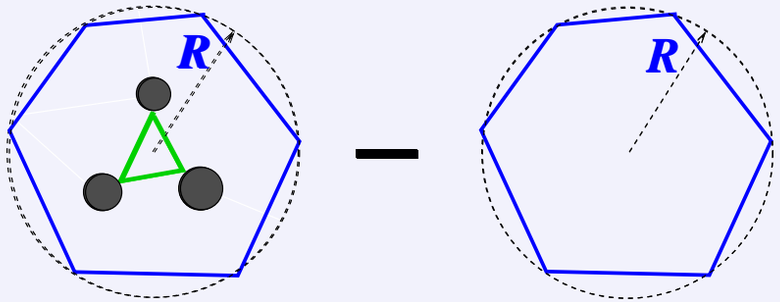

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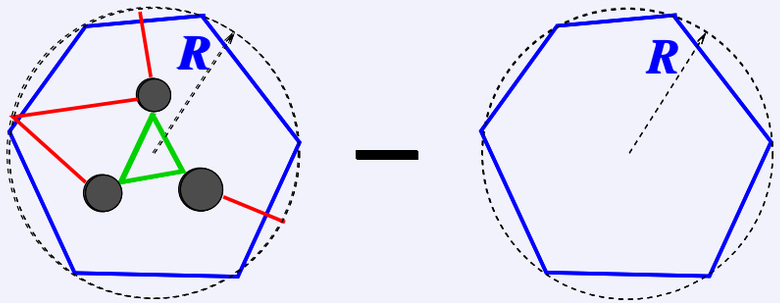

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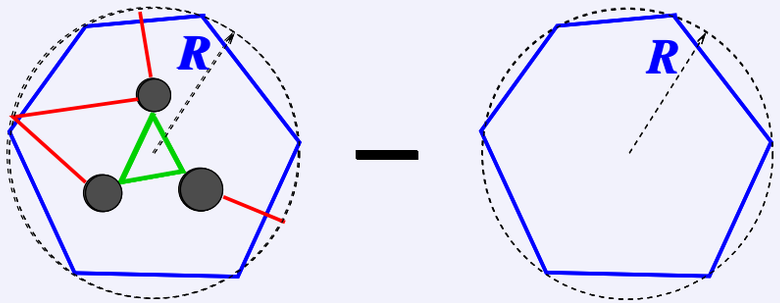

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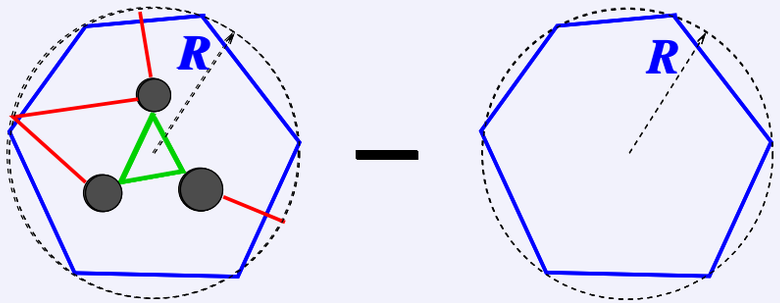

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Note the order of the limits!

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$\Rightarrow$ chem. potentials μ & μ_0 at *finite* and *infinite* separation of the scatterers differ:

$$\begin{aligned}
\mathcal{N} &= \int_0^{\mu} \rho(E)\, dE = \int_0^{\mu_0} [\rho_0(E) + \rho_W(E)]\, dE \\
\Rightarrow \quad & \int_{\mu_0}^{\mu} \rho(E)\, dE = -\int_0^{\mu_0} \underbrace{[\rho(E) - \rho_0(E) - \rho_W(E)]}_{\rho_C(E) = \frac{d}{dE}\mathcal{N}_C(E)}\, dE \\
\mathcal{E}_C|_{\mathcal{N}} &= \int_0^{\mu} E\rho(E)\, dE - \int_0^{\mu_0} E[\rho_0(E) + \rho_W(E)]\, dE
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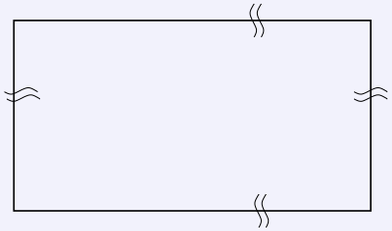
Grandcanonical Casimir energy at *fixed* $\mu = \mu_0$:

$$\widetilde{\mathcal{E}_C}|_{\mu_0} - \mu_0 \mathcal{N}_C(\mu_0) = \mathcal{E}_C|_{\mathcal{N}} + \mathcal{O}(V^{-1})$$

1. *“infinite” container:*



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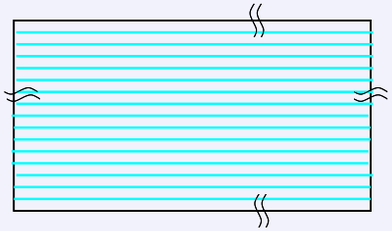

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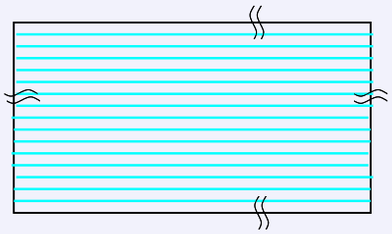

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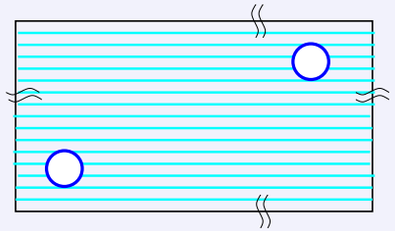
(note the excluded volume !)



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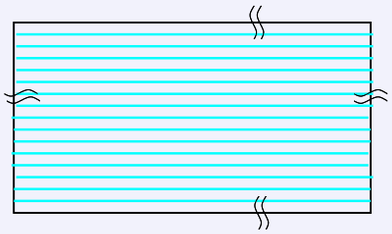

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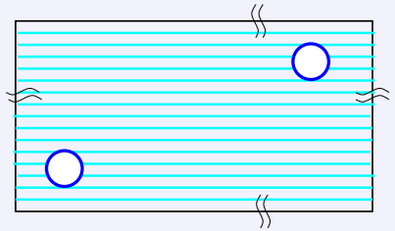

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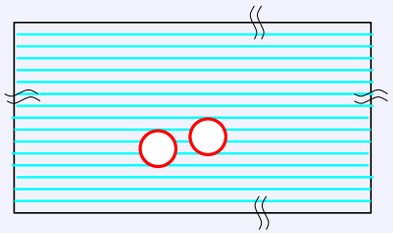

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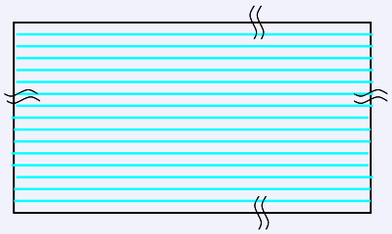

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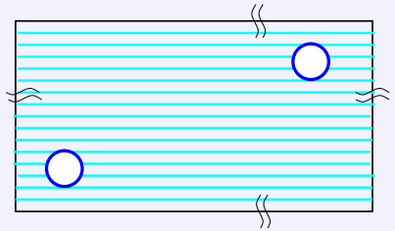

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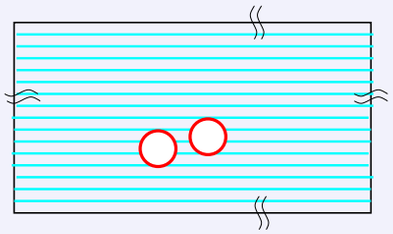

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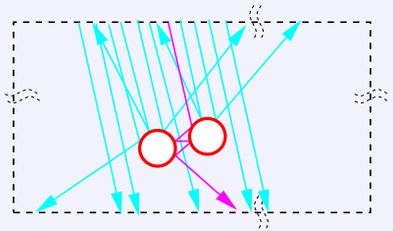

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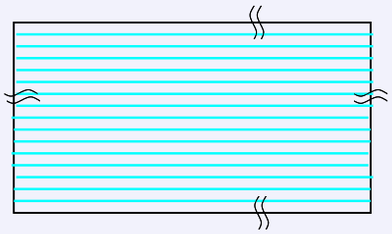

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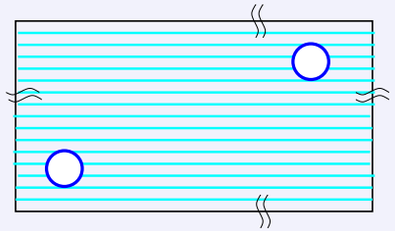

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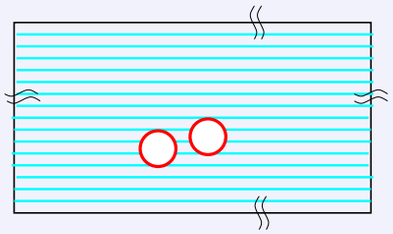

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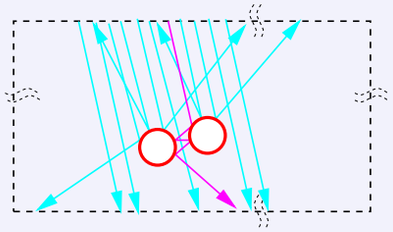

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see A.W., *Phys. Rep.* **309** (1999)

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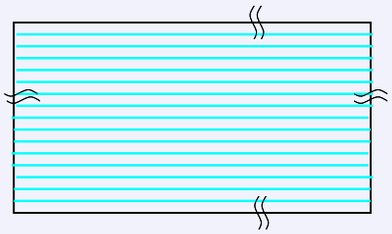

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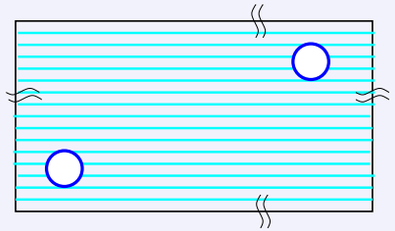

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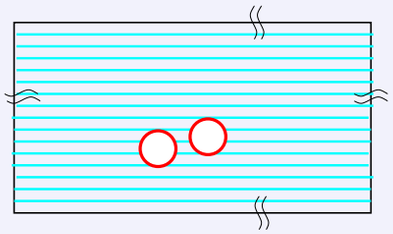

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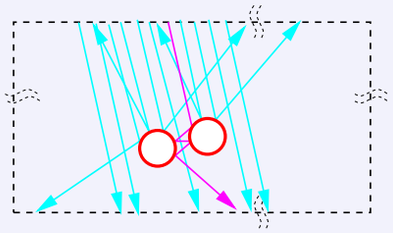

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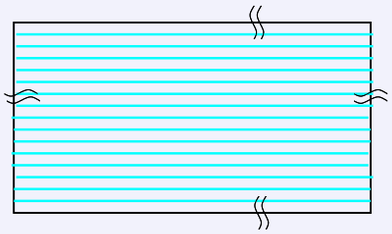

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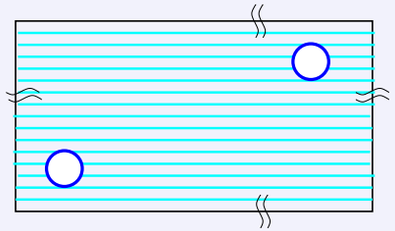

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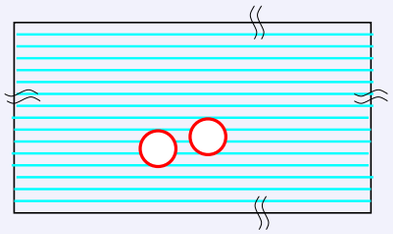

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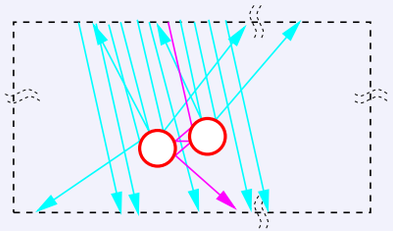

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All determinants exist (although the relevant scattering matrices are infinite dimensional) since the associated T-matrices are trace-class.

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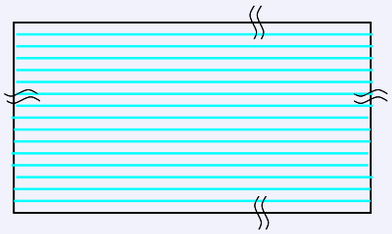

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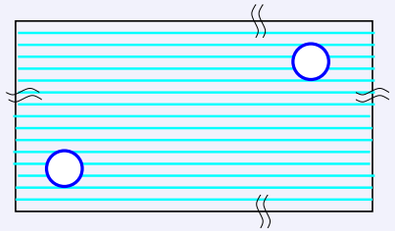

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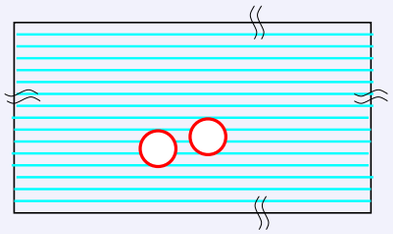

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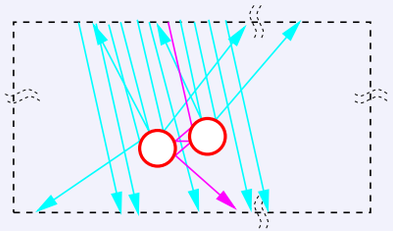

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6. Casimir energy:

$$\therefore \quad \mathcal{E}_C = \int_0^\mu dE\,(E-\mu)\,\delta\bar{\rho}_C = -\int_0^\mu dE\,\overline{\mathcal{N}}_C$$

Multi-scattering matrix for n spheres of radii a_j and distances $r_{jj'}$ $(j, j' = 1, 2, \ldots n)$

$$M^{jj'}_{lm,l'm'} = \delta^{jj'}\delta_{ll'}\delta_{mm'} + (1-\delta^{jj'})\, i^{2m+l'-l}\sqrt{4\pi(2l+1)(2l'+1)}\left(\frac{a_j}{a_{j'}}\right)^2 \frac{j_l(ka_j)}{h^{(1)}_{l'}(ka_{j'})}$$

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M. Henseler, A. Wirzba & T. Guhr, *Ann. Phys.* **258** (1997) 286.

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In the limit of small scatterers: $M^{jj'}(E) \approx \delta^{jj'} - (1-\delta^{jj'})\, \underbrace{f_j(E)}_{s\text{-wave}} \frac{\exp(ikr_{jj'})}{r_{jj'}}$ (+ p-wave)

Two spheres of radius a at distance r

A. Bulgac & A.W., *Phys. Rev. Lett.* **87** (2001) 120404

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In the limit of small scatterers:

$$M^{jj'}(E) \approx \delta^{jj'} - (1-\delta^{jj'})\, \underbrace{f_j(E)}_{s\text{-wave}}\, \frac{\exp(ikr_{jj'})}{r_{jj'}} \quad (+\ p\text{-wave})$$

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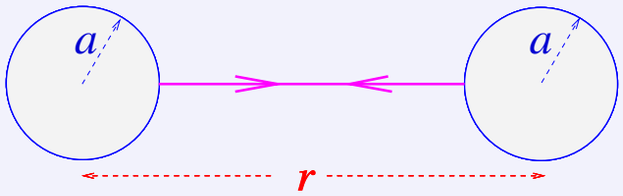

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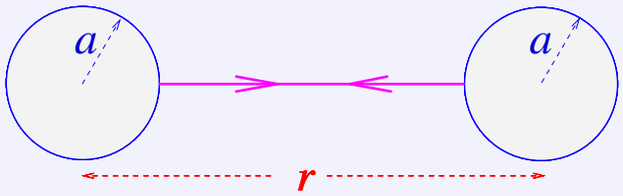

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whereas the *semiclassical* result (for the single *two-bounce periodic orbit* with no repetions) reads:

$$\mathcal{N}^{\rm oo}_{C,scl}(E) = \nu_{\rm deg}\frac{a^2}{4\pi r(r-2a)}\sin[\underbrace{2(r-2a)k}_{S_{po}(k)/\hbar}] \qquad \text{(Gutzwiller's \underline{trace} formula)}$$

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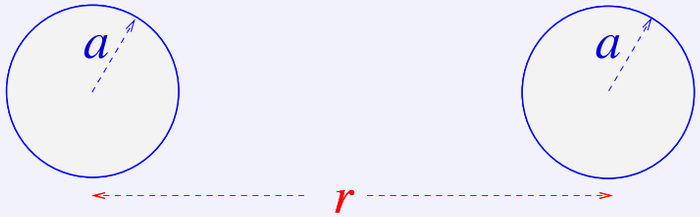

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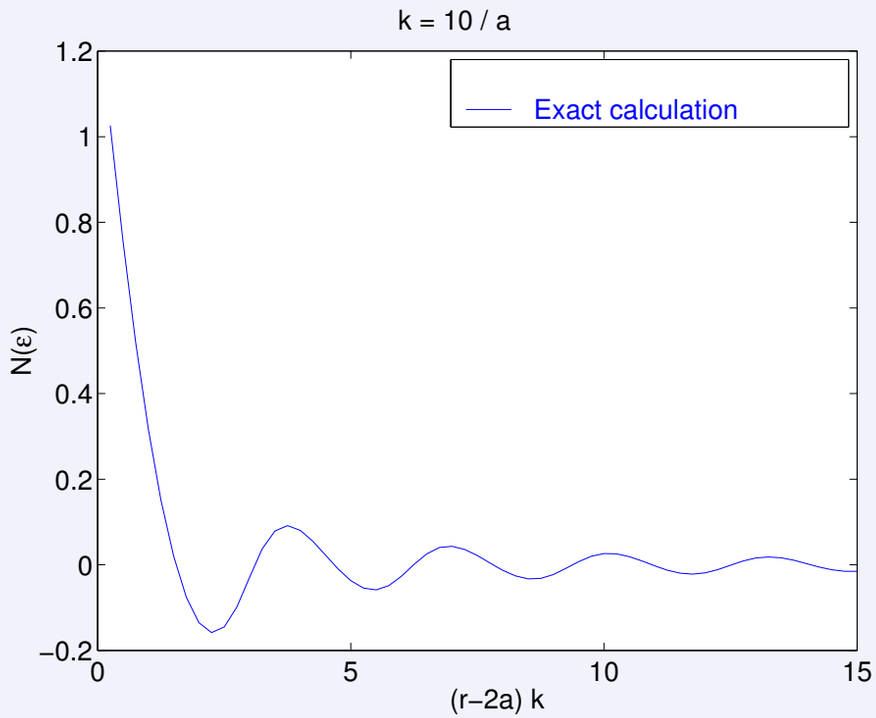

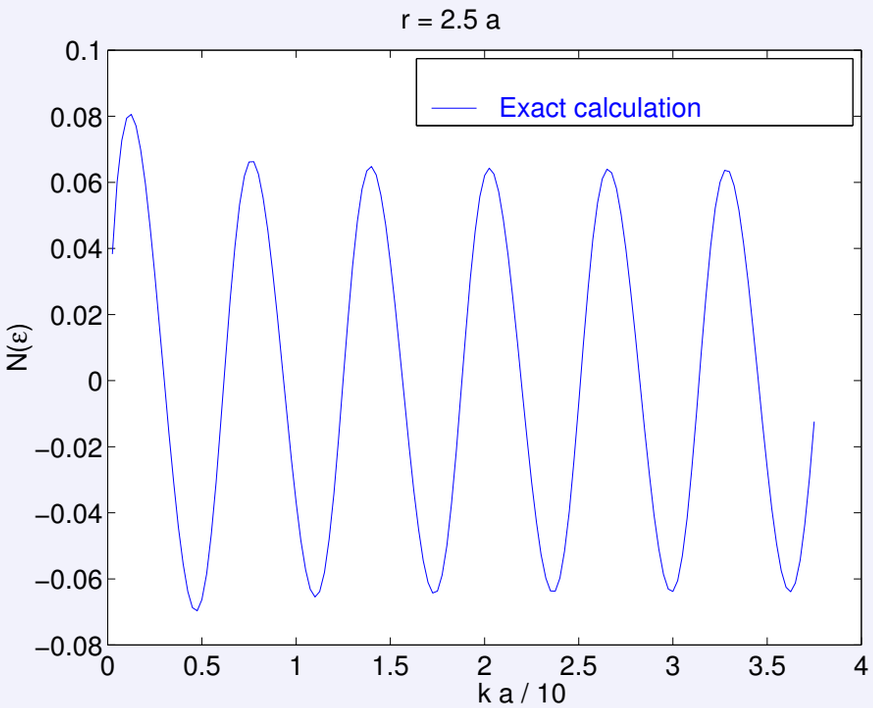

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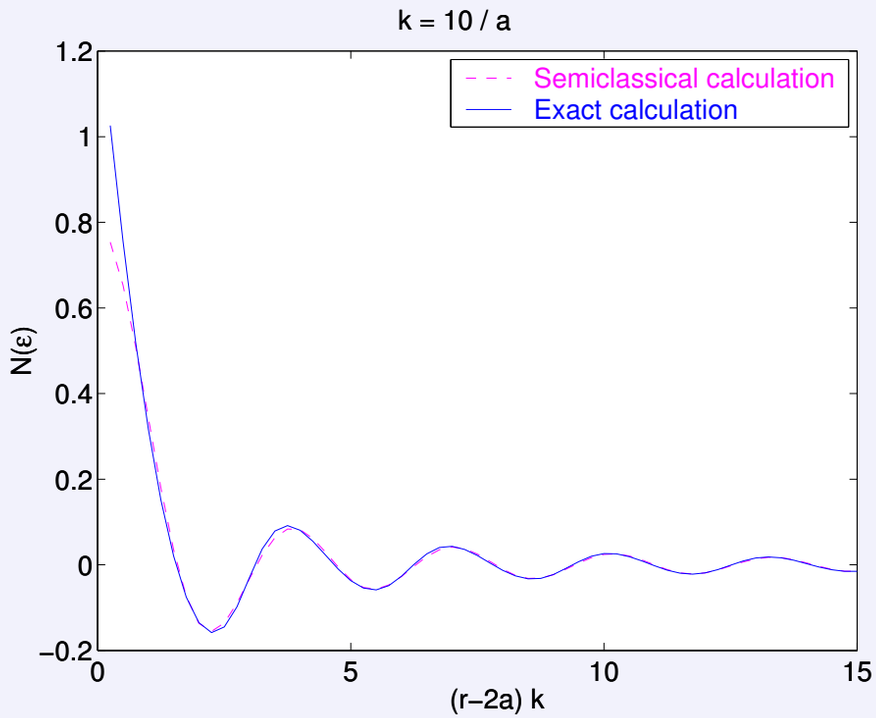

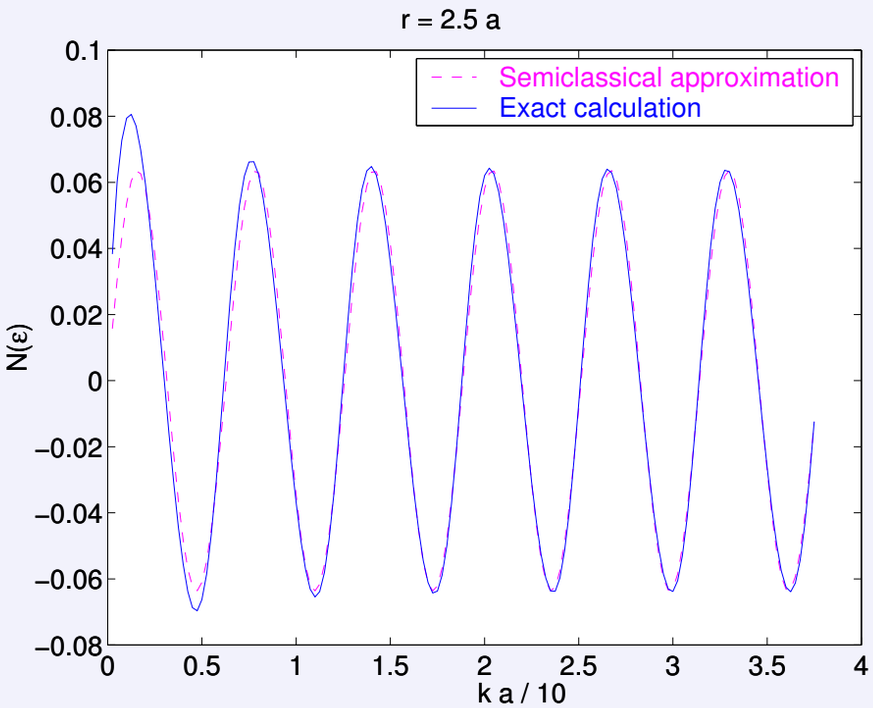

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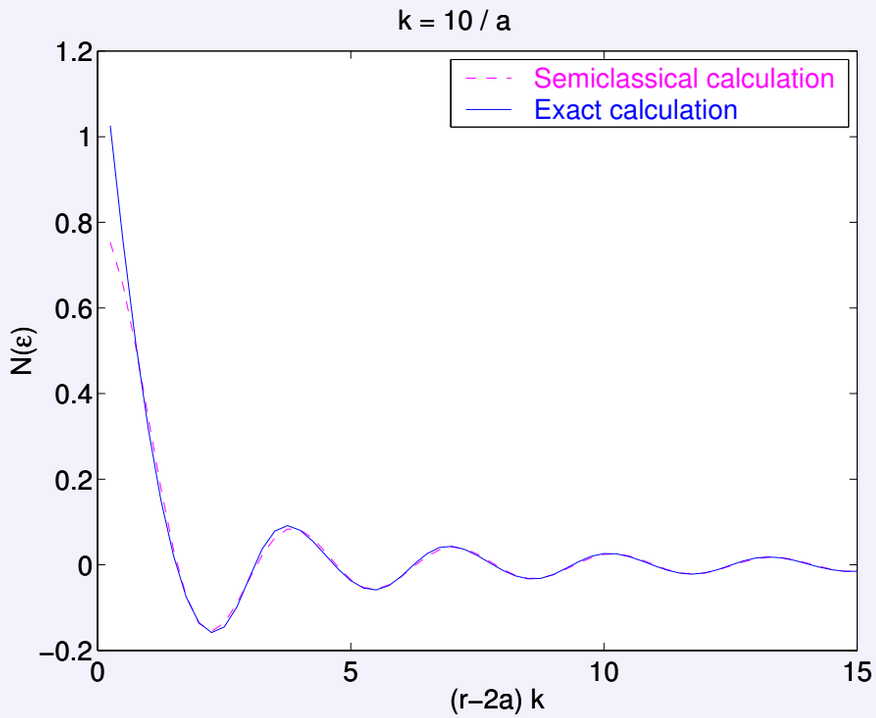

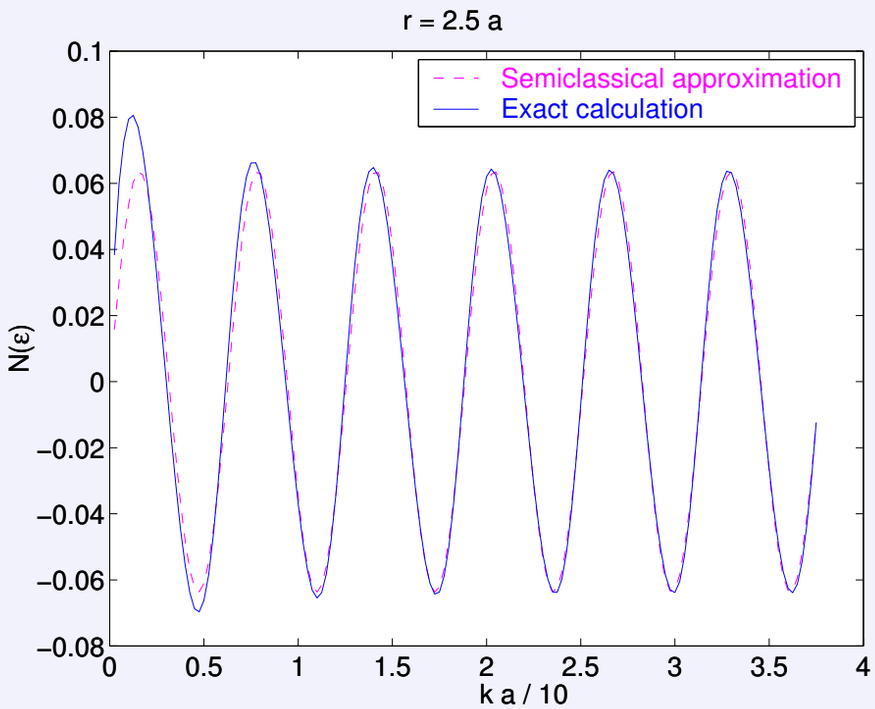

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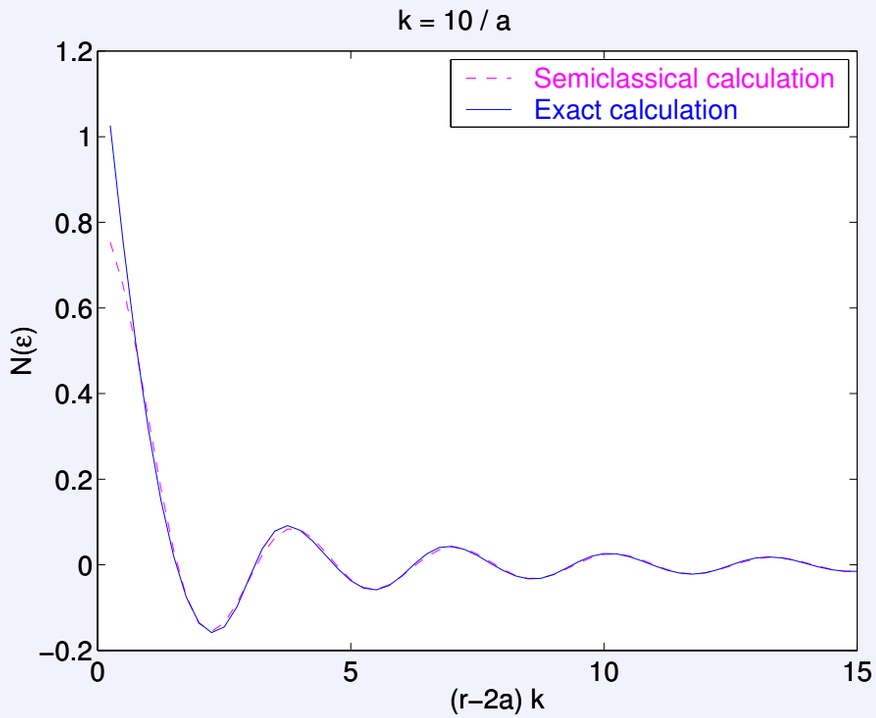

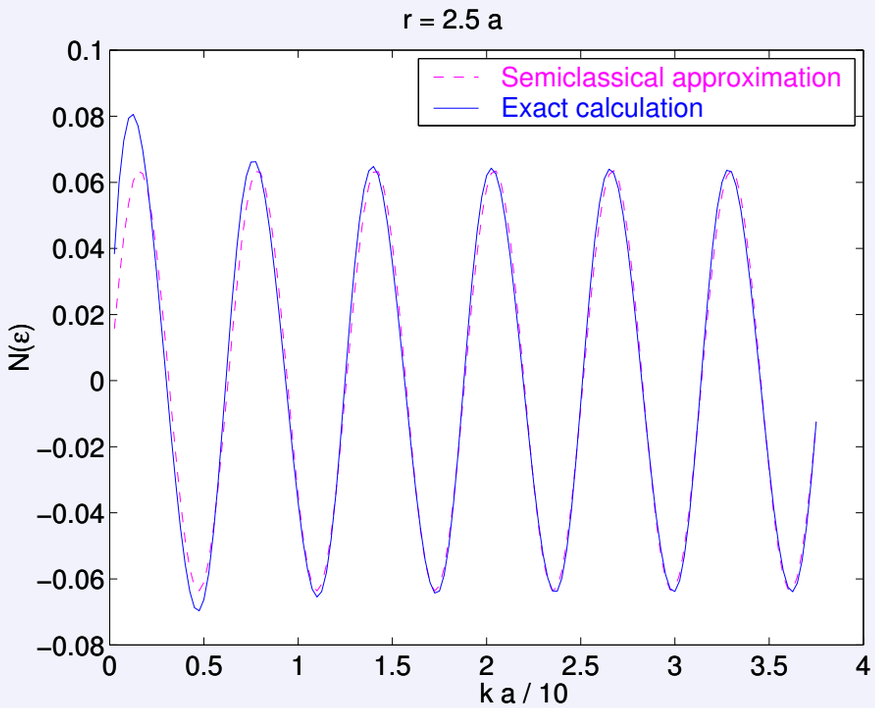

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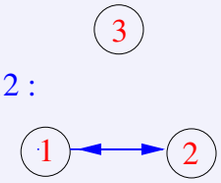

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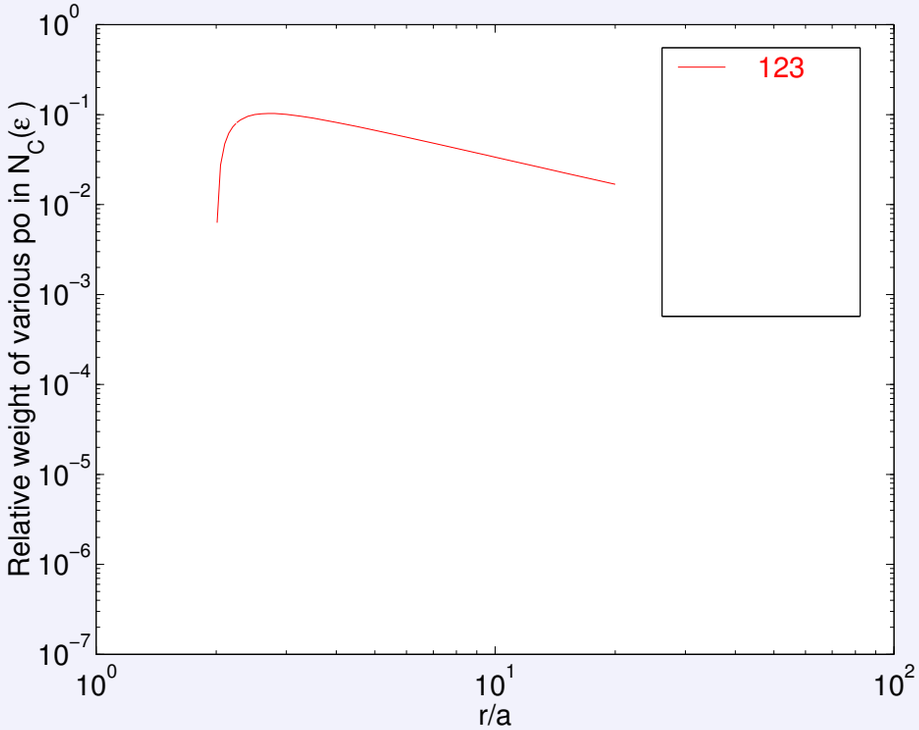

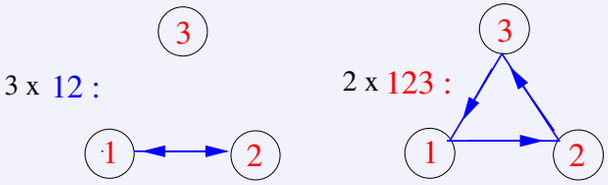

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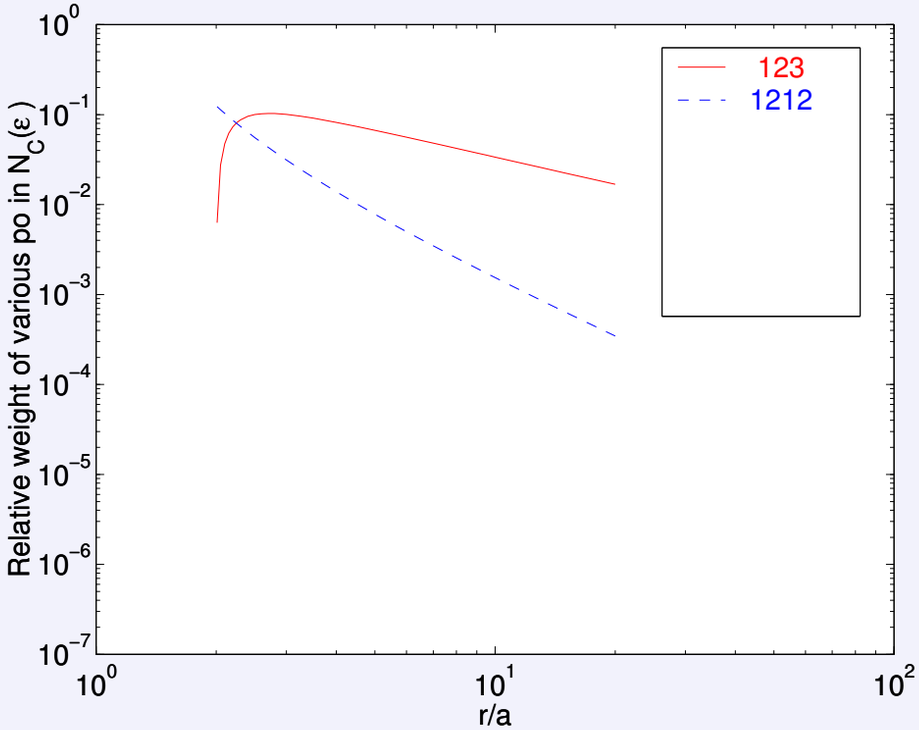

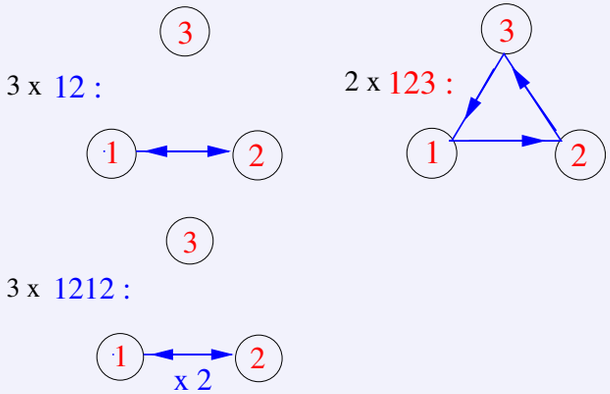

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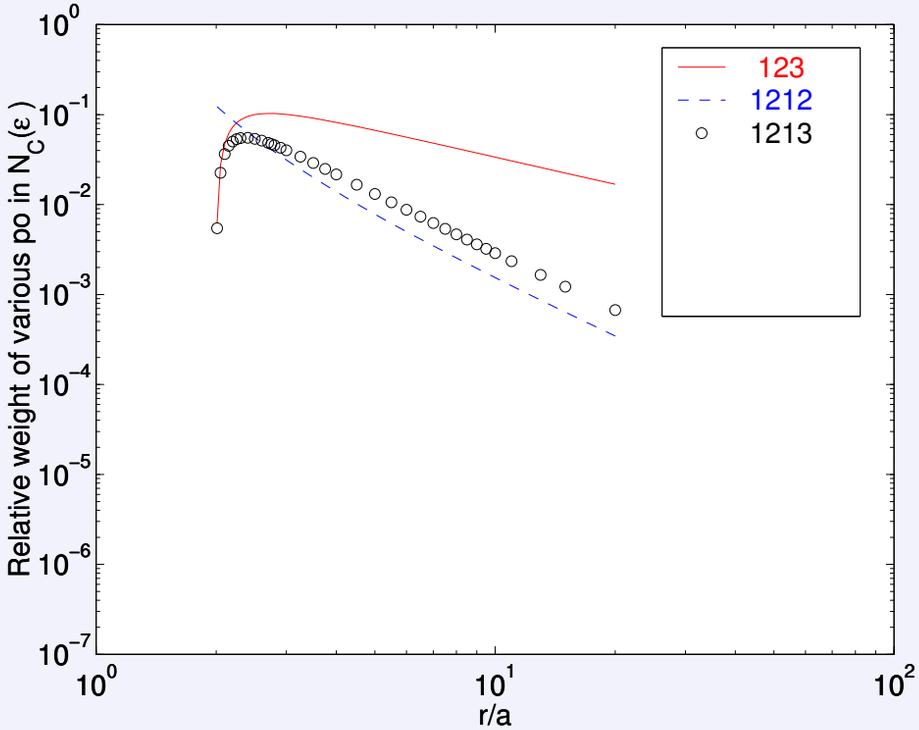

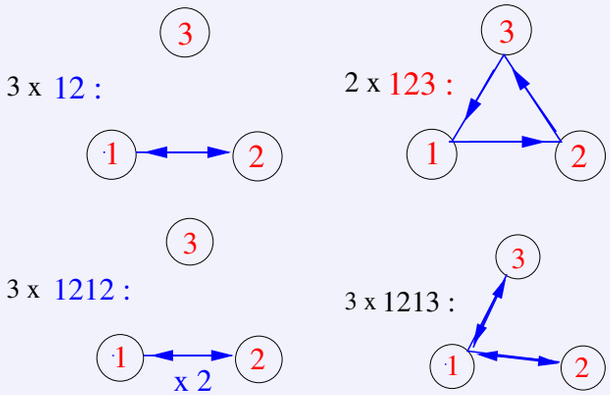

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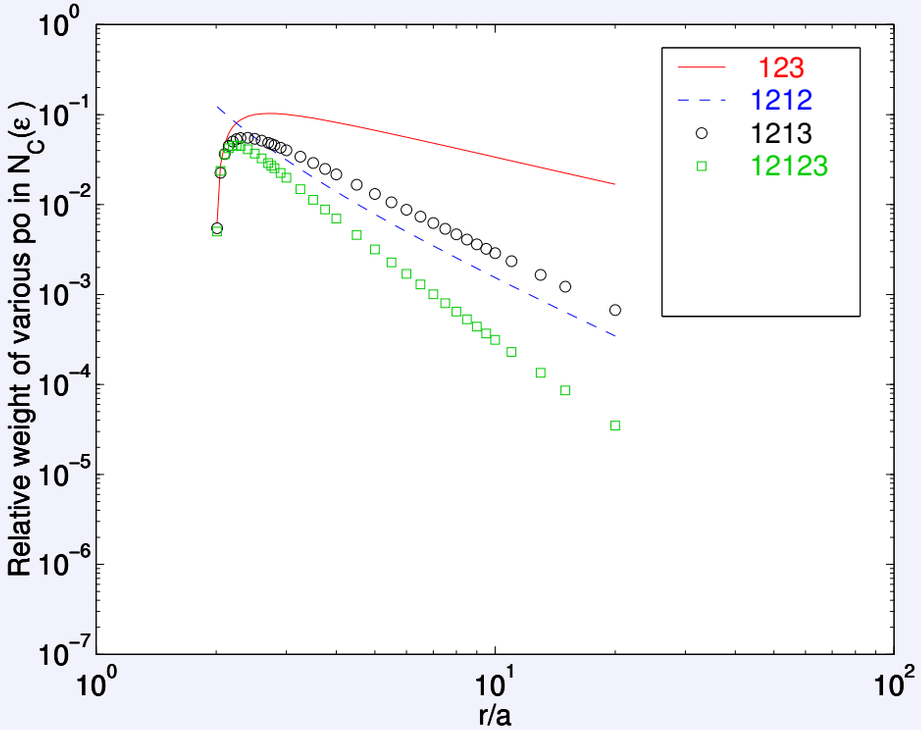

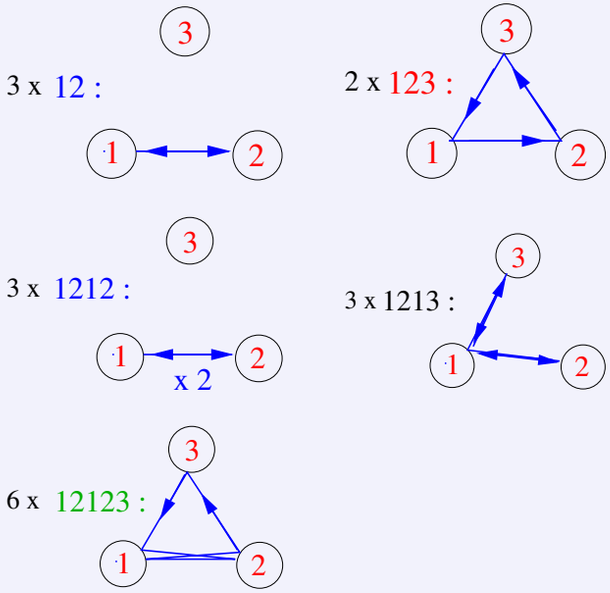

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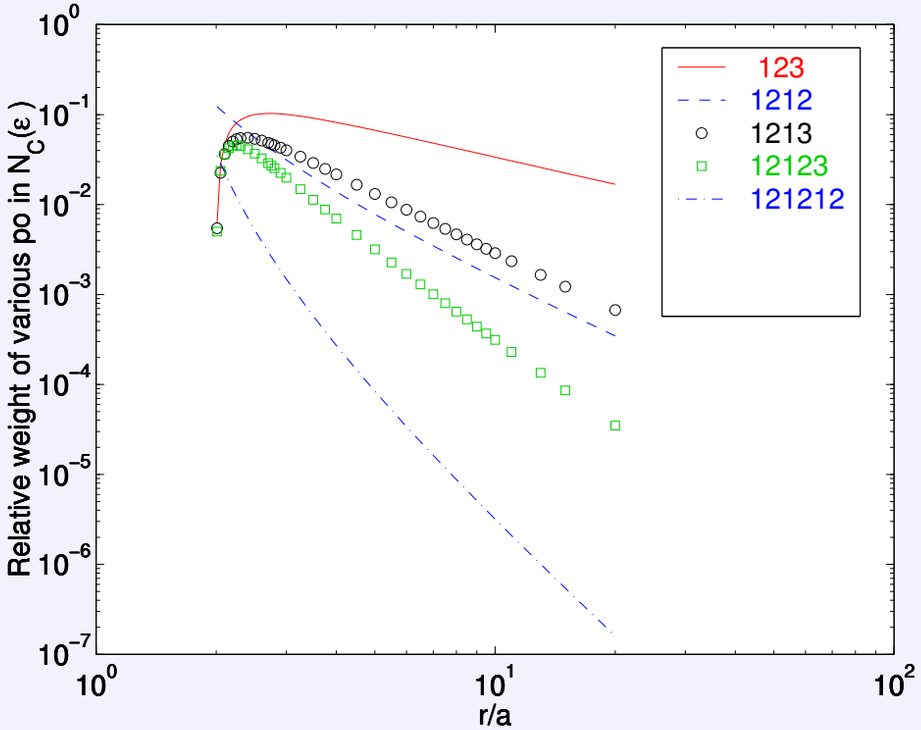

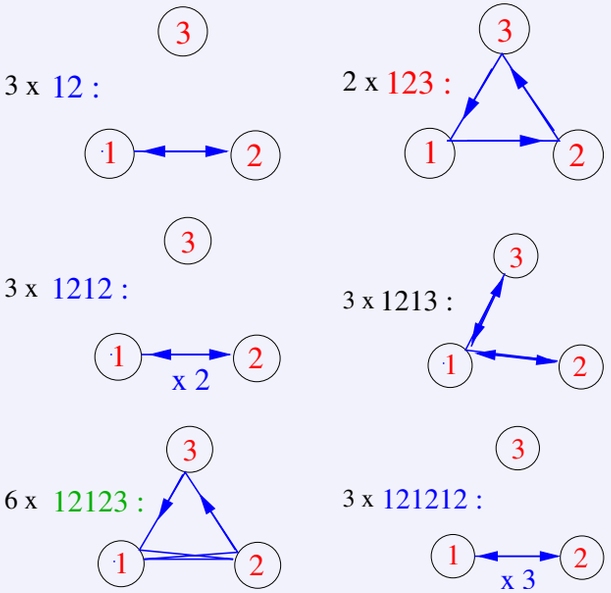

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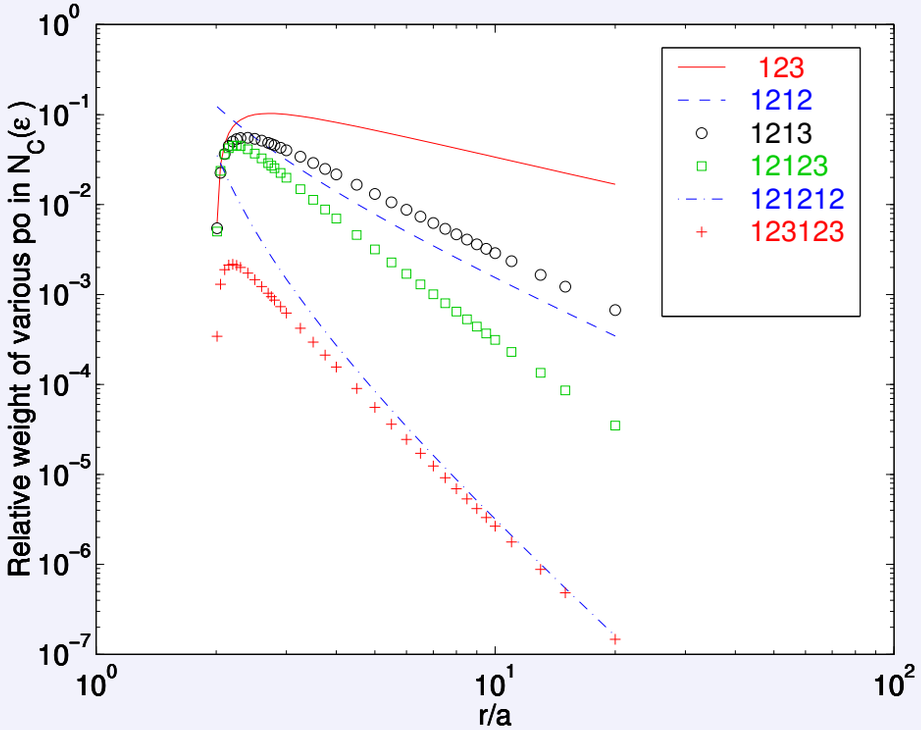

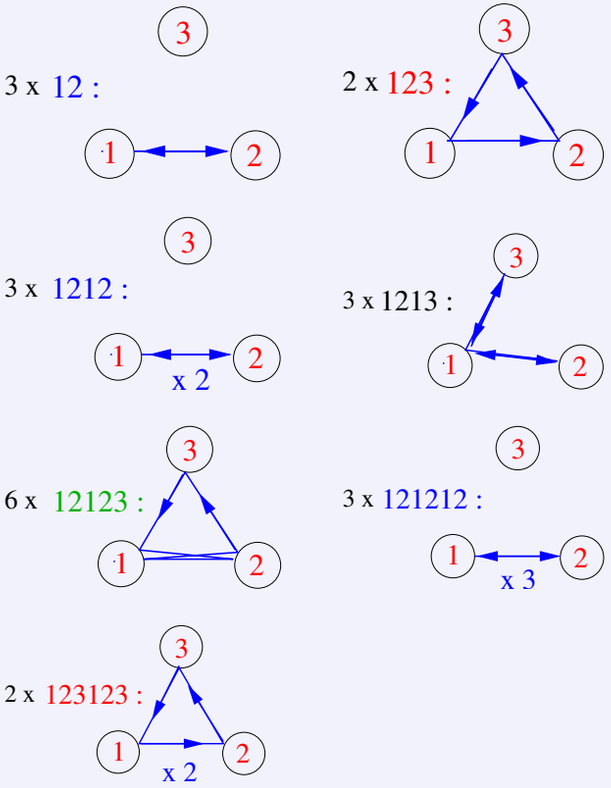

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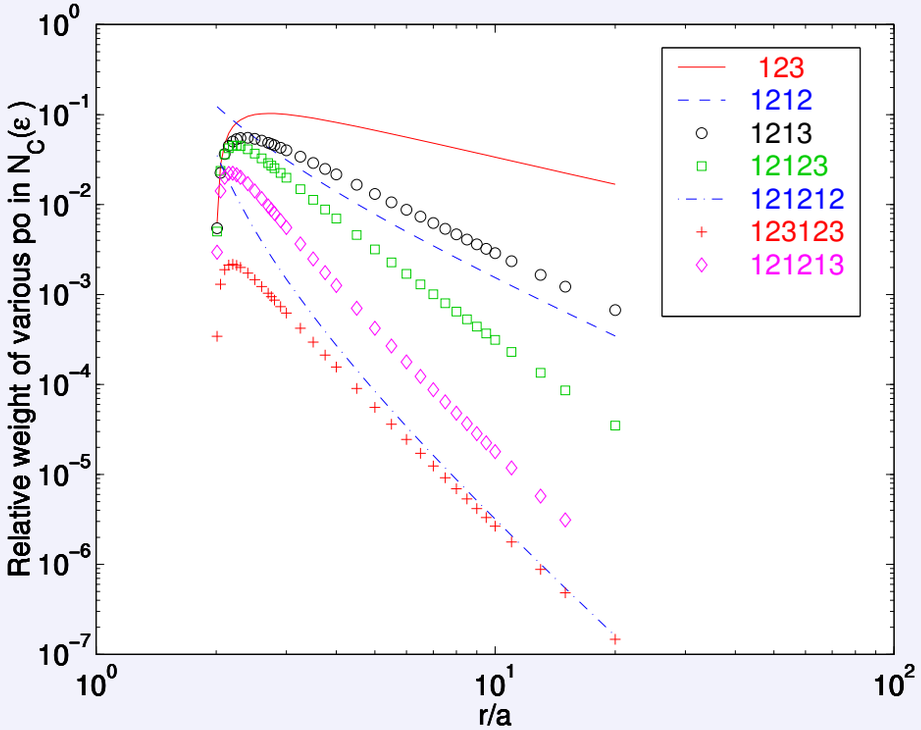

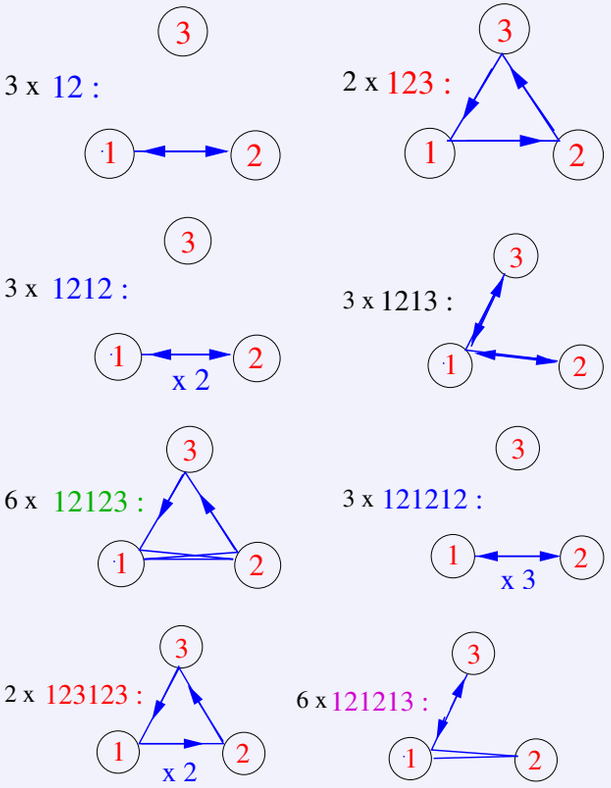

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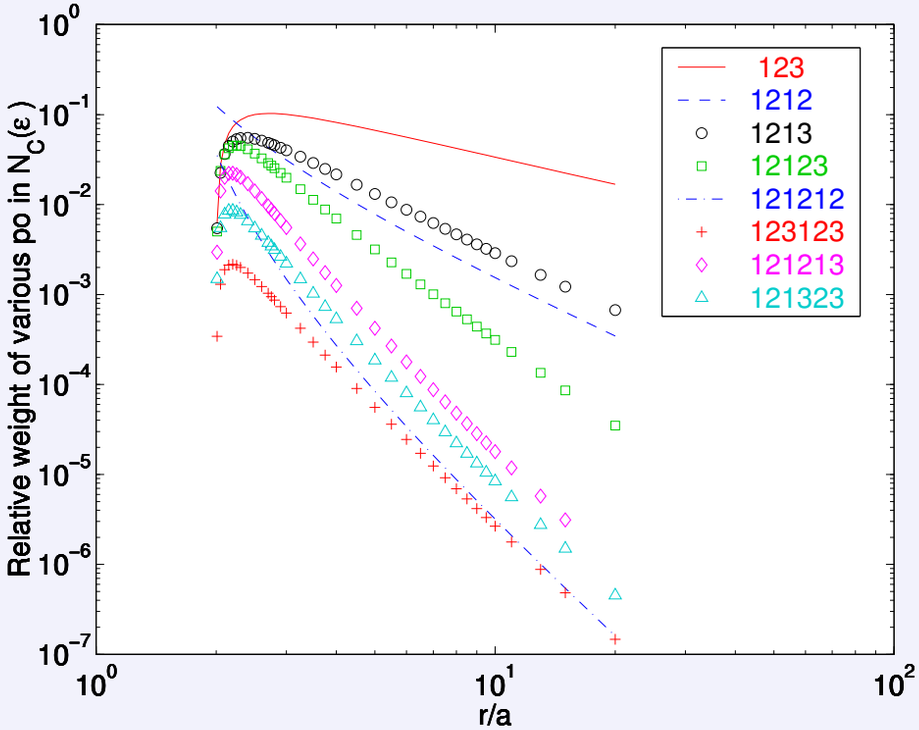

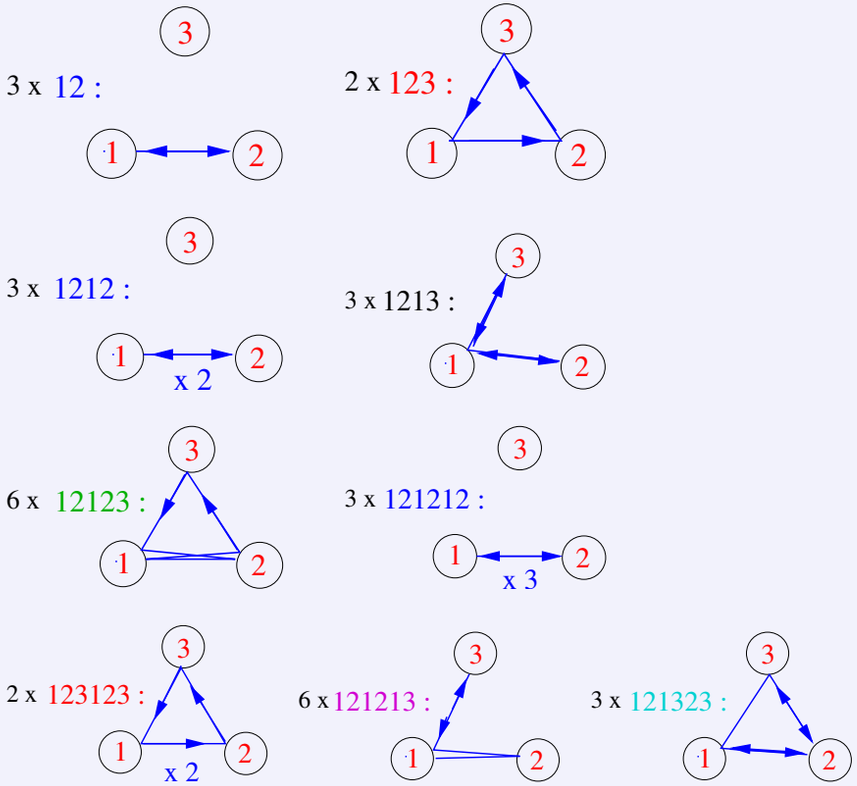

- *Billiard analogy* : difficult to make *long* shots, especially with many bounces
  – the slightest error ruins the shot.

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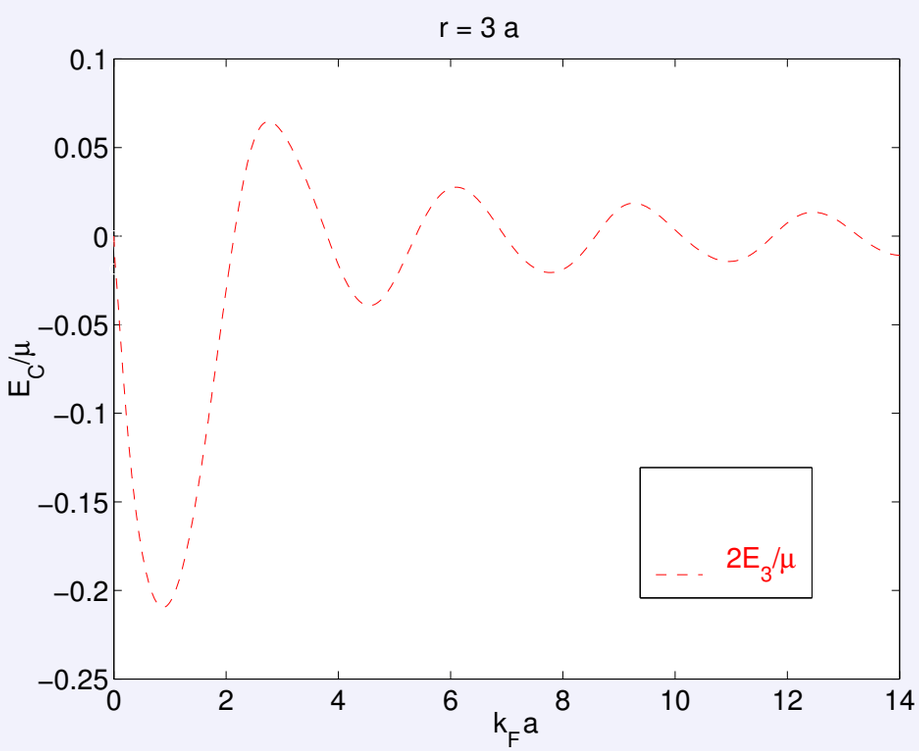
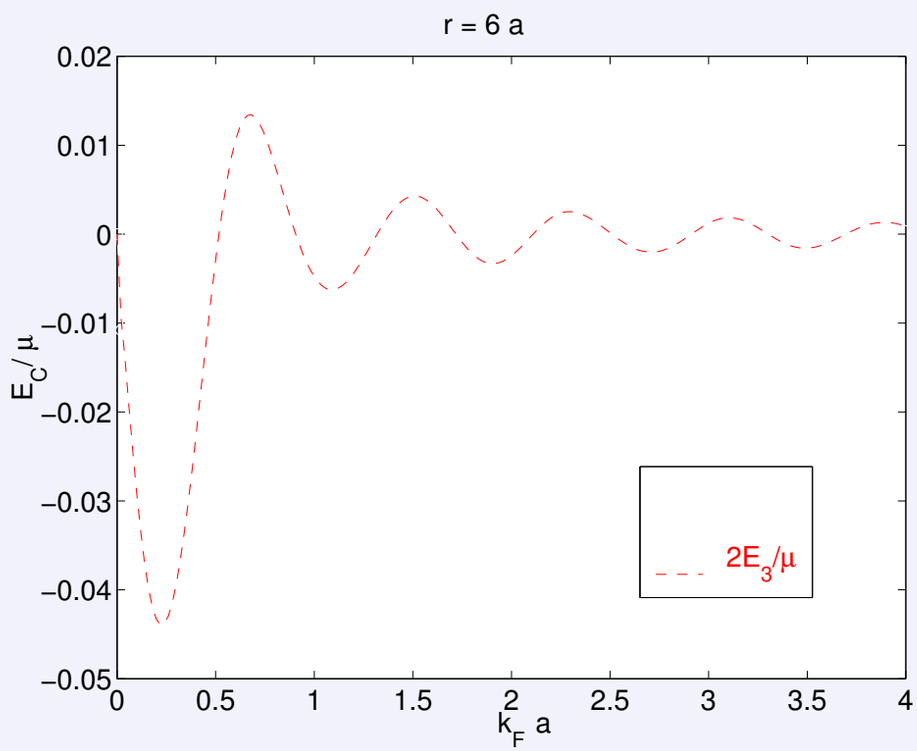
the Casimir energy for 3 ident. spheres (*equilateral triangle*)





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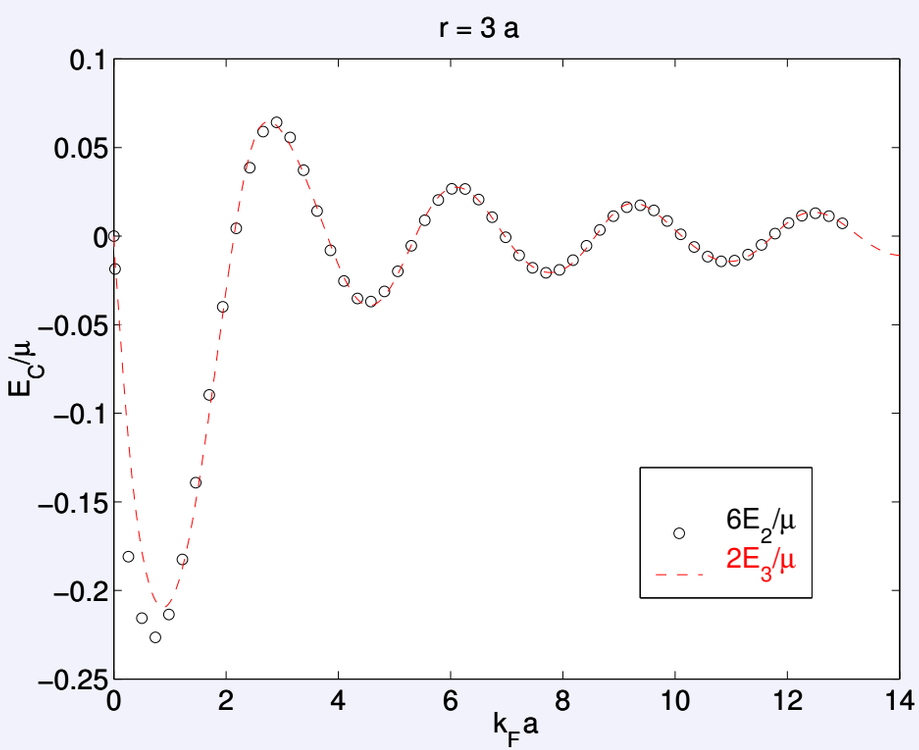
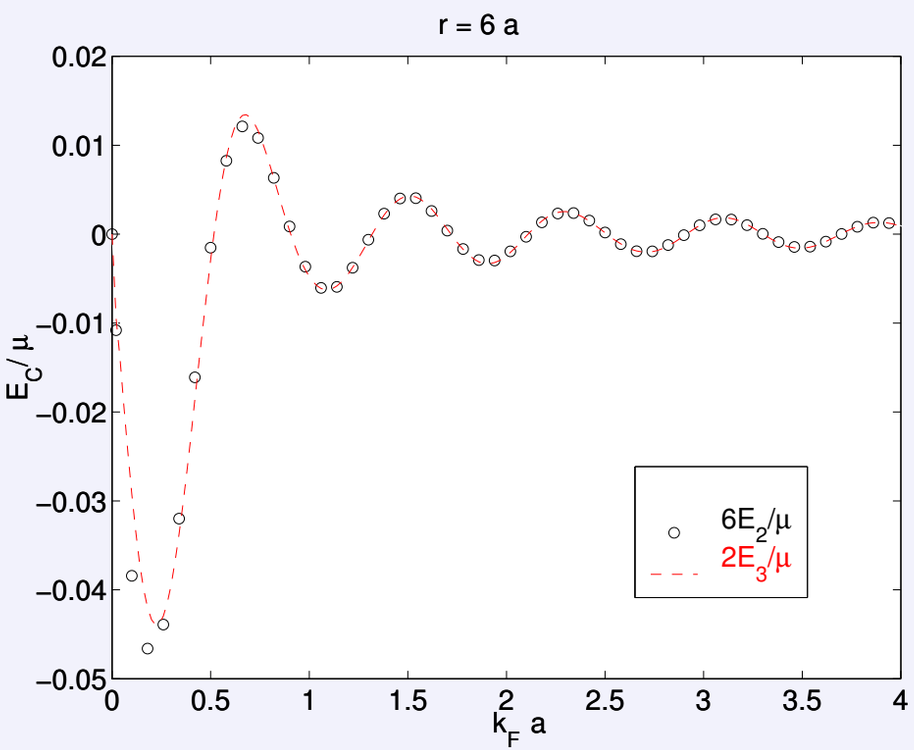
the Casimir energy satisfies the rule $E_3 \approx 3E_2$ for 3 ident. spheres (*equilateral triangle*)





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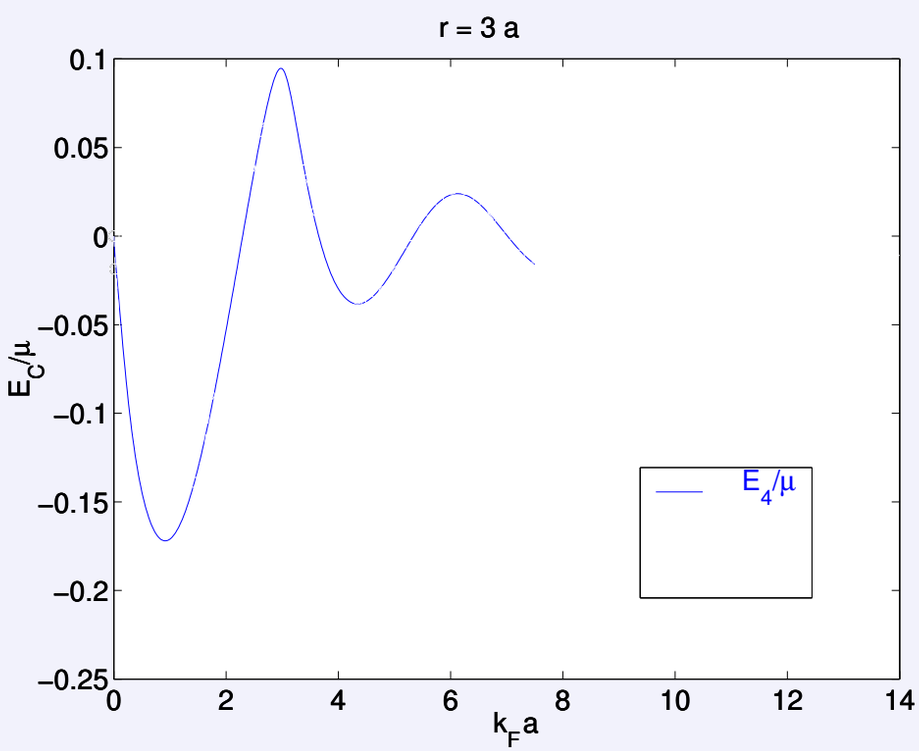
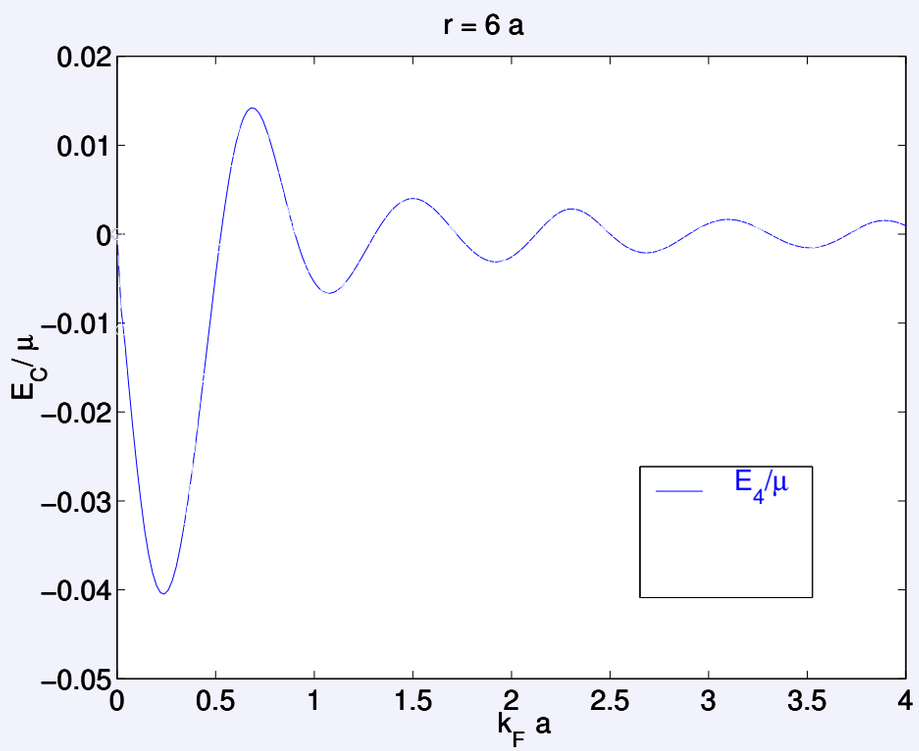
the Casimir energy

for 4 identical spheres (*symmetric tetrahedron*),





Three and four spheres: dominance of the two-body interactions:

- *period obits* summations $\Rightarrow$ genuine *three-* and *more-body* interactions,
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the Casimir energy satisfies the rule

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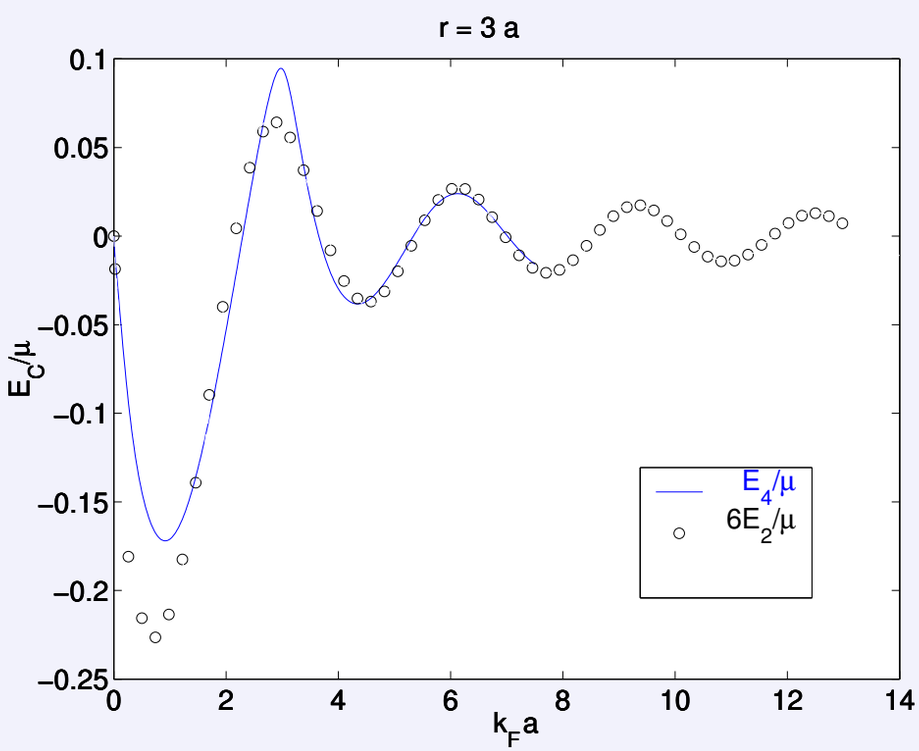

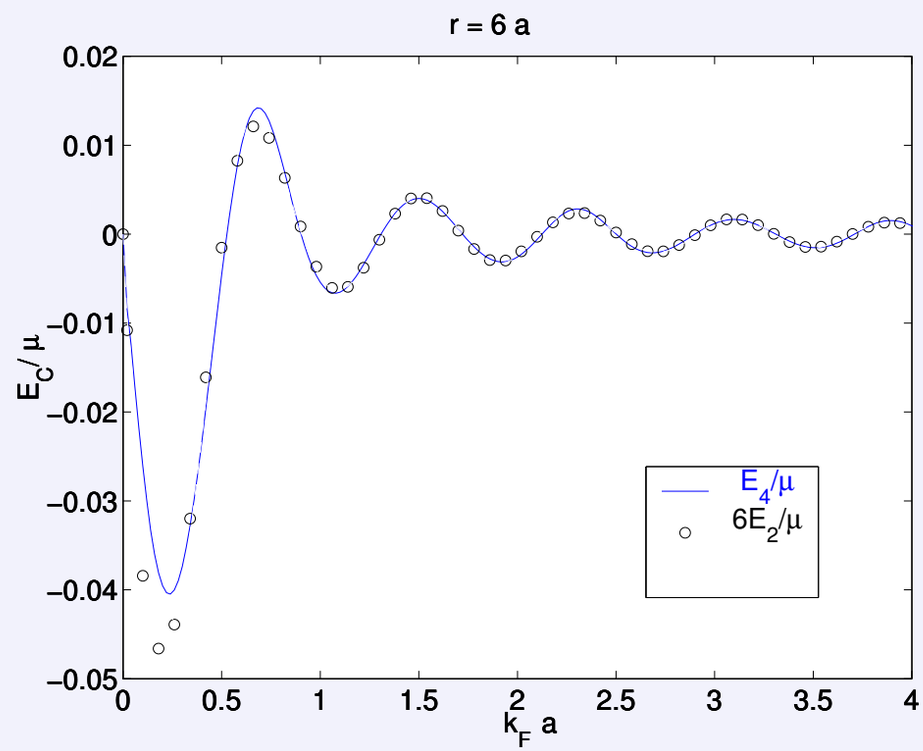

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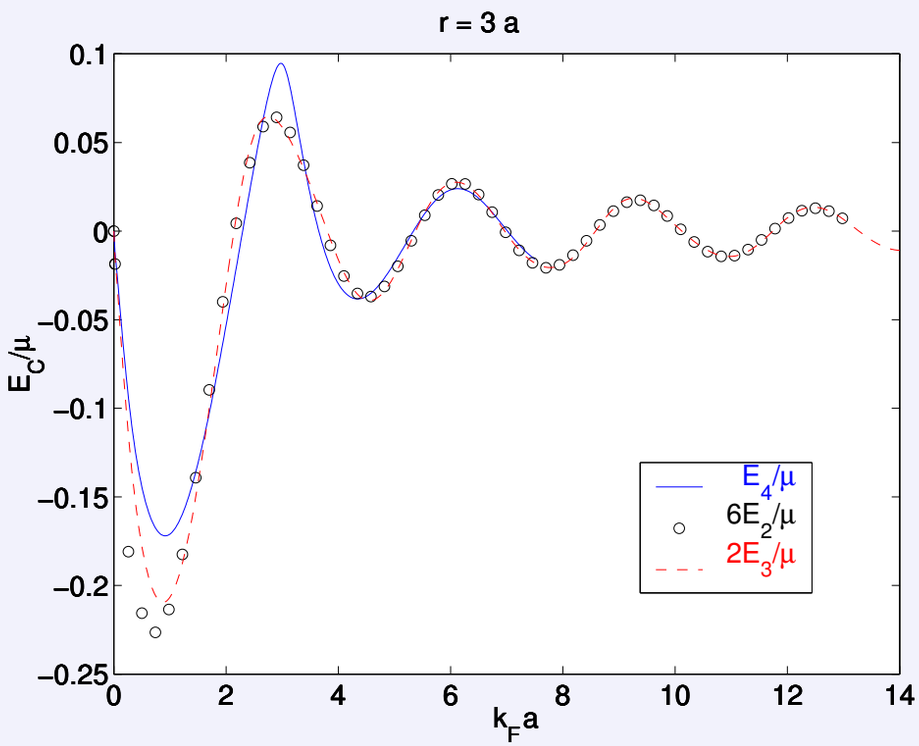
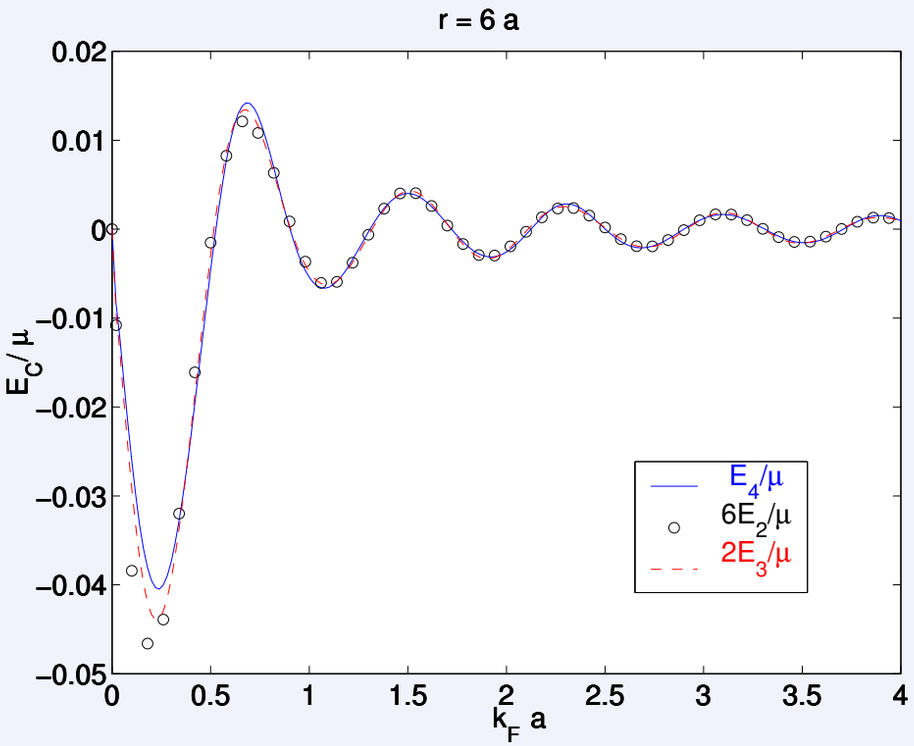
the Casimir energy satisfies the rule $E_3 \approx 3E_2$ for 3 ident. spheres (*equilateral triangle*)

and $E_4 \approx 6E_2 \approx 2E_3$ for 4 identical spheres (*symmetric tetrahedron*), if $k_F a \gg 1$.

($k_F a \leq 1$: corrections up to 10 % and 25 % for 3 & 4 spheres)





Long-range oscillating interactions between ad-atoms on metal surfaces

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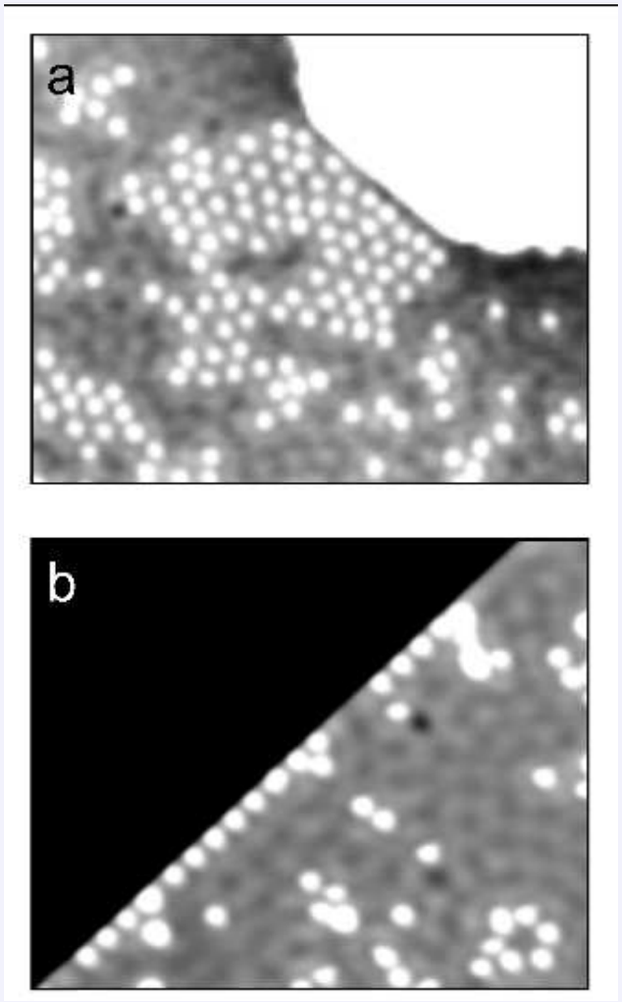
J. Repp et al., PRL 85 (2000) 2981



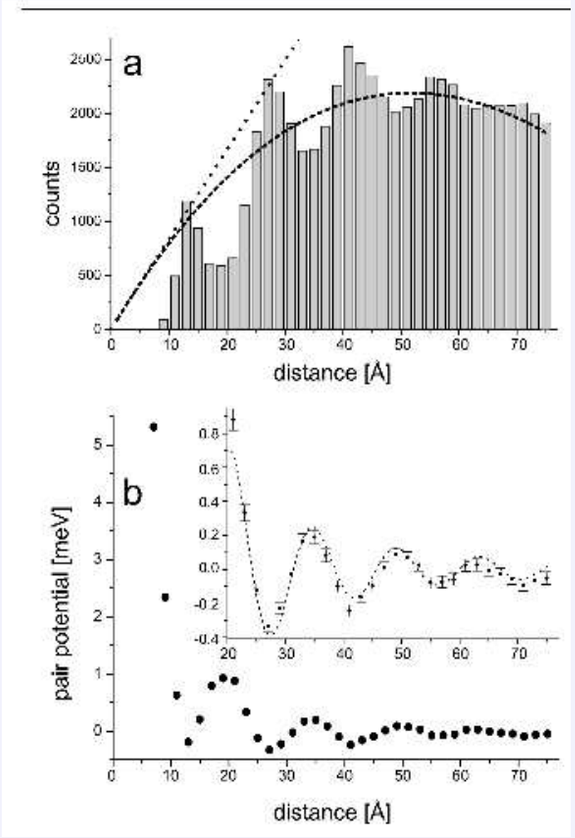
STM snap-shots (200 Å × 250 Å) of Cu-adatoms on a closed packed Cu(111)-surface (at 9 K)

– islands with local hexagonal order

– average adatom-distance 12.5 Å

– $E(R) \simeq -A \frac{4E_F \sin^2(\delta)}{\pi^2} \frac{\sin(2k_F R + 2\delta)}{(k_F R)^2}$
$E_F = 0.3$ eV, $k_F = 0.22$/Å, A = 0.08,
$\delta = 0.3\pi$.

– minima of pot.: 12.9 Å, 26.7 Å,
maximum of potential: 20.7 Å



P. Hyldgaard & M. Persson, J.Phys.:Cond.Mat. **12** (2000) L13
K.H. Lau & W. Kohn, Surf. Sci. **75** (1978) 69.

Fermionic Casimir Effect of two superfluid grains in a normal Fermi gas

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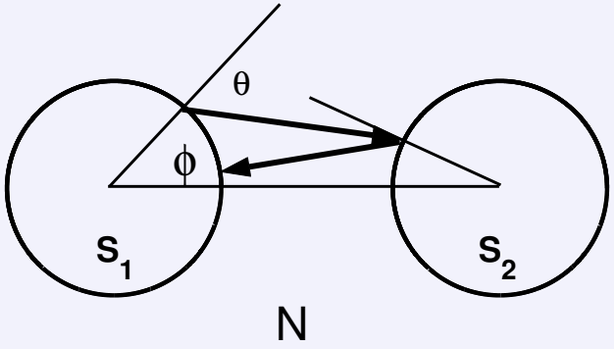

Specular reflection replaced by *Andreev reflection*

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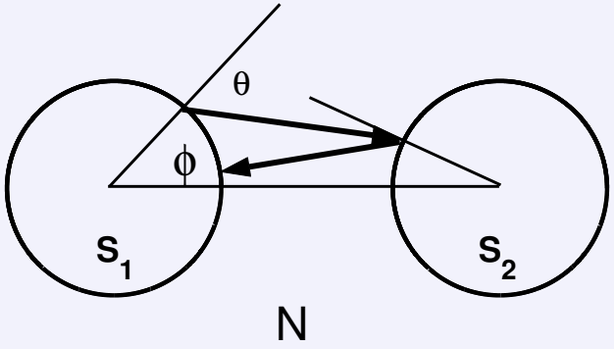

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$\Rightarrow$ particle retro-reflected as a hole.

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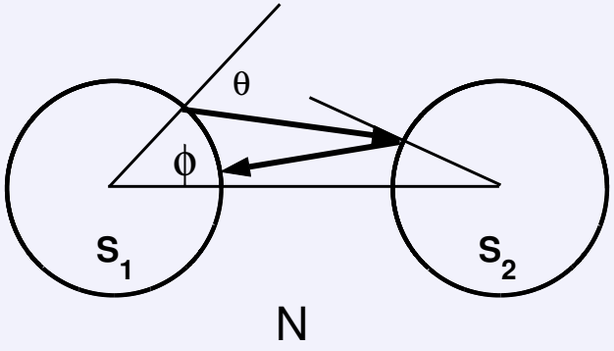

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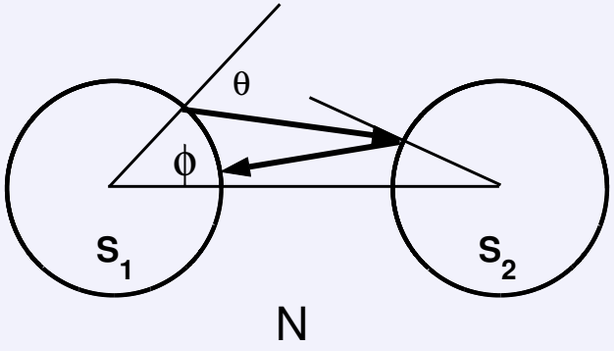

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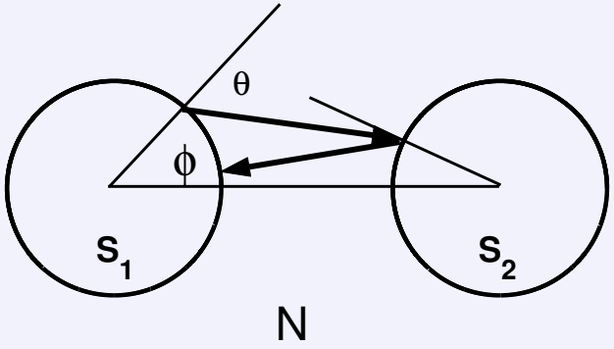

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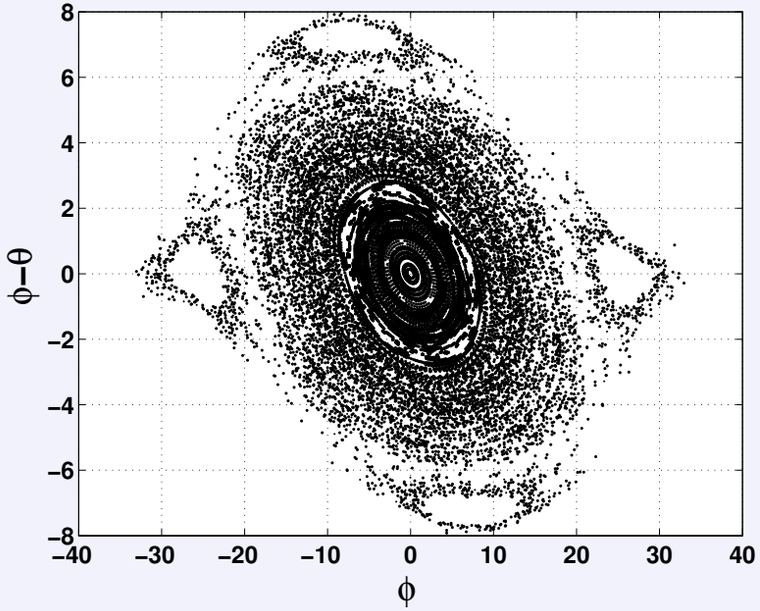

Stable orbits if $\frac{R}{2a} \leq \frac{k_p}{k_p - k_h}$ satisfied.

(here $R = 6a$ and $k_p/k_h = 1.5$)

A. Bulgac, P. Magierski & A.W., *Europhys. Lett.* **72** (2005) 327

1) Apply Bogoliubov-de Gennes equations in the p-h-space for quasi-particle energy $E = \pm \frac{k_{p/h}^2}{2m} \mp \mu$

$$\begin{pmatrix} H_0 - \mu & \Delta(\vec{r}) \\ \Delta(\vec{r}) & -(H_0 - \mu) \end{pmatrix} \begin{pmatrix} u(\vec{r}) \\ v(\vec{r}) \end{pmatrix} = E \begin{pmatrix} u(\vec{r}) \\ v(\vec{r}) \end{pmatrix}$$ with pair field $\Delta(\vec{r}) = \begin{cases} \Delta e^{+i\phi_\Delta/2} & \text{for } r \in \text{ grain 1,} \\ \Delta e^{-i\phi_\Delta/2} & \text{for } r \in \text{ grain 2,} \\ 0 & \text{otherwise.} \end{cases}$

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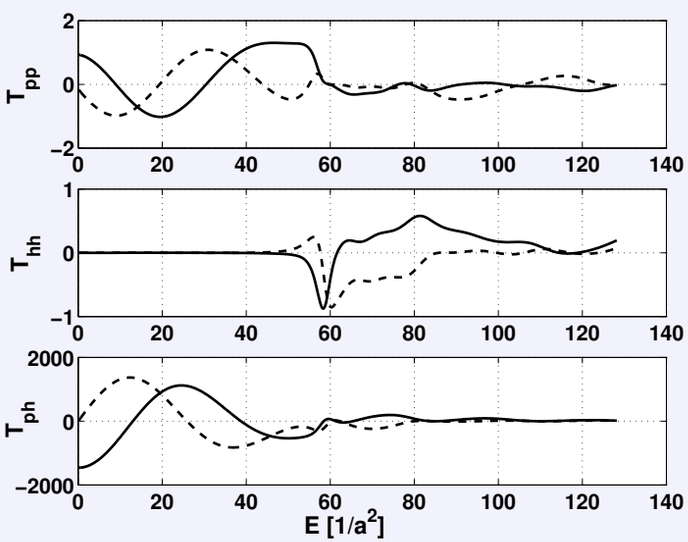

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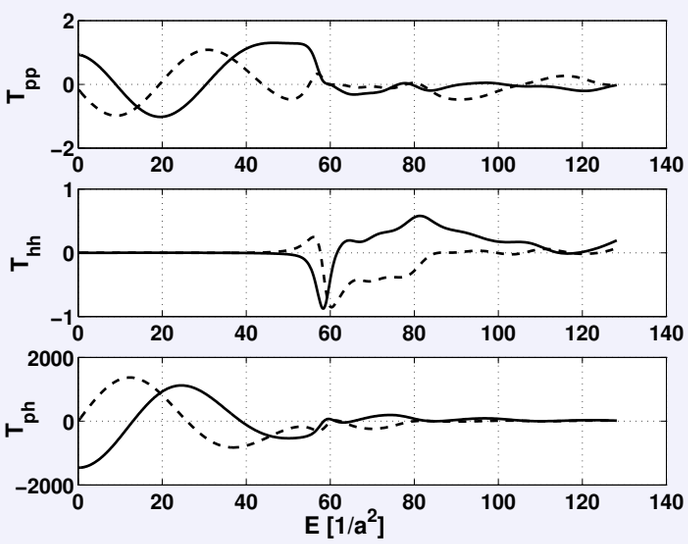

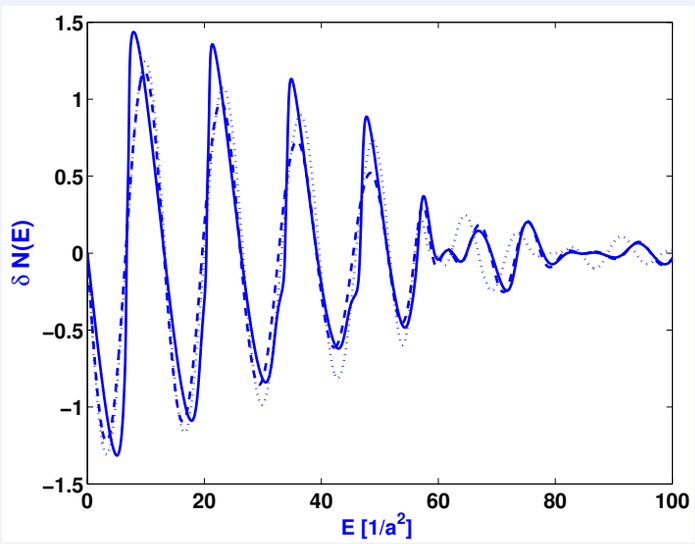

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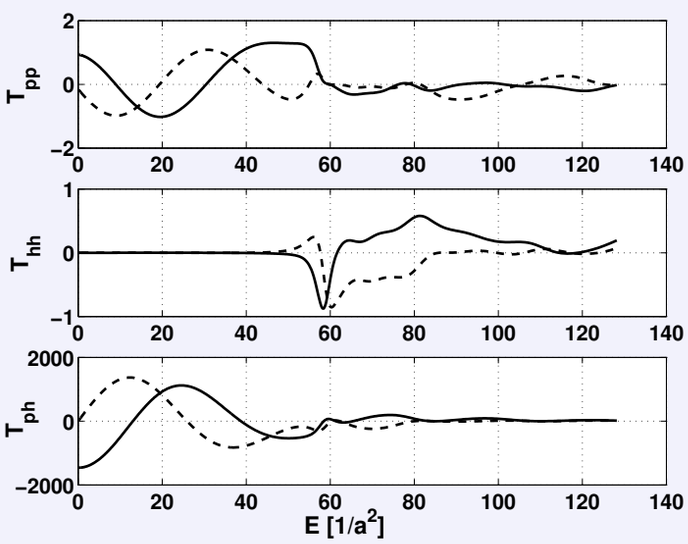

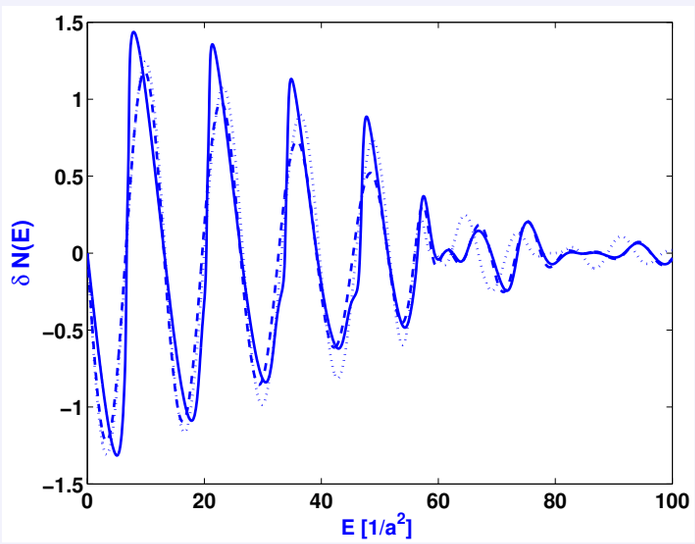

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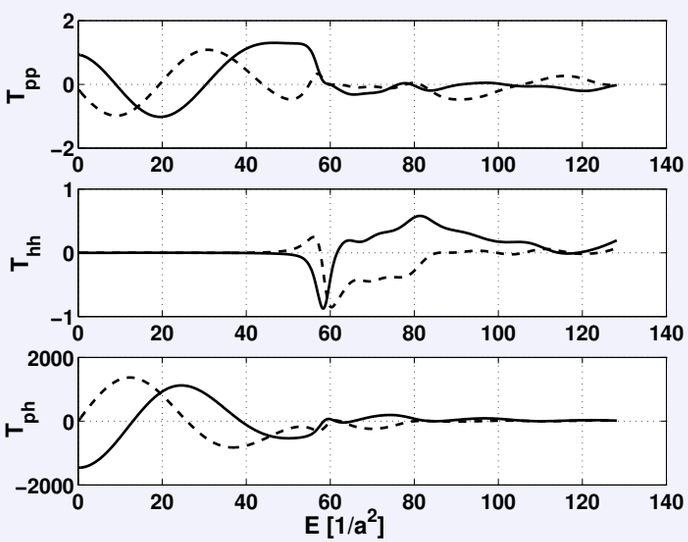

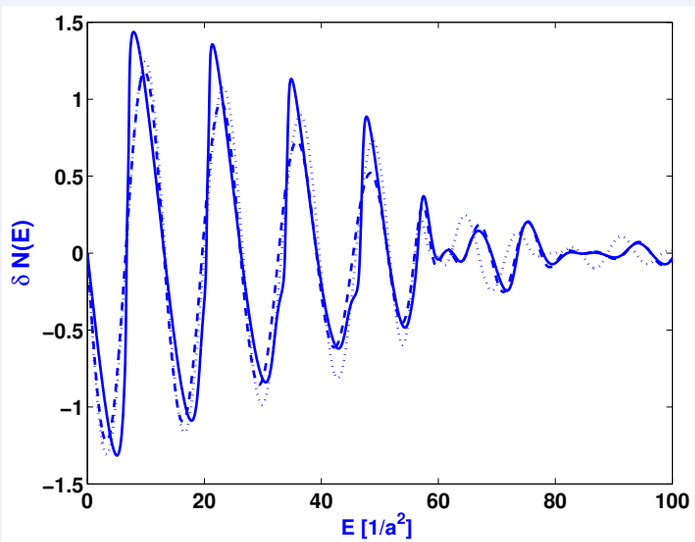

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Quantization condition of shortest periodic orbit:

$$(k_p - k_h)(R - 2a) - 2\arccos(E/\Delta) = 2\pi n, \quad n = 0, 1, 2, \cdots.$$

For $E \ll |\Delta|$:

$$\delta N(E) \quad = \quad -\mathrm{Im}\underbrace{\left[\int_{-a\,\min(k_p,k_h)}^{a\,\min(k_p,k_h)} d\nu\, \exp\left(\frac{i(k_p-k_h)\nu^2}{2k_pk_ha}\right)\right]^2}_{\to(2k_Fa)^2 \text{ if } E=0} \frac{e^{i(k_p-k_h)(R-2a)}}{\pi^2 R\sqrt{k_pk_h}}\cos(\phi_\Delta).$$

- Exponent: classical action of shortest periodic orbit (hole momentum negative!); quantization condition: $(k_p-k_h)(R-2a) - 2\arccos(E/\Delta) = 2\pi n, \quad n = 0, 1, 2, \cdots$.
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- Casimir energy of superfluid grains in the Fermi sea:

$$\mathcal{E}_{SC}(R) = \nu_{\mathrm{deg}} \int_0^\infty dE\, E\, \frac{d\,\delta N(E)}{dE} = -\nu_{\mathrm{deg}} \int_0^\infty dE\, \delta N(E)\,.$$

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- Casimir energy of superfluid grains in the Fermi sea:

$$\mathcal{E}_{SC}(R) = \nu_{\rm deg}\int_0^\infty dE\, E\,\frac{d\,\delta N(E)}{dE} = -\nu_{\rm deg}\int_0^\infty dE\,\delta N(E)\,.$$

Numerical result for two spherical three-dimensional grains at separation R :

$$\mathcal{E}_{SC}(R) \approx \nu_{\rm deg}\frac{\hbar^2k_F^2}{2m}\frac{k_Fa^4}{2\pi R^2(R-2a)}\,X \gg \left|-\nu_{\rm deg}\frac{\hbar^2k_F^2}{2m}\frac{a^2}{2\pi R(R-2a)}j_1(2k_F(R-2a))\right|$$

with $X = \mathcal{O}(1)$, positive (*i.e.* repulsive!) and weakly dependent on R, k_F, Δ.

A. Bulgac, P. Magierski & A.W., *Europhys. Lett.* **72** (2005) 327

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- Map-method onto scattering systems also applicable to traditional Casimir problems:

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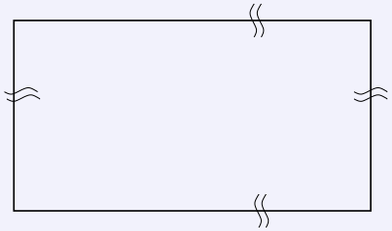

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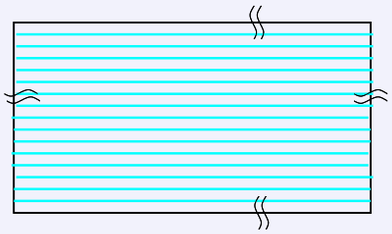

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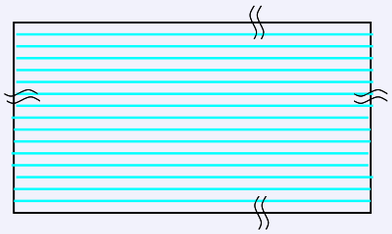

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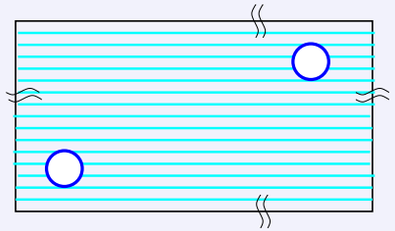

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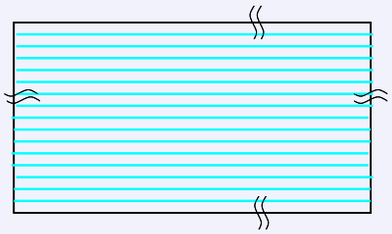

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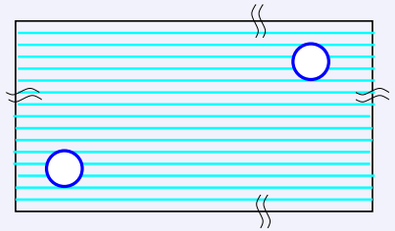

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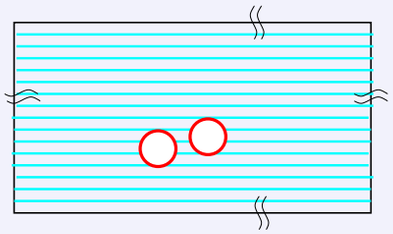

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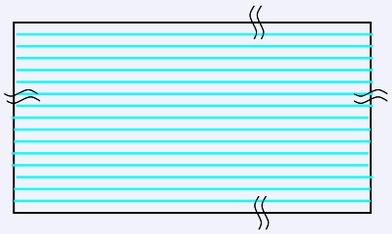

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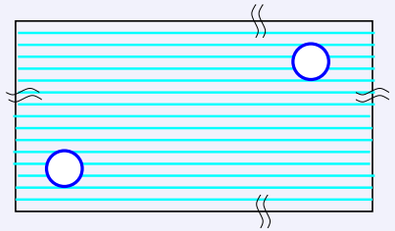

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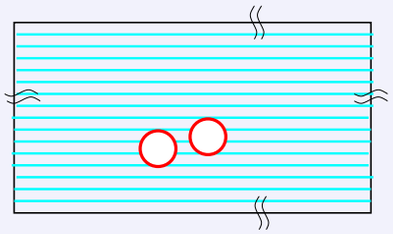

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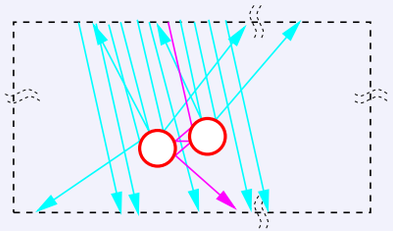

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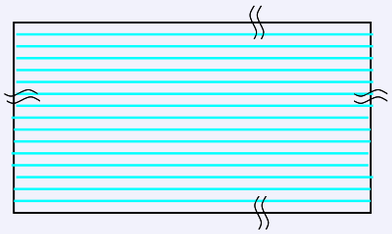

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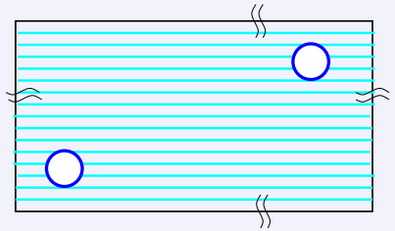

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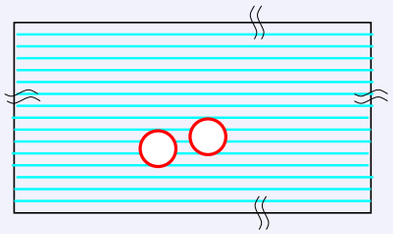

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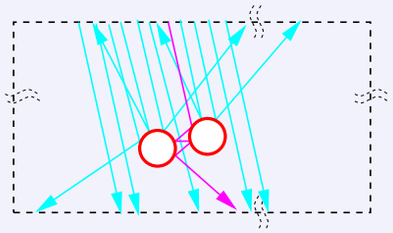

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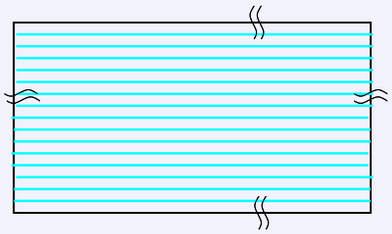

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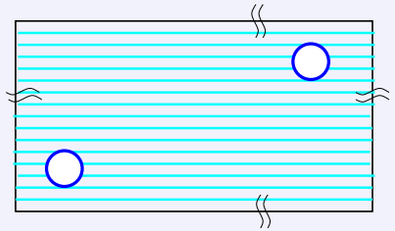

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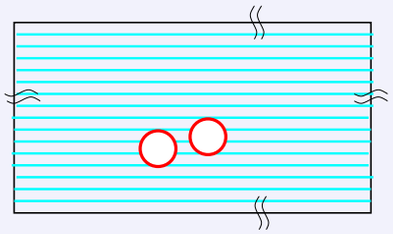

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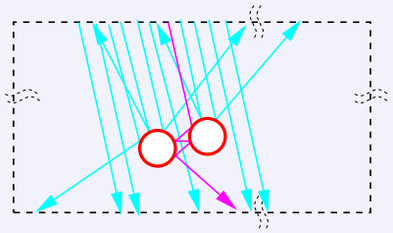

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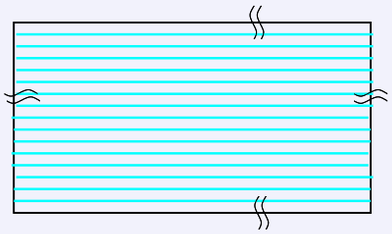

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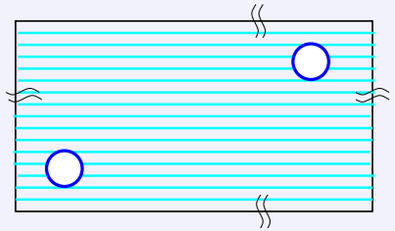

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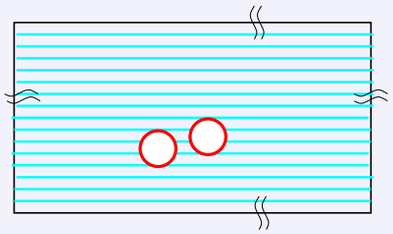

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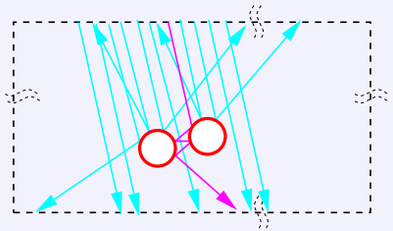

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All determinants exist (although the relevant scattering matrices are infinite dimensional) since the associated T-matrices are trace-class.

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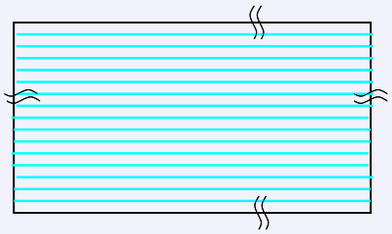

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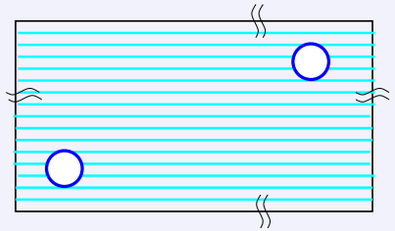
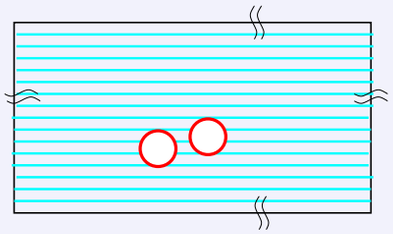


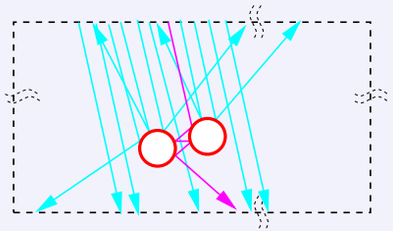
3. *geometry*-dependent arrangement of n bubbles:

$$\rho(E) = \rho_0(E) + \sum_{i=1}^{n} \rho_W(E, a_i) + \delta\rho_C(E, \{a_i\}, \{\mathbf{r}_{ij}\})$$



4. Krein trace formula (note the averaging):

$$\delta\rho(E) \quad = \quad \bar{\rho}(E) - \bar{\rho}_0(E) = \frac{1}{2\pi i}\frac{d}{dE}\overbrace{\ln\det S_n(E)}^{2i\eta_n(E)}$$



5. *Multiple scattering matrix*

$$\overbrace{\det S_n(E)} = \prod_i \det S_1(E, a_i)\,\frac{\det M^\dagger(k^*)}{\det M(k)}$$

$$\hookrightarrow \quad \delta\bar{\rho}_C(E, \{a_i\}, \{\mathbf{r}_{ij}\}) = -\frac{1}{\pi}\mathrm{Im}\left(\frac{d}{dE}\ln\det M\big(k(E)\big)\right)$$

see A.W., *Phys. Rep.* **309** (1999)

6. Casimir energy:

$$\therefore \quad \mathcal{E}_C = \int_0^\infty dE\,\tfrac{1}{2}E\,\delta\bar{\rho}_C = -\tfrac{1}{2}\int_0^\infty dE\,\overline{\mathcal{N}}_C = \frac{1}{2\pi}\int_0^\infty dE\;\mathrm{Im}\ln\det M\big(k(E)\big)$$

$$
\begin{aligned}
\mathcal{E}_C &= \int_0^\infty dE\, \tfrac{1}{2}\hbar c k\, \delta\bar{\rho}_C\big(k(E)\big) = -\frac{\hbar c}{2}\int_0^\infty dk\, \overline{\mathcal{N}}_C(k) = \frac{\hbar c}{2\pi}\int_0^\infty dk\, \mathrm{Im}\ln\det M\big(k\big) \\
&= \frac{\hbar c}{4\pi i}\left[\int_0^{\infty(1+i0_+)} dk\ \ln\det M\big(k\big) - \int_0^{\infty(1-i0_+)} dk\ \ln\det M\big(k\big)^\dagger\right] \\
&= \frac{\hbar c}{2\pi}\int_0^\infty dk_4 \ln\det M(ik_4) \quad \text{for relativistic disp. } E=\hbar c k \text{ and after Wick-rotation}
\end{aligned}
$$

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$$

Note that $\det M(ik_4)^\dagger = \det M(ik_4)$ since $\det M(k) = \det M\big((-k^*)\big)^\dagger$;

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\end{aligned}$$

Note that $\det M(ik_4)^\dagger = \det M(ik_4)$ since $\det M(k) = \det M\big((-k^*)\big)^\dagger$; therefore corollary:

$$\frac{\hbar c}{2\pi}\int_0^\infty dk\, k^{2n+1}\mathrm{Im}\ln\det M(k) = i(-1)^n\frac{\hbar c}{4\pi}\int_0^\infty dk_4\, k_4^{2n+1}\left[\ln\det M(ik_4) - \ln\det M(ik_4)^\dagger\right] = 0$$

e.g. Casimir energy over modes with non-relativistic dispersion $E=\hbar^2k^2/2m$ integrates to zero, unless there is a *finite upper cutoff*, as e.g. the *Fermi momentum* k_F in the case of the so-called *fermionic Casimir effect*

– see A. Bulgac & AW., Phys. Rev. Lett. 87 (2001).

Scalar Casmir effect of Dirichlet-spheres and -plates – or *How good is the PFA?*

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M. Schaden, L. Spruch, PRA **58** (1998): *short-distance PFA confirmed by semiclassics (periodic orbits)*

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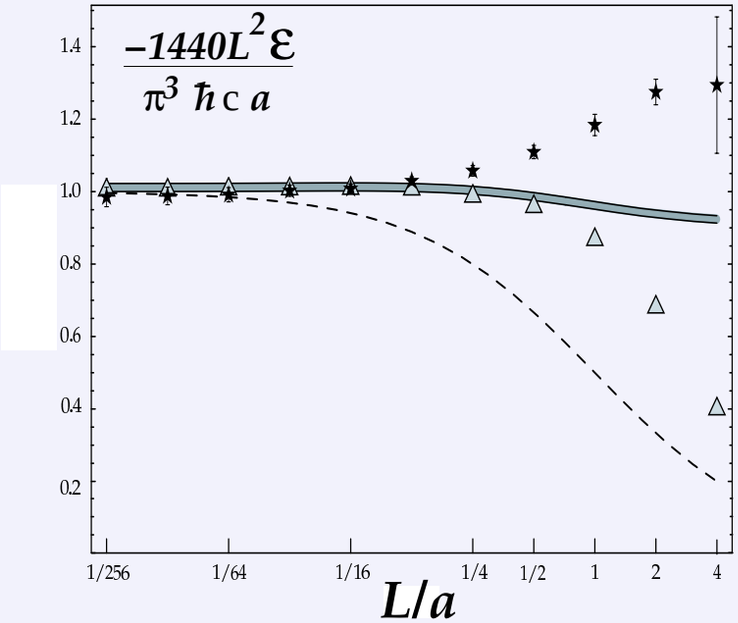
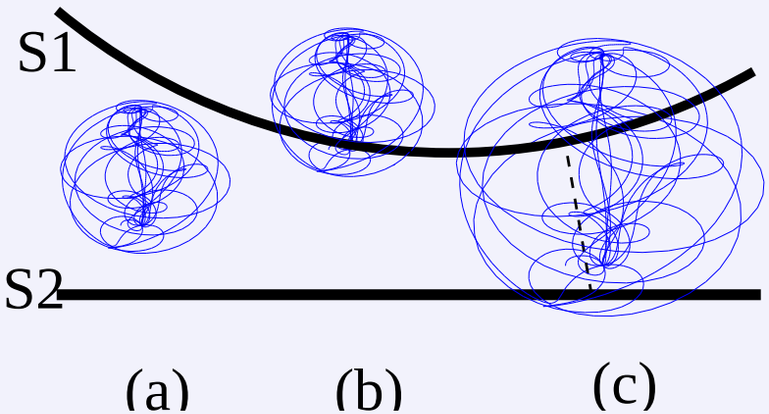
H. Gies, K. Langfeld, L. Moyaerts, JHEP **0306** (2003): *world-line approach (Feynman integral in x-space)*

A. Scardicchio, R.L. Jaffe, NPB **704** (2005): *optical ansatz (summation over closed orbits)*





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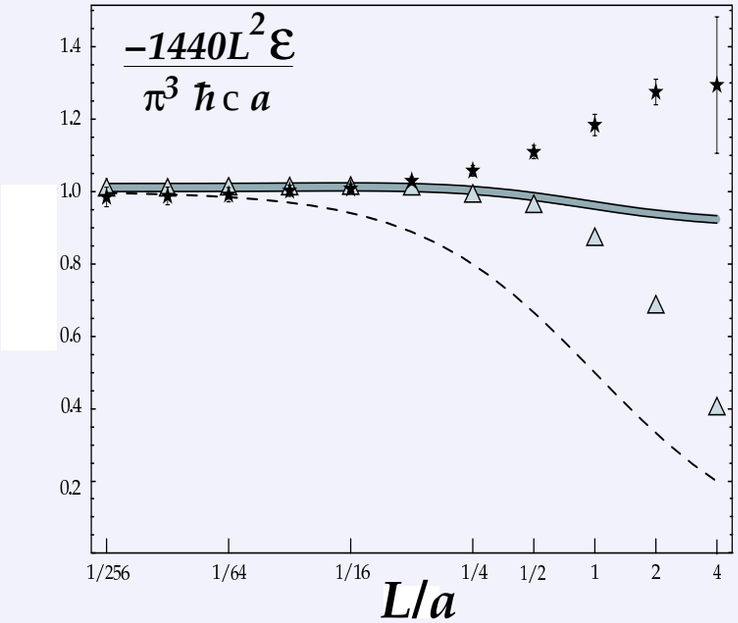

Note: asymptotically ($L/a \gg 1$) s-wave scattering dominates:

$$\mathcal{E}(L) \quad \sim \quad -\frac{\pi^3\hbar c\, a}{1440L^2}\,\frac{90}{\pi^4}\,\frac{2}{(1+a/L)(1+a/2L)} \rightarrow -\frac{\pi^3\hbar c\, a}{1440L^2}\,\frac{90}{\pi^4}\times 2 = -\frac{\pi^3\hbar c\, a}{1440L^2}\times 1.847\cdots$$

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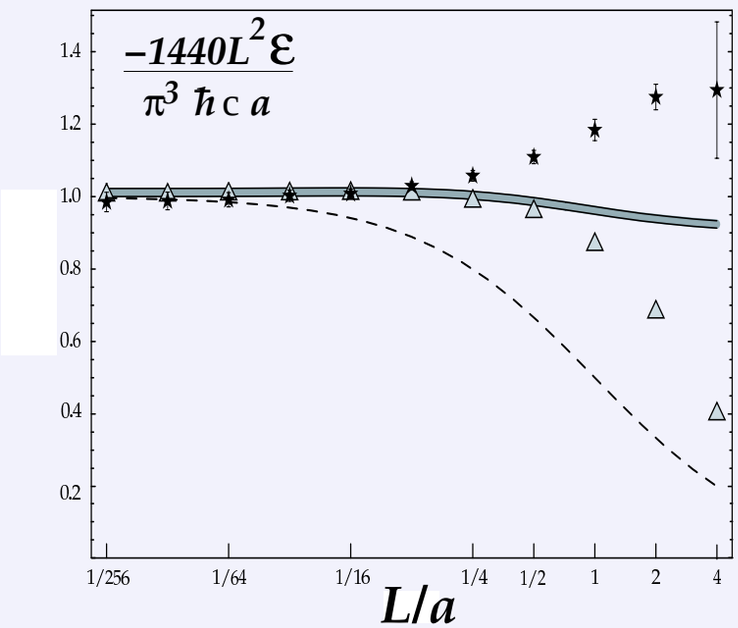

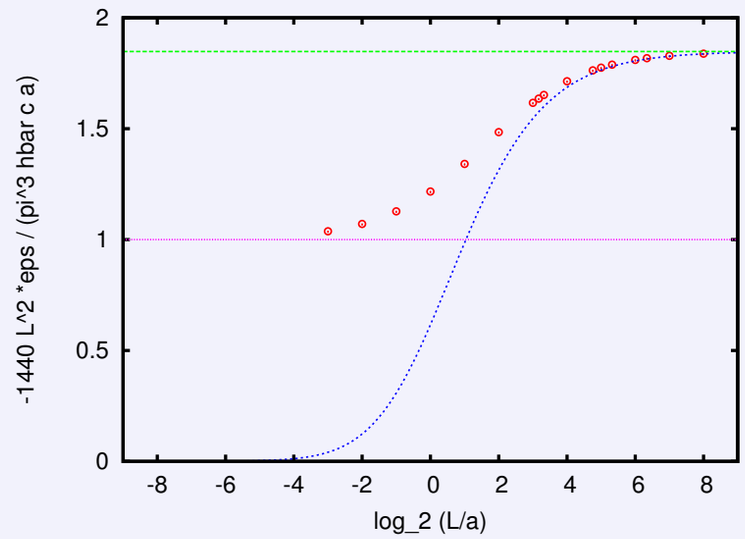

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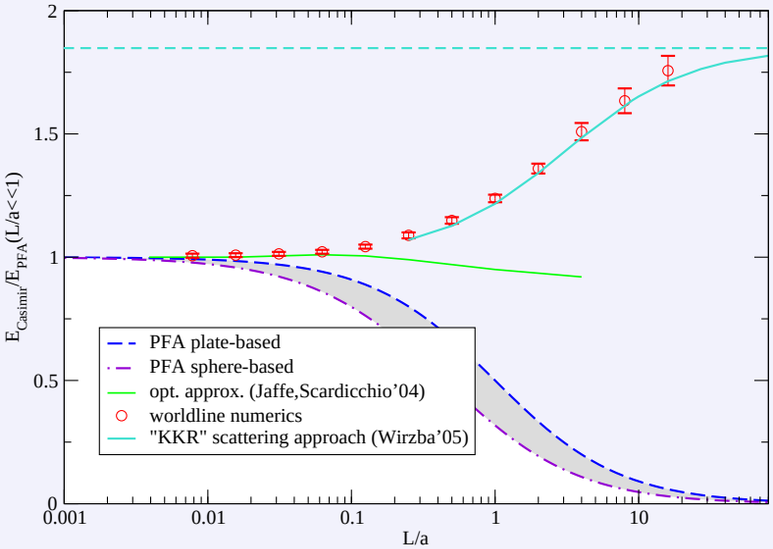
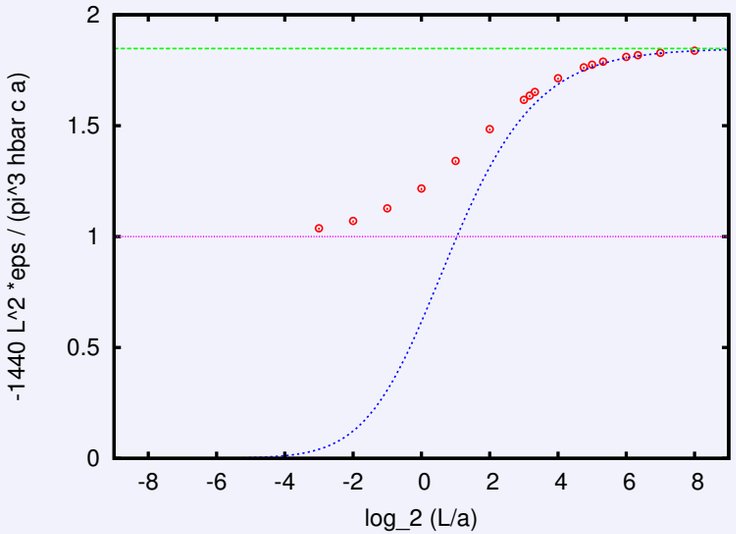
Scalar Casmir effect of Dirichlet-spheres and -plates – or *How good is the PFA?*



H. Gies & K. Klingmüller, 2006



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while

$$\mathcal{E}_{\text{p-wave}}(L) \quad \sim \quad -\frac{5\pi^3 \hbar c\, a^3}{1440 L^4}\,\frac{90}{\pi^4}$$

(i.e. no Casimir-Polder behavior in the scalar case)

A. Bulgac, P. Magierski & A.W., *Phys. Rev. D* **73** (2006) 025007

Additional Conclusions:

- Casimir energy re-defined as vacuum energy of the geometry-dependent part of the level density (connected to the multi-scattering phase shift by a modified Krein trace formula).
- The non-overlapping (i.g. non-separable) N-sphere, sphere-plate, N-disk (and N-cylinder) Casimir problems can be solved exactly in the scalar (and also in the fermionic) case.
- Calculation not plagued by subtraction of single-sphere contributions or by the removal of diverging ultraviolet contributions; all involved determinants exist and are finite since the pertinent *T*-matrices are trace-class.
- Large-distance behavior dominated by *s*-wave scattering in the case of the scalar Casimir effect and by *p*-wave scattering for the EM Casimir effect.
- The presented method can easily be applied to any number of spheres or cylinders with or without planes (in 2D disks with or without lines).
- Moreover, the Dirichlet boundary conditions can be replaced by Neumann or mixed boundary conditons.
- The Casimir energy is dominated by momenta $k \sim 1/L$ where L is the separation scale. For the scalar sphere-plate case the integration can be truncated at $k_{\max} \sim 10/L$ corresponding to a truncation in the angular momentum $l_{\max} \geq (e/2)k_{\max}a \approx 14a/L$.

Publications:

Aurel Bulgac & A.W., *Phys. Rev. Lett.* **87** (2001) 120404
Physics News Update, No. 556, #1, Sep. 13, 2001.
Aurel Bulgac, Piotr Magierski & A.W., *Europhys. Lett.* **72** (2005) 372.
Aurel Bulgac, Piotr Magierski & A.W., *Phys. Rev. D* **73** (2006) 025007.
A.W., Aurel Bulgac & Piotr Magierski, *J. Phys. A: Math. Gen.* **39** (2006) 6815.
A.W., *J. Phys. A: Math. Theor.* **41** (2008) 164003.