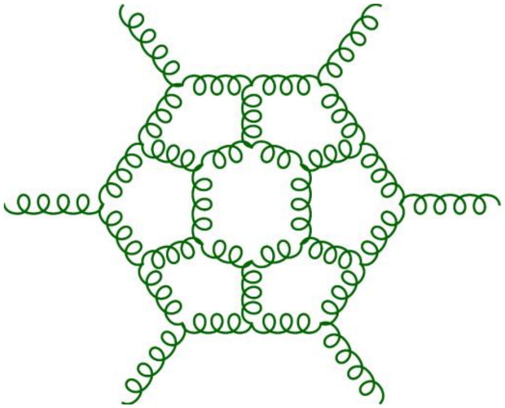
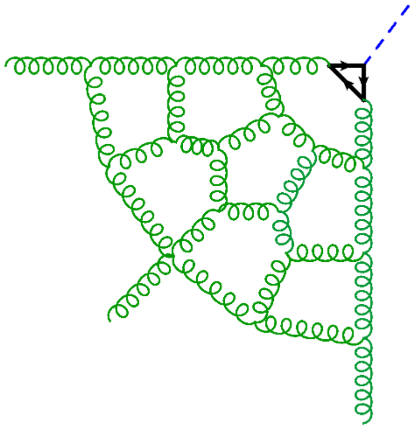
A New Duality in Planar N=4 SYM and Possible Flux Tube Implications





Lance Dixon (SLAC)

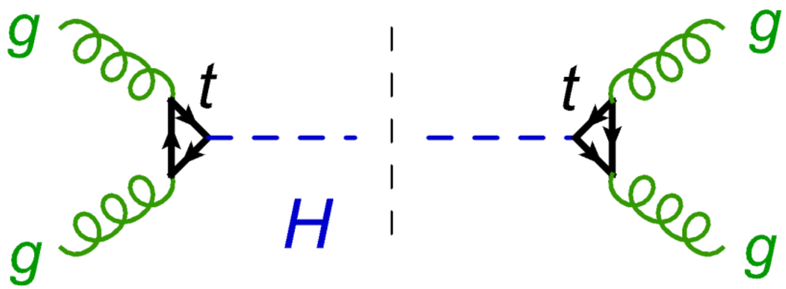
LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, 2112.06243
+ Y.-T. Liu, in progress

KITP Program on
Flux Tubes and Confinement
January 26, 2022

SLAC NATIONAL ACCELERATOR LABORATORY

Total cross section for producing Higgs boson at LHC via gluon fusion

Leading Order (LO)



- Higgs production at LHC is dominantly via gluon fusion, mediated by a top quark loop.
- Since $\boldsymbol{2m_{top}} = 350\ \mathrm{GeV}$
  $\gg \boldsymbol{m_{Higgs}} = 125\ \mathrm{GeV}$,

we can integrate out the top quark to get a leading operator $H\,G^a_{\mu\nu}G^{\mu\nu\,a}$

State of Art: N3LO

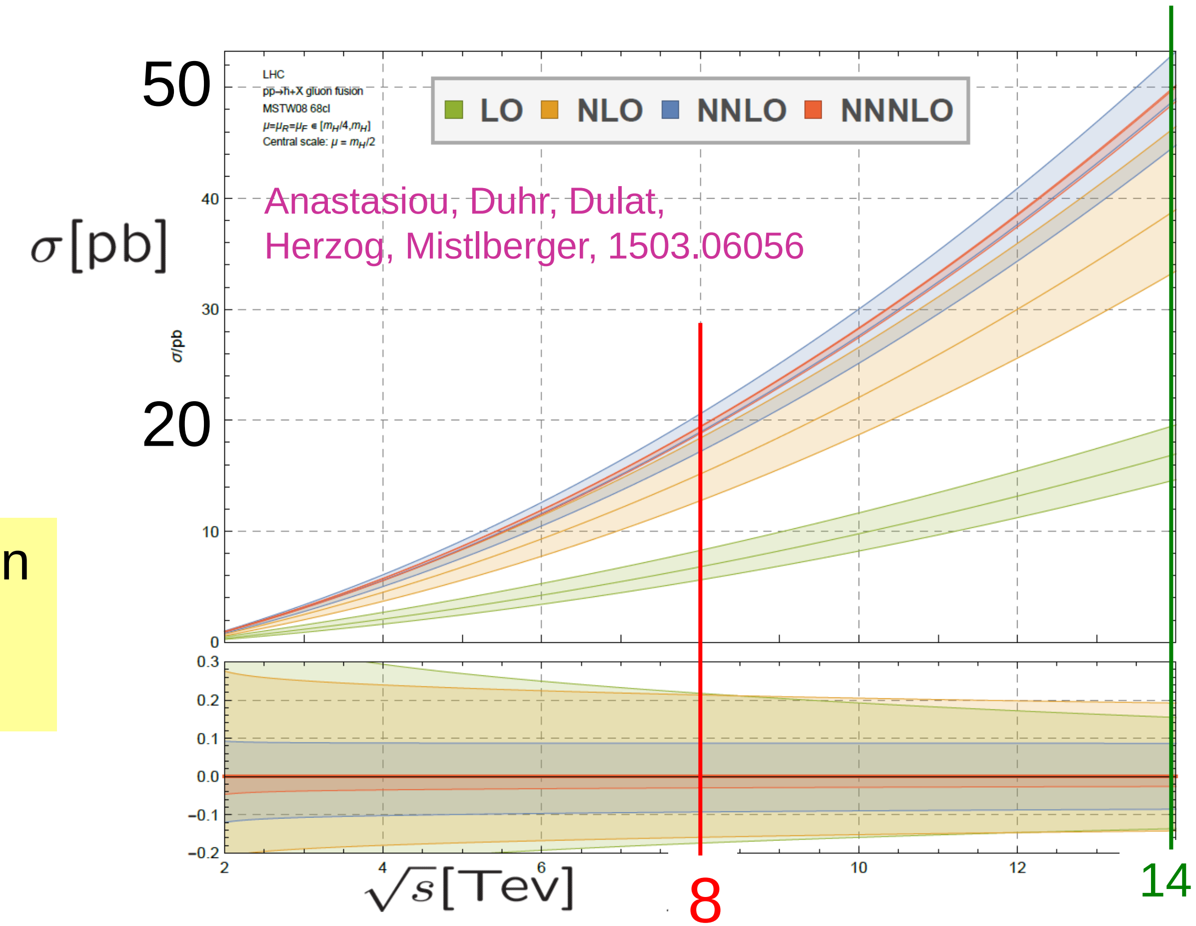
$$\hat{\sigma}(\alpha_s, \mu_F, \mu_R) \;=\; [\alpha_s(\mu_R)]^{n_\alpha} \left[\hat{\sigma}^{(0)} + \frac{\alpha_s}{2\pi} \hat{\sigma}^{(1)}(\mu_F, \mu_R) + \left(\frac{\alpha_s}{2\pi}\right)^2 \hat{\sigma}^{(2)}(\mu_F, \mu_R) + \cdots \right]$$

LO   NLO   NNLO

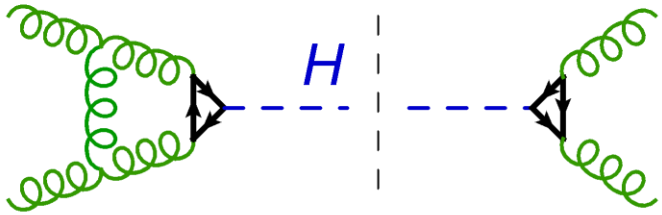
Leading-order (LO) predictions qualitative:
poor convergence
of expansion in $\alpha_s(\mu)$
Uncertainty bands from varying $\mu_R = \mu_F = \mu$

Example: Higgs gluon fusion cross section at LHC vs. CM energy $\sqrt{s}$

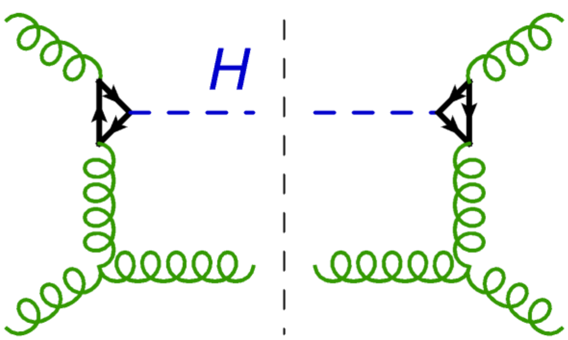
LO → NNNLO
→ factor of 2.7 increase!



NLO QCD topologies

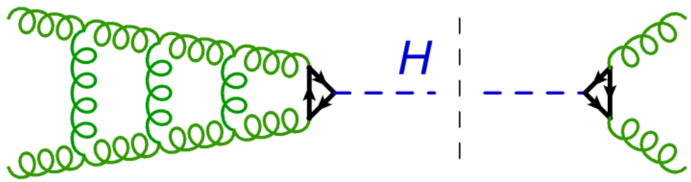


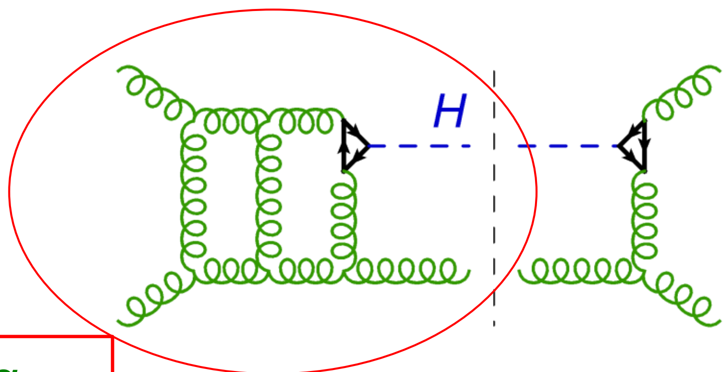
virtual *gg* → *H*

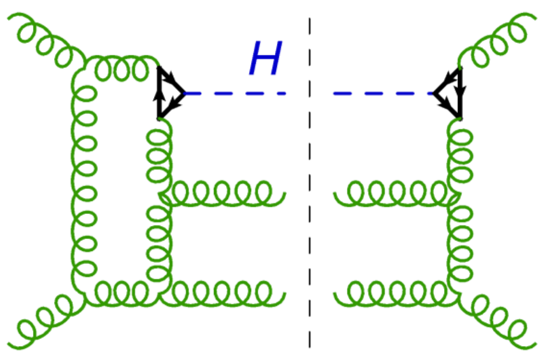


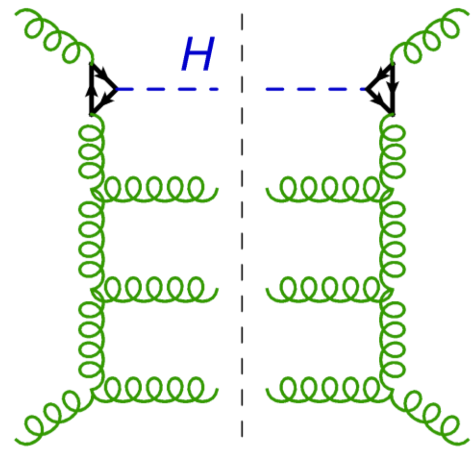
real, *gg* → *Hg*

N3LO QCD topologies









gg → *Hg*
@ 2 loops,
state of art
in QCD

+ …
+ quarks
+ operator renormalization
+ $1/m_t^2$ corrections
+ parton distributions

Multi-loop complexities

- Multi-loop multiscale integrals typically very difficult
- All 1 loop integrals with external legs in D=4 are reducible to scalar box integrals + simpler

→ combinations of

$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t)$$

+ simpler

Brown-Feynman (1952), Melrose (1965), Passarino-Veltman (1979), van Neerven-Vermaseren (1984), Bern, LD, Kosower (1992)

- At L loops, get special functions with up to $2L$ integrations
  Hannesdottier, McLeod, Schwartz, Vergu, 2109.09744
- Weight $2L$ iterated integrals, generalized polylogarithms, or worse

Planar N=4 SYM, toy model for QCD amplitudes

- QCD's maximally supersymmetric cousin, N=4 super-Yang-Mills theory (SYM), gauge group $SU(N_c)$, in the large N_c (planar) limit
- Structure very rigid:

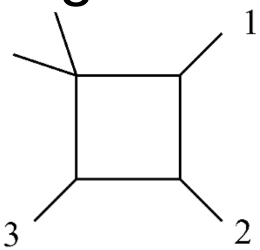
$$\text{Amplitudes} = \sum_i rational_i \times transcendental_i$$

- For planar N=4 SYM, we understand rational structure quite well, focus on the transcendental functions.
- Furthermore, at least three dualities hold:

1. AdS/CFT
2. Amplitudes dual to Wilson loops
3. New "antipodal" duality between amplitudes and form factors

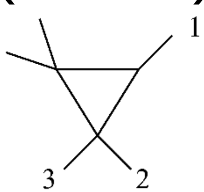
N=4 SYM very special

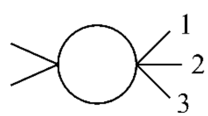
- At one loop, cancellation of loop momenta in numerator → only scalar box integrals
  Bern, LD, Dunbar, Kosower, hep-ph/9403226
- Weight 2 functions – dilogs. E.g., *gg* → *Hg* @ 1 loop ⊃



$$= \mathrm{Li}_2\left(1-\frac{s_{123}}{s_{12}}\right)+\mathrm{Li}_2\left(1-\frac{s_{123}}{s_{23}}\right)+\frac{1}{2}\ln^2\left(\frac{s_{12}}{s_{23}}\right)+\cdots$$

- QCD results also contain single log's and rational parts from (tensor) triangle + bubble integrals





$$=\frac{1}{\epsilon}-\ln(s_{123})$$

Higher loops

- Much evidence that N=4 SYM amplitudes have "uniform weight (transcendentality)" $2L$ at loop order L
- Weight ~ number of integrations, e.g.

$$\ln(s) = \int_1^s \frac{dt}{t} = \int_1^s d\ln t \qquad 1$$

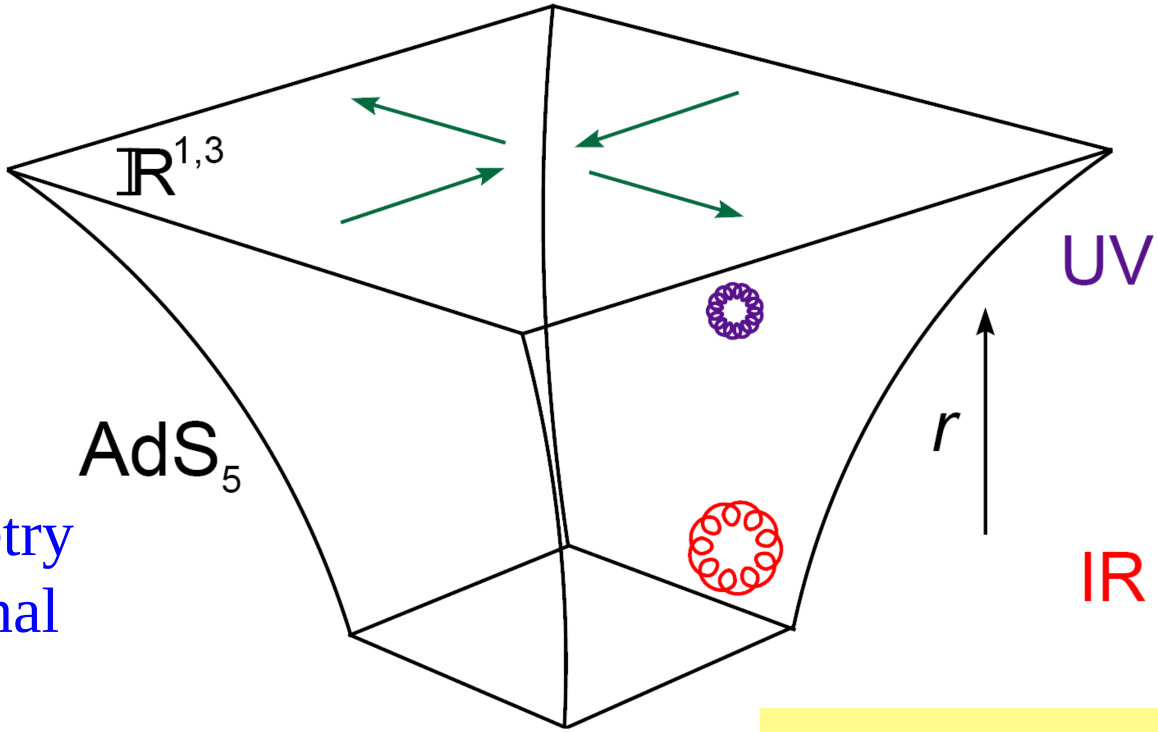
$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t) = \int_0^x d\ln t \cdot [-\ln(1-t)] \qquad 2$$

$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t) \qquad n$$

AdS/CFT

Maldacena (1997)
Gubser, Klebanov, Polyakov; Witten (1998)

Conformal field theory (like N=4 SYM) is dual to a theory of gravity in anti-de Sitter space (like strings in $AdS_5 \times S^5$)



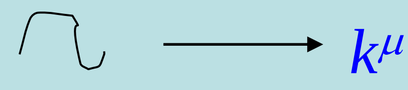
SO(4,2) isometry
of 5 dimensional
space-time
←→ 4d conformal symmetry

A weak-strong duality

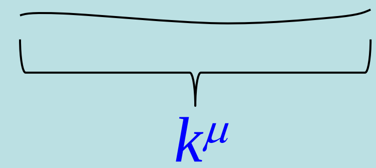
T-duality symmetry of string theory

Alday, Maldacena, 0705.0303

- Exchanges string world-sheet variables σ, τ
- $X^\mu(\tau,\sigma) = x^\mu + k^\mu \tau$ + oscillators



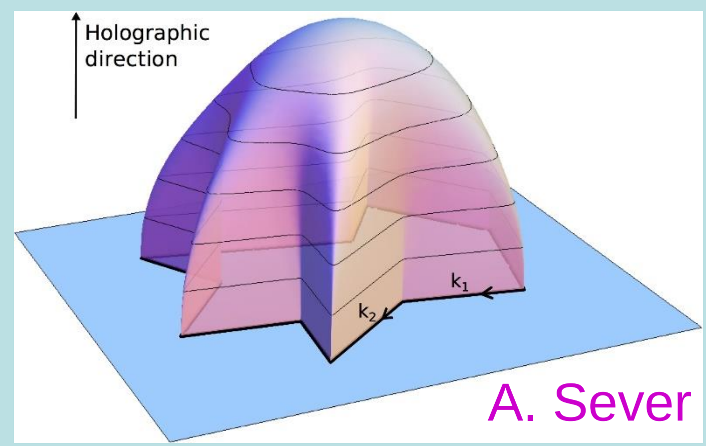
→ $X^\mu(\tau,\sigma) = x^\mu + k^\mu \sigma$ + oscillators



- Strong coupling limit of planar N=4 SYM

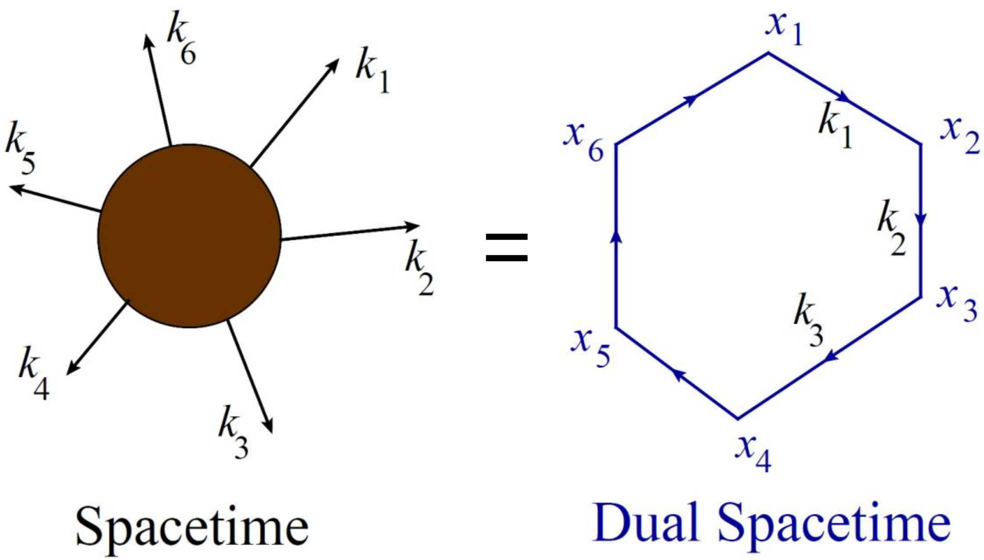
is semi-classical limit of string theory: world-sheet stretches tight around minimal area surface in AdS.

- Boundary determined by momenta of external states: light-like polygon with null edges = momenta k^μ



A. Sever

Amplitudes = Wilson loops



- Polygon vertices x_i are not positions but dual momenta, $x_i - x_{i+1} = k_i$
- Transform like positions under dual conformal symmetry

Alday, Maldacena, 0705.0303
Drummond, Korchemsky, Sokatchev, 0707.0243
Brandhuber, Heslop, Travaglini, 0707.1153
Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223, 0803.1466;
Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465

Duality verified to hold at weak coupling too

weak-weak duality, holds order-by-order

Dual conformal invariance

- Wilson n-gon invariant under inversion: $x_i^\mu \to \frac{x_i^\mu}{x_i^2}, \quad x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}$

$$x_{ij}^2 = (k_i + k_{i+1} + \cdots + k_{j-1})^2 \equiv s_{i,i+1,\cdots,j-1}$$

- Fixed, up to functions of invariant cross ratios:

$$u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

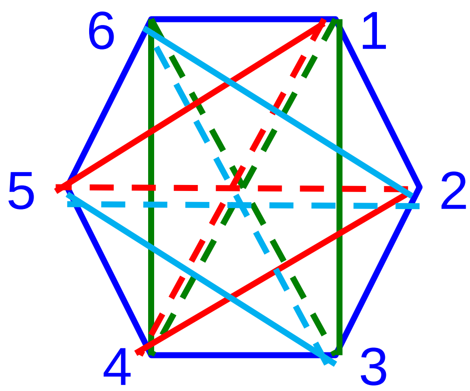
- $x_{i,i+1}^2 = k_i^2 = 0$ → no such variables for $n = 4,5$

$n = 6$ → precisely 3 ratios:

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}}$$

$$v = \frac{s_{23}s_{56}}{s_{234}s_{123}}$$

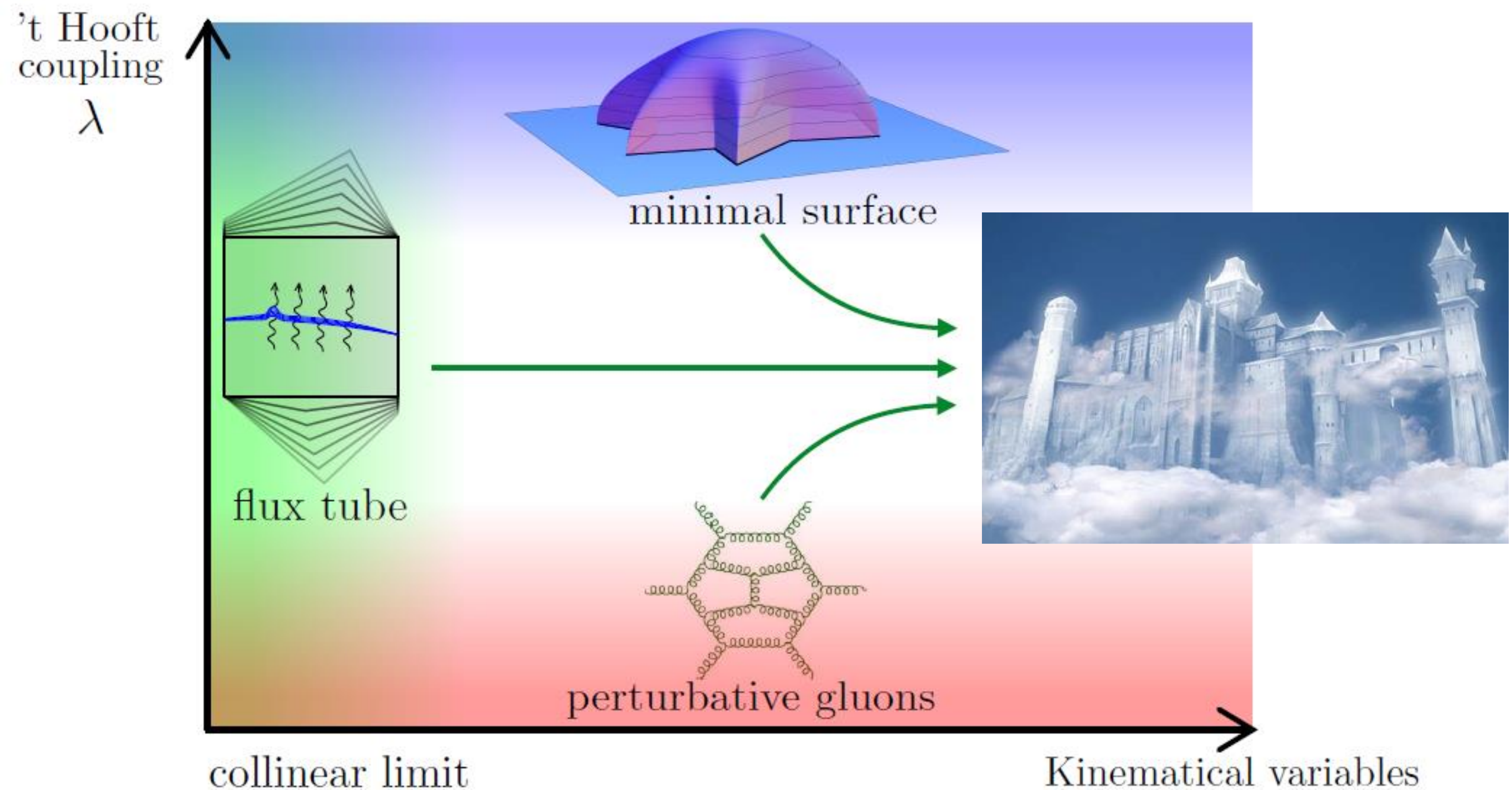
$$w = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$



$n = 7$ → 6 ratios.
In general, $3n$-15 ratios.

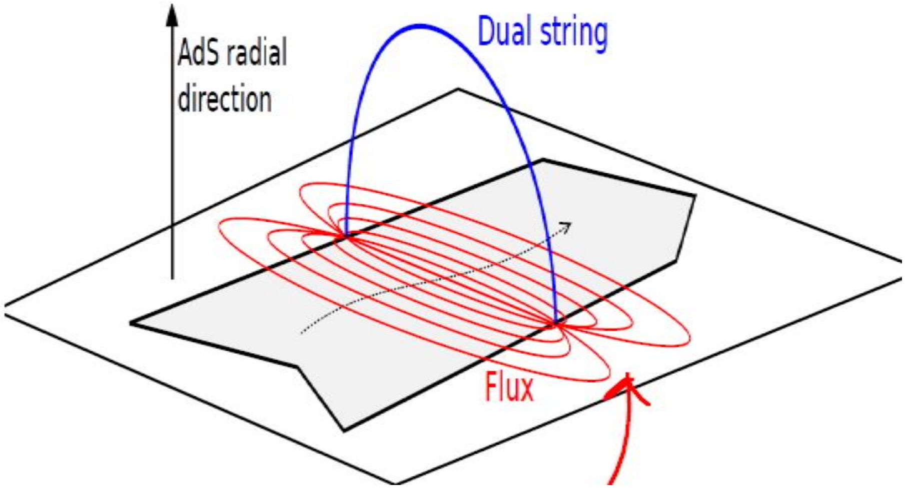
Solving for Planar N=4 SYM Amplitudes

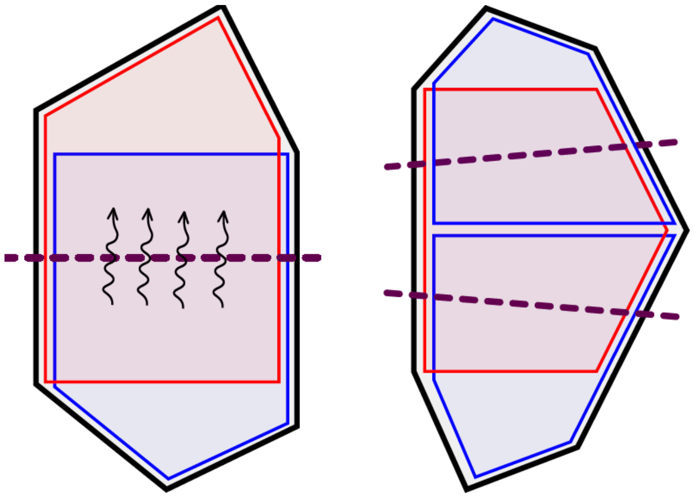
Images: A. Sever, N. Arkani-Hamed



Flux tubes at finite coupling

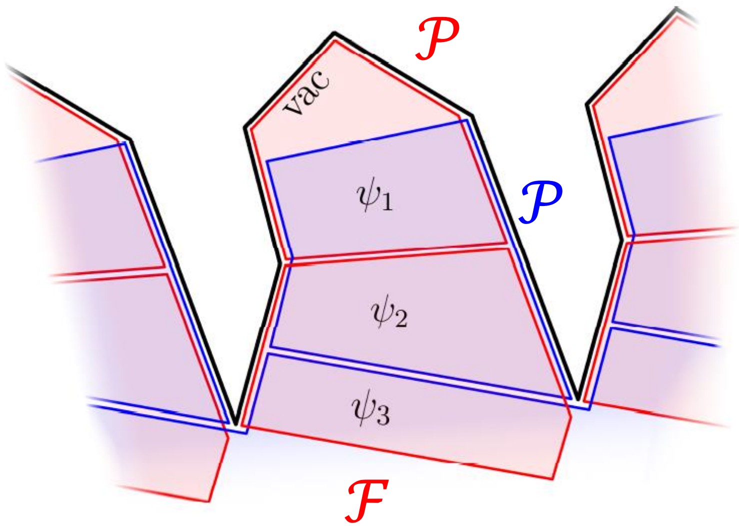
Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788;
Basso, Sever, Vieira, 1303.1396, 1306.2058, 1402.3307, 1407.1736, 1508.03045
BSV+Caetano+Cordova, 1412.1132, 1508.02987





- Tile n-gon with pentagon transitions.
- Quantum integrability → compute pentagons exactly in 't Hooft coupling
- 4d S-matrix as expansion (OPE) in number of flux-tube excitations = expansion around near collinear limit

The new FFOPE



- Form factors are Wilson loops in a periodic space, due to injection of operator momentum

Alday, Maldacena, 0710.1060; Maldacena, Zhiboedov, 1009.1139; Brandhuber, Spence, Travaglini, Yang, 1011.1899

- Besides pentagon transitions $\mathcal{P}$, this program needs an additional ingredient, the form factor transition $\mathcal{F}$

Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367, 2112.10569

OPE representation

- 6-gluon amplitude:

$$\mathcal{W}_{\rm hex} = \sum_{\mathbf{a}} \int d\mathbf{u}\, P_{\mathbf{a}}(0|\mathbf{u}) P_{\mathbf{a}}(\bar{\mathbf{u}}|0)\, e^{-E(\mathbf{u})\tau + ip(\mathbf{u})\sigma + im\phi}$$

$T = e^{-\tau}, S = e^{-\sigma}, F = e^{i\phi}$. $\quad v = \frac{T^2}{1+T^2} \to 0,$

weak-coupling, $E = k + \mathcal{O}(g^2)$ → expansion in T^k

- 3-gluon form factor: $\psi = helicity\ 0\ pairs\ of\ states$

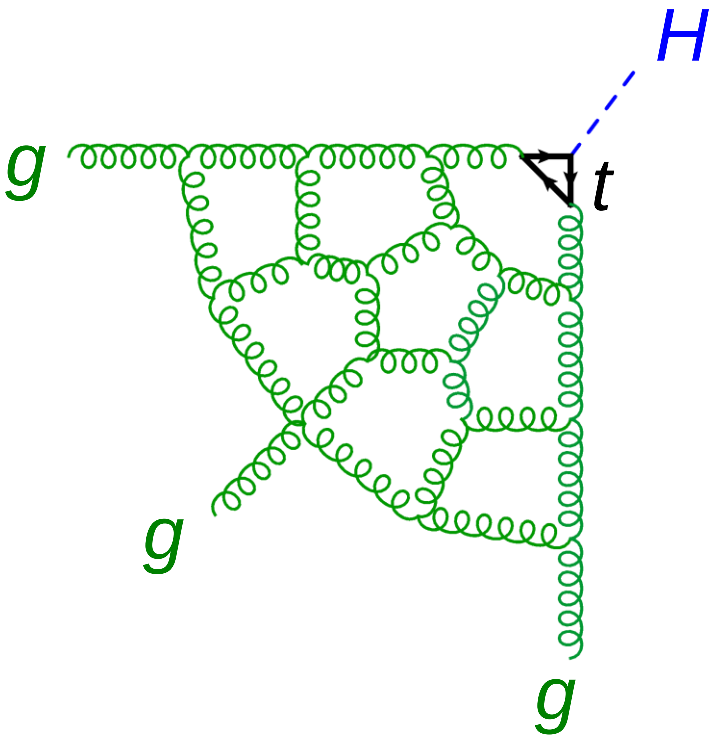
$$\mathcal{W}_3 = \sum\!\!\!\!\!\!\!\int_{\psi} e^{-E_\psi \tau + ip_\psi \sigma} \mathcal{P}(0|\psi) \mathcal{F}(\psi)$$

weak-coupling → expansion in T^{2k} (no azimuthal angle ϕ)

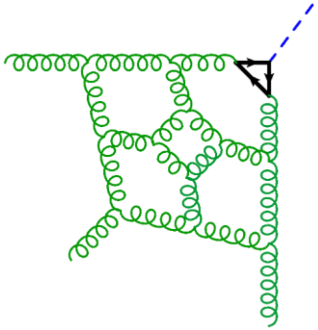
“Higgs” amplitudes and N=4 SYM form factors

LD, A. McLeod, M. Wilhelm, 2012.12286 — 3,4,5 loops

+ Ö. Gürdoğan, to appear — 6,7,8 loops



- At leading order in $\mathbf{1/m_{top}}$, Higgs boson couples to gluons via the operator $H\,G^{a}_{\mu\nu}G^{\mu\nu\,a}$



Form factors (cont.)

- Higgs is a scalar, color singlet. In QCD its amplitudes with gluons are matrix elements of $G_{\mu\nu}^{a} G^{\mu\nu\, a}$ with on-shell gluons: "form factors"
- In N=4, this operator is part of the (BPS-protected) stress tensor supermultiplet, which also includes for example $\phi_1^\dagger \phi_1 - \phi_2^\dagger \phi_2$ ($\in$ $\mathbf{20}$ of $SU(4)_R$)
- *Hgg* "Sudakov" form factor is "too simple"; it has no kinematic dependence beyond overall $(-s_{12})^{-L\epsilon}$
- *Hggg* is "just right", depends on 2 dimensionless ratios

Hggg kinematics is two-dimensional

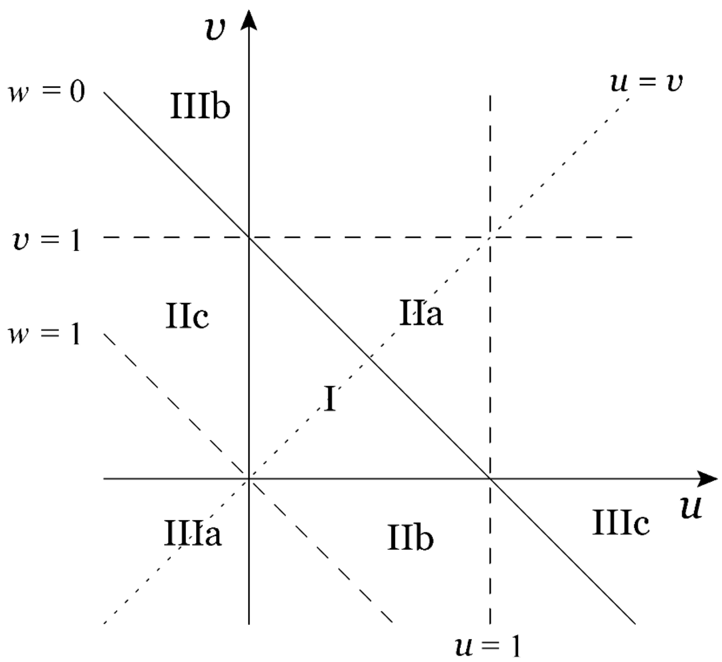
$$k_1 + k_2 + k_3 = -k_H$$

$$s_{123} = s_{12} + s_{23} + s_{31} = m_H^2$$

$$s_{ij} = (k_i + k_j)^2 \qquad k_i^2 = 0$$

$$u = \frac{s_{12}}{s_{123}} \qquad v = \frac{s_{23}}{s_{123}} \qquad w = \frac{s_{31}}{s_{123}}$$

not cross ratios!



$$u + v + w = 1$$

I = decay / Euclidean

IIa,b,c = scattering / spacelike operator

IIIa,b,c = scattering / timelike operator

N=4 amplitude is S_3 invariant

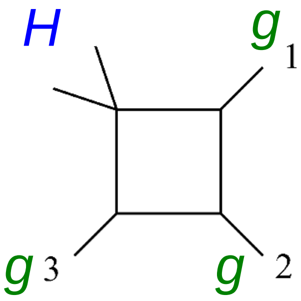
$D_3 \equiv S_3$ dihedral symmetry generated by:

a. cycle: $i \to i + 1 \pmod 3$, or

$$u \to v \to w \to u$$

b. flip: $u \leftrightarrow v$

One loop integrals/amplitudes



$$= \mathrm{Li}_2\left(1 - \frac{s_{123}}{s_{12}}\right) + \mathrm{Li}_2\left(1 - \frac{s_{123}}{s_{23}}\right) + \frac{1}{2}\ln^2\left(\frac{s_{12}}{s_{23}}\right) + \cdots$$

$$= \mathrm{Li}_2\left(1 - \frac{1}{u}\right) + \mathrm{Li}_2\left(1 - \frac{1}{v}\right) + \frac{1}{2}\ln^2\left(\frac{u}{v}\right) + \cdots$$

A two-loop story

- Gehrmann et al. computed *Hggg* in QCD at 2 loops
  Gehrmann, Jaquier, Glover, Koukoutsakis, 1112.3554
- Soon after, Brandhuber et al. computed stress tensor 3-point form factor $\mathcal{F}_3$ in N=4 SYM,
  Brandhuber, Travaglini, Yang, 1201.4170
  saw that "maximally transcendental part" of QCD result (both (+++) and (-++)) was **same as N=4 result!!**
- This "principle of maximal transcendentality"
  Kotikov, Lipatov, Velizhanin, hep-ph/0301021, hep-ph/0611204
  was known to work for DGLAP and BFKL anomalous dimensions, but not for generic scattering amplitudes, so this one is very special

2d HPLs

Gehrmann, Remiddi, hep-ph/0008287

Space graded by weight n. Every function F obeys:

$$\frac{\partial F(u,v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{1-u-v} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{u+v}$$

$$\frac{\partial F(u,v)}{\partial v} = \frac{F^v}{v} - \frac{F^w}{1-u-v} - \frac{F^{1-v}}{1-v} + \frac{F^{1-w}}{u+v}$$

$$w = 1-u-v$$

where $F^u, F^v, F^w, F^{1-u}, F^{1-v}, F^{1-w}$ are weight n-1 2d HPLs.

To bootstrap *Hggg* amplitude beyond 2 loops, find as small a subspace of 2d HPLs as possible, construct it to high weight.

Generalized polylogarithms

Chen, Goncharov, Brown,…

- Can be defined as iterated integrals, e.g.

$$G(a_1, a_2, \ldots, a_n, x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n, t)$$

- Or define differentially: $dF = \sum_{s_k \in \mathcal{S}} F^{s_k} \, d \ln s_k$

- There is a Hopf algebra that "co-acts" on the space of polylogarithms, $\Delta: F \to F \otimes F$
- The derivative dF is one piece of Δ: $\Delta_{n-1,1} F = \sum_{s_k \in \mathcal{S}} F^{s_k} \otimes \ln s_k$
- so we refer to F^{s_k} as a $\{n\text{-}1,1\}$ coproduct of F
- $\boldsymbol{s_k}$ **are letters in the symbol alphabet** $\mathcal{S}$

Generalized polylogarithms (cont.)

- The $\{n\text{-}1,1\}$ coaction can be applied iteratively.
- Define the $\{n\text{-}2,1,1\}$ double coproducts, F^{s_k,s_j},

via the derivatives of the $\{n\text{-}1,1\}$ single coproducts F^{s_j}:

$$dF^{s_j} \equiv \sum_{s_k \in \mathcal{S}} F^{s_k,s_j}\, d \ln s_k$$

- And so on for the $\{n\text{-}m,1,\ldots,1\}$ $m^{\rm th}$ coproducts of F.
- The maximal iteration, n times for a weight n function, is

the symbol,

$$\mathcal{S}[F] = \sum_{s_{i_1},\ldots,s_{i_n} \in \mathcal{S}} F^{s_{i_1},\ldots,s_{i_n}}\, d \ln s_{i_1} \ldots d \ln s_{i_n} \quad \equiv \sum_{s_{i_1},\ldots,s_{i_n} \in \mathcal{S}} F^{s_{i_1},\ldots,s_{i_n}}\, s_{i_1} \otimes \ldots \otimes s_{i_n}$$

where now $F^{s_{i_1},\ldots,s_{i_n}}$ are just rational numbers

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Example: The classical polylogarithms

$$\mathrm{Li}_1(x) = -\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

- Regular at $x = 0$, branch cut starts at $x = 1$.
- Iterated differentiation gives the symbol:

$$\mathcal{S}[Li_n(x)] = \mathcal{S}[Li_{n-1}(x)] \otimes x$$
$$= \cdots = -(1-x) \otimes x \otimes \ \ldots \otimes x$$

- Branch cut discontinuities displayed in first entry of symbol, e.g clip off leading $(1-x)$ to compute discontinuity at $x = 1$.
- Derivatives computed from symbol by clipping last entry, multiplying by that $d \ln(\ldots)$.

Example: Harmonic Polylogarithms in one variable (HPLs {0,1})

Remiddi, Vermaseren, hep-ph/9905237

- Generalize the classical polylogs:

$$\text{Li}_n(u) \;=\; \int_0^u \frac{dt}{t}\text{Li}_{n-1}(t), \quad \text{Li}_1(t) = -\ln(1-t)$$

- Define HPLs by iterated integration:

$$H_{0,\vec{w}}(u) = \int_0^u \frac{dt}{t} H_{\vec{w}}(t), \quad H_{1,\vec{w}}(u) = \int_0^u \frac{dt}{1-t} H_{\vec{w}}(t)$$

- Or by derivatives:

$$dH_{0,\vec{w}}(u) = H_{\vec{w}}(u)\ d\ln u \quad dH_{1,\vec{w}}(u) = -H_{\vec{w}}(u) d\ln(1-u)$$

- Symbol letters: $\mathcal{S} \;=\; \{u, 1-u\}$
- Weight n = length of binary string $\vec{w}$
- Number of functions at weight $n = 2L$: 2^{2L}
- Branch cuts dictated by first integration/entry in symbol
- Derivatives dictated by last integration/entry in symbol

Symbol alphabet for *Hggg*

Gehrmann, Remiddi, hep-ph/0008287

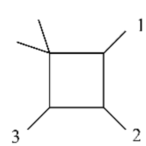
- Comparing
$$\frac{\partial F(u,v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{1-u-v} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{u+v}$$
$$\frac{\partial F(u,v)}{\partial v} = \frac{F^v}{v} - \frac{F^w}{1-u-v} - \frac{F^{1-v}}{1-v} + \frac{F^{1-w}}{u+v}$$

with $dF = \sum_{s_k \in \mathcal{S}} F^{s_k}\, d\ln s_k$

we see that $\mathcal{S} = \{u, v, w, 1-u, 1-v, 1-w\}$ $\quad w = 1-u-v$

$\exists$ dihedral symmetry $D_3 \equiv S_3$, permutations of $\{u, v, w\}$

For example, all permutations of (finite part of) box integral are in this space.



$$= \mathrm{Li}_2\left(1-\frac{1}{u}\right) + \mathrm{Li}_2\left(1-\frac{1}{v}\right) + \frac{1}{2}\ln^2\left(\frac{u}{v}\right) + \cdots$$

A better alphabet

- Motivated by a similar change of variables in the 6 gluon case Caron-Huot, LD, von Hippel, McLeod, 1609.00669 (which exposes the Steinmann relations there), we also switch to the alphabet

$$\mathcal{S}' = \{\, a = \frac{u}{vw}, b = \frac{v}{wu}, c = \frac{w}{uv}, d = \frac{1-u}{u}, e = \frac{1-v}{v}, f = \frac{1-w}{w} \,\}$$

- We find that the symbols of the (suitably normalized) form factor $F_3^{(L)}$ at one and two loops simplify remarkably, down to just 1 and 2 terms, plus dihedral images(!!!):

$$S\left[F_3^{(1)}\right] = (-1)\; b \otimes d + \text{dihedral}$$

$$S\left[F_3^{(2)}\right] = 4\; b \otimes d \otimes d \otimes d + 2\; b \otimes b \otimes b \otimes d + \text{dihedral}$$

Simplest analytic form is for $v \to \infty$

→ Harmonic polylogarithms $H_{\vec{w}} \equiv H_{\vec{w}}(1-\frac{1}{u})$

$$
\begin{aligned}
F_3^{(1)}(v\to\infty) &= 2H_{0,1} + 6\zeta_2 \\
F_3^{(2)}(v\to\infty) &= -8H_{0,0,0,1} - 4H_{0,1,1,1} + 12\zeta_2 H_{0,1} + 13\zeta_4 \\
F_3^{(3)}(v\to\infty) &= 96H_{0,0,0,0,0,1} + 16H_{0,0,0,1,0,1} + 16H_{0,0,0,1,1,1} + 16H_{0,0,1,0,0,1} + 8H_{0,0,1,0,1,1} \\
&\quad + 8H_{0,0,1,1,0,1} + 16H_{0,1,0,0,0,1} + 8H_{0,1,0,0,1,1} + 12H_{0,1,0,1,0,1} + 4H_{0,1,0,1,1,1} \\
&\quad + 8H_{0,1,1,0,0,1} + 4H_{0,1,1,0,1,1} + 4H_{0,1,1,1,0,1} + 24H_{0,1,1,1,1,1} \\
&\quad - \zeta_2(32H_{0,0,0,1} + 8H_{0,0,1,1} + 4H_{0,1,0,1} + 52H_{0,1,1,1}) \\
&\quad - \zeta_3(8H_{0,0,1} - 4H_{0,1,1}) - 53\zeta_4 H_{0,1} - \frac{167}{4}\zeta_6 + 2(\zeta_3)^2
\end{aligned}
$$

8 loop result has $\sim 2^{2\times 8-2} = 16{,}384$ terms

6-gluon amplitude is simplest for $(\hat{u}, \hat{v}, \hat{w}) = (1, \hat{v}, \hat{v})$

- Let $H_{\vec{w}} \equiv H_{\vec{w}}(1 - \frac{1}{\hat{v}})$

$$A_6^{(1)}(1,\hat{v},\hat{v}) = 2H_{0,1}$$

$$A_6^{(2)}(1,\hat{v},\hat{v}) = -8H_{0,1,1,1} - 4H_{0,0,0,1} - 4\zeta_2 H_{0,1} - 9\zeta_4$$

$$\begin{aligned} A_6^{(3)}(1,\hat{v},\hat{v}) = \; & 96H_{0,1,1,1,1,1} + 16H_{0,1,0,1,1,1} + 16H_{0,0,0,1,1,1} + 16H_{0,1,1,0,1,1} + 8H_{0,0,1,0,1,1} \\ & + 8H_{0,1,0,0,1,1} + 16H_{0,1,1,1,0,1} + 8H_{0,0,1,1,0,1} + 12H_{0,1,0,1,0,1} + 4H_{0,0,0,1,0,1} \\ & + 8H_{0,1,1,0,0,1} + 4H_{0,0,1,0,0,1} + 4H_{0,1,0,0,0,1} + 24H_{0,0,0,0,0,1} \\ & + \zeta_2(8H_{0,0,0,1} + 8H_{0,1,0,1} + 48H_{0,1,1,1}) \\ & + 42\zeta_4 H_{0,1} + 121\zeta_6 \end{aligned}$$

There is an exact map at symbol level, with $\frac{1}{\hat{v}} = 1 - \frac{1}{u}$,
$0 \leftrightarrow 1$, if you also reverse the order of the symbol entries!!!
It works to 7 loops, where $\sim 2^{2\times 7-2} = 4{,}096$ terms agree

Antipodal duality in 2d

weak-weak duality

$$F_3^{(L)}(u,v,w) = S\left(A_6^{(L)}(\hat{u},\hat{v},\hat{w})\right)$$

where the antipode S, at symbol level, reverses the order of all letters:

$$S(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = (-1)^m \, x_m \otimes \cdots \otimes x_2 \otimes x_1$$

and the kinematic map is

$$\hat{u} = \frac{vw}{(1-v)(1-w)}, \qquad \hat{v} = \frac{wu}{(1-w)(1-u)}, \qquad \hat{w} = \frac{uv}{(1-u)(1-v)}$$

which maps $u+v+w=1$ to the parity-preserving surface

$$\Delta \equiv (1-\hat{u}-\hat{v}-\hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0$$

corresponding to $\hat{k}_{i+n}^{\mu} = -\hat{k}_i^{\mu}, i = 1,2,\ldots,n \quad (n = 3\ here)$

6-gluon alphabet and symbol map

Goncharov, Spradlin, Vergu, Volovich, 1006.5703; LD, Drummond, Henn, 1108.4461; Caron-Huot, LD, von Hippel, McLeod, 1609.00669

- $\mathcal{S}_6 = \{\, \hat{u}, \hat{v}, \hat{w}, 1-\hat{u}, 1-\hat{v}, 1-\hat{w}, \hat{y}_u, \hat{y}_v, \hat{y}_w \}$ (the $\hat{y}_u, \hat{y}_v, \hat{y}_w$ are crossed out, with an arrow to:) 1 for $\Delta = 0$

$\rightarrow \mathcal{S}_6' = \{\, \hat{a} = \frac{\hat{u}}{\hat{v}\hat{w}}, \hat{b} = \frac{\hat{v}}{\hat{w}u}, \hat{c} = \frac{\hat{w}}{\hat{u}\hat{v}}, \hat{d} = \frac{1-\hat{u}}{\hat{u}}, \hat{e} = \frac{1-\hat{v}}{\hat{v}}, \hat{f} = \frac{1-\hat{w}}{\hat{w}} \}$

- Kinematic map on letters:

$$\sqrt{\hat{a}} = d, \qquad \hat{d} = a, \qquad \textit{plus cyclic relations}$$

$$\mathcal{S}\left[A_6^{(1)}\right] = (-\frac{1}{2})\hat{b}\otimes\hat{d} + \text{dihedral}$$

$$\mathcal{S}\left[A_6^{(2)}\right] = b\otimes d\otimes d\otimes d + \frac{1}{2} b\otimes b\otimes b\otimes d + \text{dihedral}$$

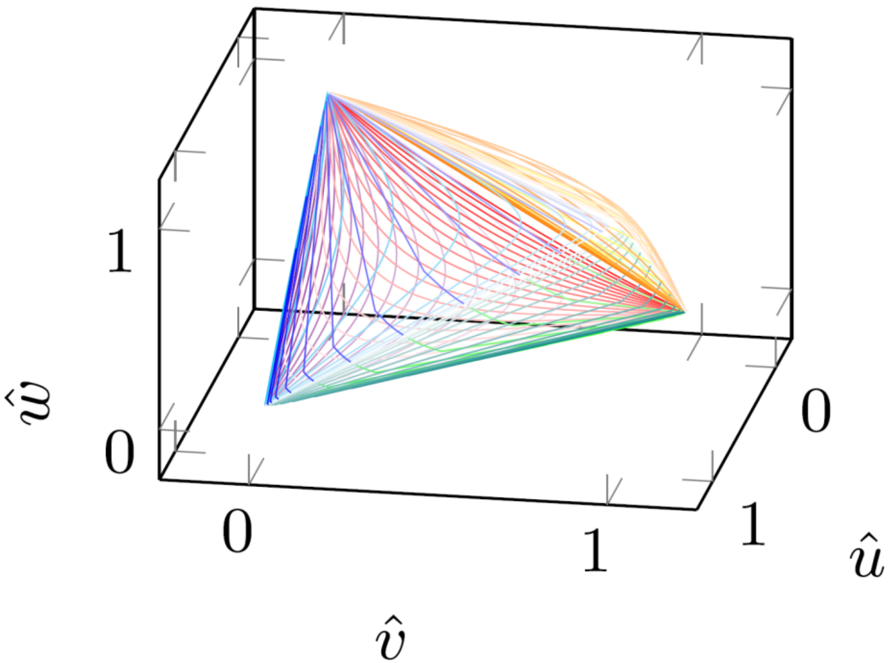
$$\ldots$$

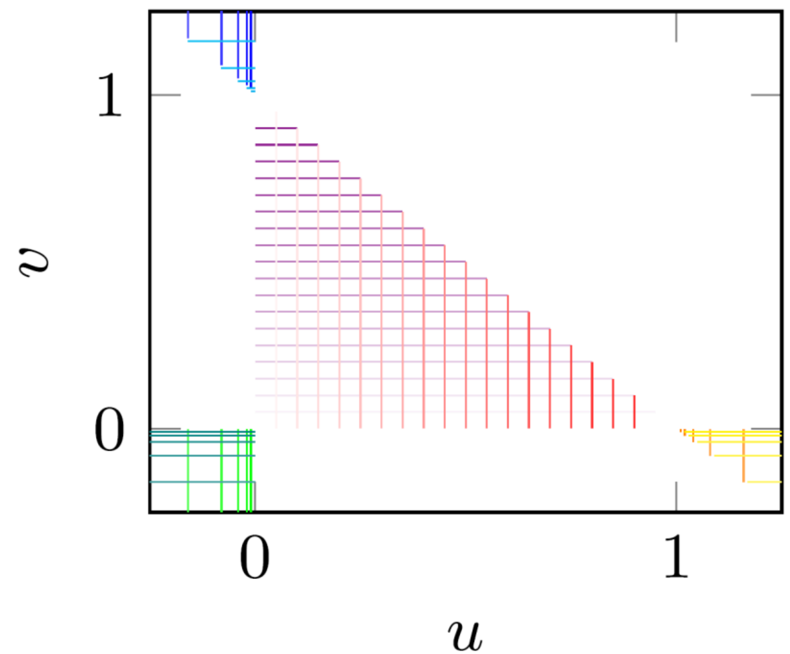
L loop symbol

L	number of terms
1	6
2	12
3	636
4	11,208
5	263,880
6	4,916,466
7	92,954,568
8	1,671,656,292

- Works through 7 loops! (arrow to the $L = 7$ row)

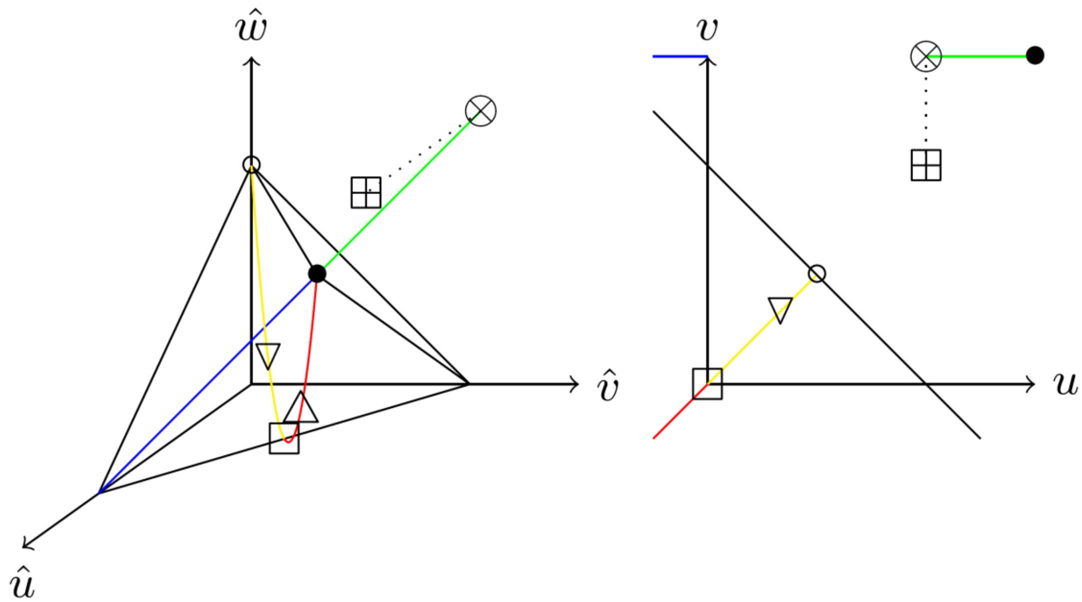
Map covers entire phase space for 3-gluon form factor





- Soft is dual to collinear; collinear is dual to soft
- White regions in (u, v) map to some of $\hat{u}, \hat{v}, \hat{w} > 1$

Many special dual points



There is an "f" alphabet at all of these points, which is a way of writing multiple zeta values (MZV's) so that the coaction is manifest.

F. Brown, 1102.1310; O. Schnetz, HyperlogProcedures

	$(\hat{u},\hat{v},\hat{w})$	(u,v,w)	functions
$\triangledown$	$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$\sqrt[6]{1}$
$\square$	$(\frac{1}{2},\frac{1}{2},0)$	$(0,0,1)$	$\mathrm{Li}_2(\frac{1}{2})$ + logs
$\bullet$	$(1,1,1)$	$\lim_{u\to\infty}(u,u,1-2u)$	MZVs
$\circ$	$(0,0,1)$	$(\frac{1}{2},\frac{1}{2},0)$	MZVs + logs
$\triangle$	$(\frac{3}{4},\frac{3}{4},\frac{1}{4})$	$(-1,-1,3)$	$\sqrt[6]{1}$
$\boxplus$	(∞,∞,∞)	$(1,1,-1)$	alternating sums
$\otimes$	$\lim_{\hat{v}\to\infty}(1,\hat{v},\hat{v})$	$\lim_{v\to\infty}(1,v,-v)$	MZVs
green–blue line	$(1,\hat{v},\hat{v})$	$\lim_{v\to\infty}(u,v,1-u-v)$	HPL$\{0,1\}$
red–yellow line	$(\hat{u},\hat{u},(1-2\hat{u})^2)$	$(u,u,1-2u)$	HPL$\{-1,0,1\}$

The simplest point

- $(\hat{u},\hat{v},\hat{w}) = (1,1,1) \Leftrightarrow u,v \to \infty$
- At this point,

$$A_6^{(1)}(\cdot) = 0 \qquad F_3^{(1)}(\cdot) = 8\zeta_2$$

$$A_6^{(2)}(\cdot) = -9\zeta_4 \qquad F_3^{(2)}(\cdot) = 31\zeta_4$$

$$A_6^{(3)}(\cdot) = 121\zeta_6 \qquad F_3^{(3)}(\cdot) = -145\zeta_6$$

$$A_6^{(4)}(\cdot) = 120 f_{3,5} - 48\zeta_2 f_{3,3} - \frac{6381}{4}\zeta_8 \qquad F_3^{(4)}(\cdot) = 120 f_{5,3} + \frac{6381}{4}\zeta_8$$

$$A_6^{(5)}(\cdot) = -2688 f_{3,7} - 1560 f_{5,5} + \mathcal{O}(\pi^2) \qquad F_3^{(5)}(\cdot) = -2688 f_{7,3} - 1560 f_{5,5} + \mathcal{O}(\pi^2)$$

$$A_6^{(6)}(\cdot) = 48528 f_{3,9} + 37296 f_{5,7} + 21120 f_{7,5} + \mathcal{O}(\pi^2) \qquad F_3^{(6)}(\cdot) = 48528 f_{9,3} + 37296 f_{7,5} + 21120 f_{5,7} + \mathcal{O}(\pi^2)$$

- Reversing ordering of words in *f*-alphabet, the blue values show that antipodal duality holds at these points beyond symbol level, modulo $i\pi$
- modulo $i\pi$ seems to be the best we can get from the antipode

OPE parametrizations

- Amplitude:

$$\hat{u} = \frac{1}{1 + (\hat{T} + \hat{S}\hat{F})(\hat{T} + \hat{S}/\hat{F})} \,,$$

$$\hat{v} = \hat{u}\hat{w}\hat{S}^2/\hat{T}^2 \,, \qquad \hat{w} = \frac{\hat{T}^2}{1 + \hat{T}^2}$$

$(\hat{F} = 1 \text{ for } \Delta = 0)$

- Form factor:

$$u = \frac{1}{1 + S^2 + T^2} \,, \qquad v = \frac{T^2}{1 + T^2} \,,$$

$$w = \frac{1}{(1 + T^2)(1 + S^{-2}(1 + T^2))} \,,$$

- Apply the kinematic map →

$$\hat{T} = \frac{T}{S} \,, \qquad \hat{S} = \frac{1}{TS}$$

- There is apparently a correspondence between single flux tube excitations for the amplitude (T^1) and double (or bound state) excitations for the form factor (T^2)

8-gluon Amp ←→ 4-gluon FF

- We have a candidate kinematic map for a 4-dimensional surface (4-gluon FF is 5d).
- $\mathcal{S}[R_8^{(2)}]$ is known S. Caron-Huot, 1105.5606
- The kinematic+antipodal maps take it to a symbol with 40 letters, the first 8 of which are "right": $u_i = \frac{s_{i,i+1}}{s_{1234}}, \quad v_i = \frac{s_{i,i+1,i+2}}{s_{1234}}$
- But we still have to run more checks on this candidate 2-loop 4-gluon form factor

8-4 Kinematic Map in OPE Parametrization

- 8-point amplitude has D_8 dihedral symmetry; change it to D_4 of the form factor by requiring

$$\hat{T}_3 = \hat{T}_1, \quad \hat{S}_3 = \hat{S}_1, \quad \hat{F}_3 = \hat{F}_1$$

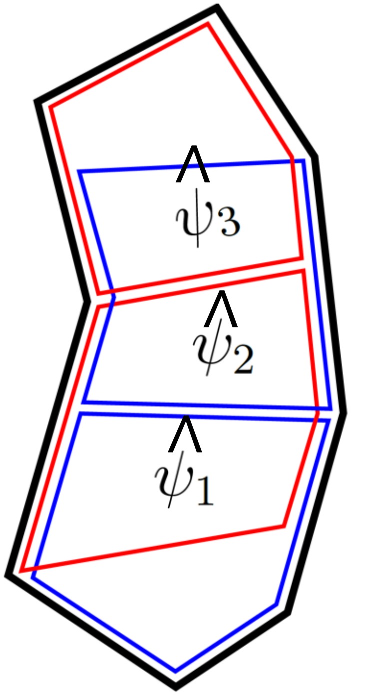
- To get $\mathcal{S}[R_8^{(2)}]$ to have only 8 final

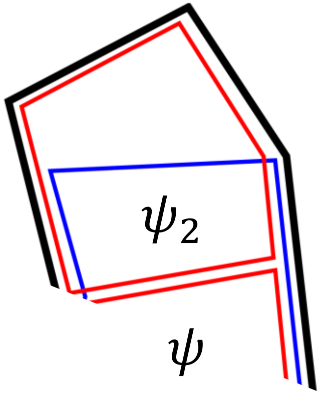
entries, we also fix $\hat{F}_1 = \hat{F}_2 = 1$.

- The kinematic map becomes

$$\hat{T}_1 = \frac{T}{S}, \hat{S}_1 = \frac{1}{TS},$$

$$\hat{T}_2 = \frac{T_2}{S_2}, \hat{S}_2 = \frac{1}{T_2 S_2}$$ and requires $F_2 = i$





Beyond 8-4

- The map $\hat{T}_1 = \frac{T}{S}, \hat{S}_1 = \frac{1}{TS}, \ \hat{T}_2 = \frac{T_2}{S_2}, \hat{S}_2 = \frac{1}{T_2 S_2}$

seems likely to generalize to give rise to a $2(n-2)$ parameter subspace of the full $3n-7$ dimensional n-point form factor kinematics, presumably from setting $F_2 = \cdots = F_{n-2} = \ i$

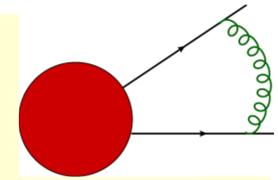
- We can conjecture that antipodal duality applies on this subspace
- But there is still a lot to be checked!

Summary & Outlook

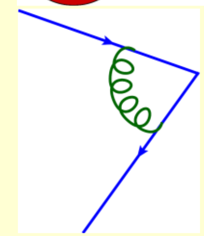
- Form factors as well as scattering amplitudes in planar N=4 SYM can now be **bootstrapped** to high loop order
- By comparing the 3-gluon form factor to the 6-gluon amplitude, we found a strange new antipodal duality, which swaps the role of branch cuts and derivatives, and seems to map single flux-tube excitations (amplitude) to doubles (form factor).
- What is the underlying physical reason for this duality?
- (How) does it hold at strong coupling?
- (How much) can we verify of it at the 8-4 level, and beyond?
- How much can we exploit it to learn more about both amplitudes and form factors?

Extra Slides

Removing Amplitude (or Form Factor) Infrared Divergences



- On-shell amplitudes IR divergent due to long-range gluons



- Polygonal Wilson loops UV divergent at cusps, anomalous dimension $\Gamma_{\rm cusp}$

  – known to all orders in planar N=4 SYM:
  Beisert, Eden, Staudacher, hep-th/0610251

- Both removed by dividing by BDS-like ansatz
  Bern, LD, Smirnov, hep-th/0505205, Alday, Gaiotto, Maldacena, 0911.4708
- Normalized [MHV] amplitude is finite, dual conformal invariant, also uniquely (up to constant) maintains important symbol adjacency relations due to causality (Steinmann relations for 3-particle invariants):

$$\mathcal{E}(u_i) = \lim_{\epsilon\to 0} \frac{\mathcal{A}_6(s_{i,i+1},\epsilon)}{\mathcal{A}_6^{\rm BDS-like}(s_{i,i+1},\epsilon)} = \exp[\frac{\Gamma_{\rm cusp}}{4}\mathcal{E}^{(1)} + R_6]$$

remainder function

BDS & BDS-like normalization for $\mathcal{F}_3$

$$\boxed{\frac{\mathcal{F}_3}{\mathcal{F}_3^{\text{MHV, tree}}} = \exp\Bigg\{\underbrace{\sum_{L=1}^{\infty} g^{2L}\Bigg[\Big(\frac{\Gamma_{\text{cusp}}^{(L)}}{4} + \mathcal{O}(\epsilon)\Big)M^{1-\text{loop}}(L\epsilon) + C^{(L)}}_{\text{BDS ansatz}} + R^{(L)}(u,v,w)\Bigg]\Bigg\}}$$

remainder function only a function of u, v, w; vanishes in all collinear limits, but no adjac... y constraints

split 1-loop amplitude judiciously:

$$\boxed{\frac{\mathcal{F}_3^{1-\text{loop}}}{\mathcal{F}_3^{\text{MHV, tree}}} \equiv M^{1-\text{loop}}(\epsilon) = M(\epsilon) + \mathcal{E}^{(1)}(u,v,w)}$$

$$M(\epsilon) = -\frac{1}{\epsilon^2}\sum_{i=1}^{3}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{\epsilon} - \frac{7}{2}\zeta_2 + \sum^{3} \ldots$$

$$\mathcal{E}^{(1)}(u,v \ldots \Big(1-\frac{1}{v}\Big) + \mathrm{Li}_2\Big(1-\frac{1}{w}\Big)\Big]$$

$$\mathcal{E}^{(1),u} + \mathcal{E}^{(1),1-u} = 0$$

$\mathcal{E}$ obeys "adjacency constraints"

Now div... by:

$$\frac{\mathcal{F}_3^{\text{BDS-like}}}{\mathcal{F}_3^{\text{MHV, tree}}} = \exp\Bigg\{\sum_{L=1}^{\infty} g^{2L}\Bigg[\Big(\frac{\Gamma_{\text{cusp}}}{4} + \mathcal{O}(\epsilon)\Big)M(L\epsilon) + C^{(L)}\Bigg]\Bigg\} \Rightarrow \boxed{\mathcal{E} = \exp\left[\frac{\Gamma_{\text{cusp}}}{4}\mathcal{E}^{(1)} + R\right]}$$

Different routes to perturbative amplitudes

Draw all Feynman graphs G_i

↓

Evaluate all Feynman rules: I_i

↓

Perform all loop integrations: A_i

↓

$$A = \sum_i A_i$$

Evaluate all unitarity cuts C_α

↓

Construct **local**: integrand I

↓

Perform all loop integrations: A_α

↓

$$A = \sum_\alpha A_\alpha$$

Bootstrap: Guess
$A = \sum_m c_m F_m$
F_m known functions
$c_m \in \mathbb{Q}$ unknown constants

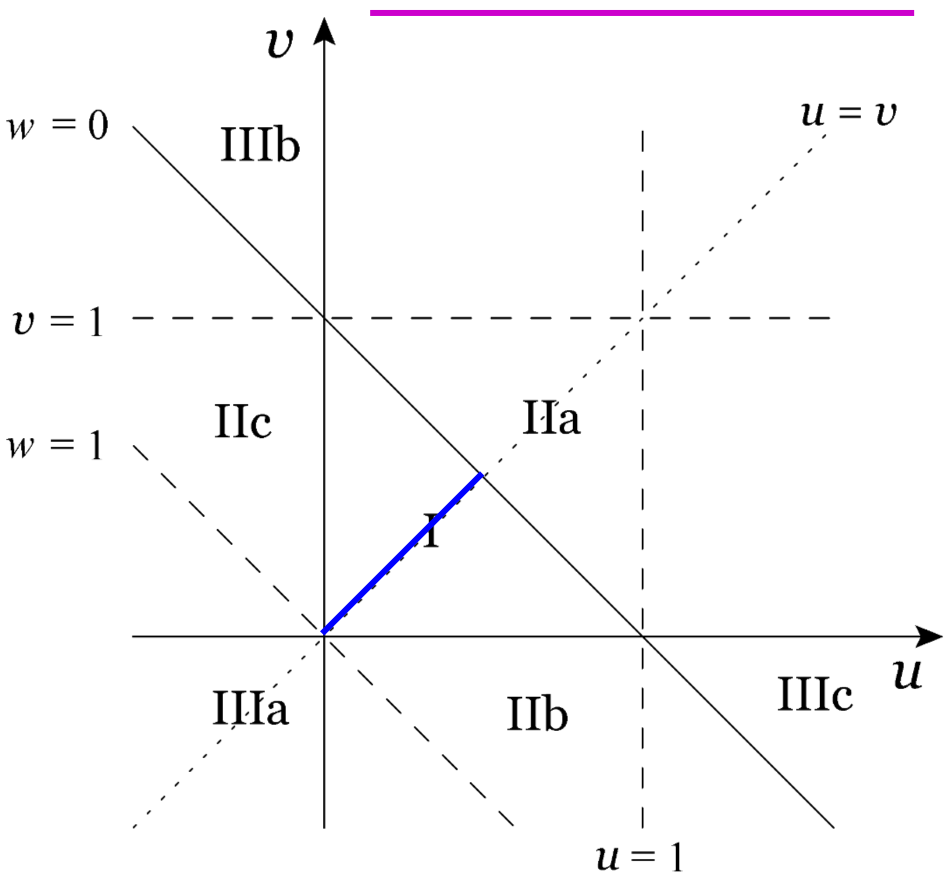
↓

Solve constraints, linear equations for $c_m \to r_m$

↓

$$A = \sum_m r_m F_m$$

Some numerics

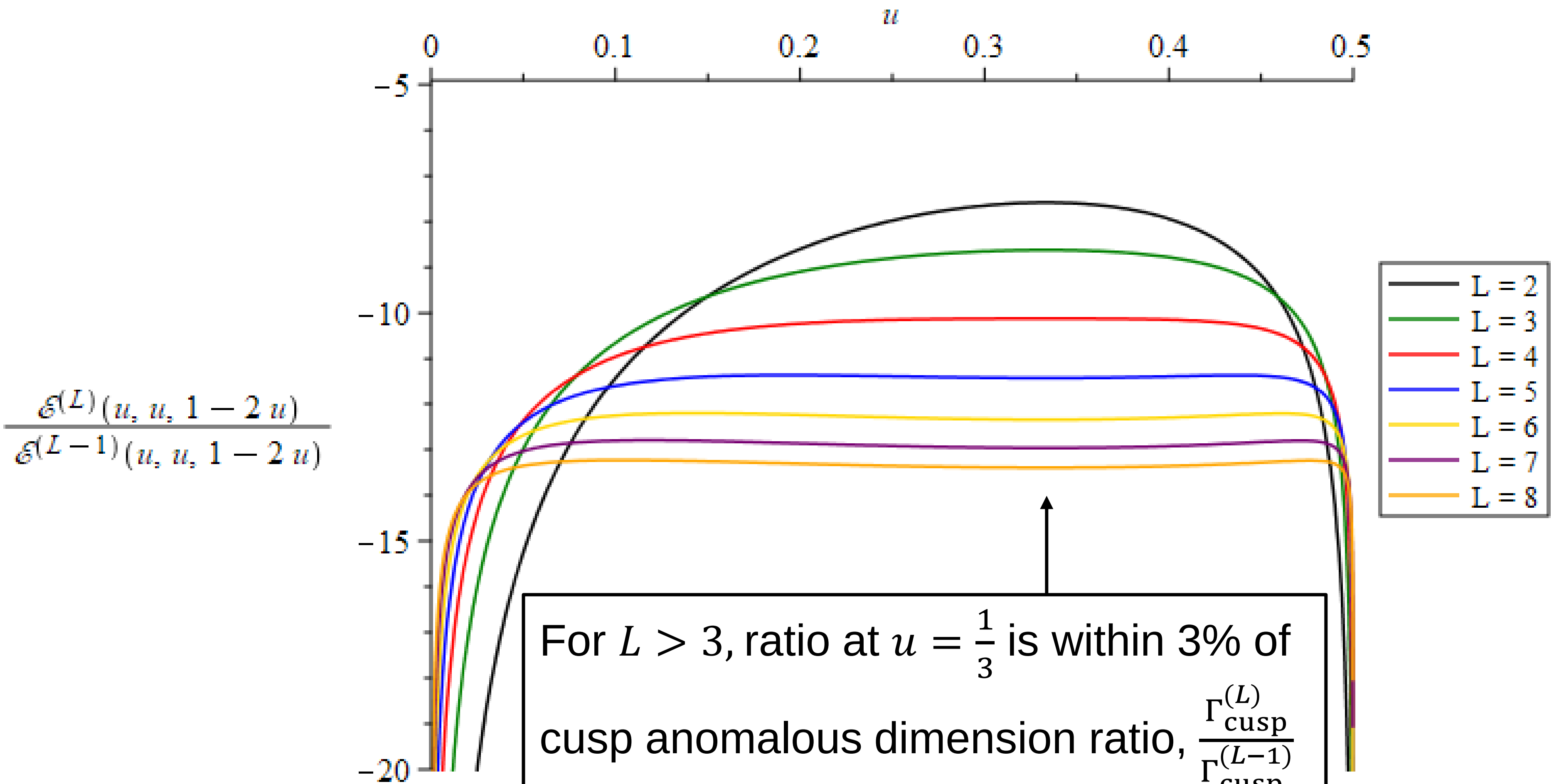


I = decay / Euclidean

IIa,b,c = scattering / spacelike operator

IIIa,b,c = scattering / timelike operator

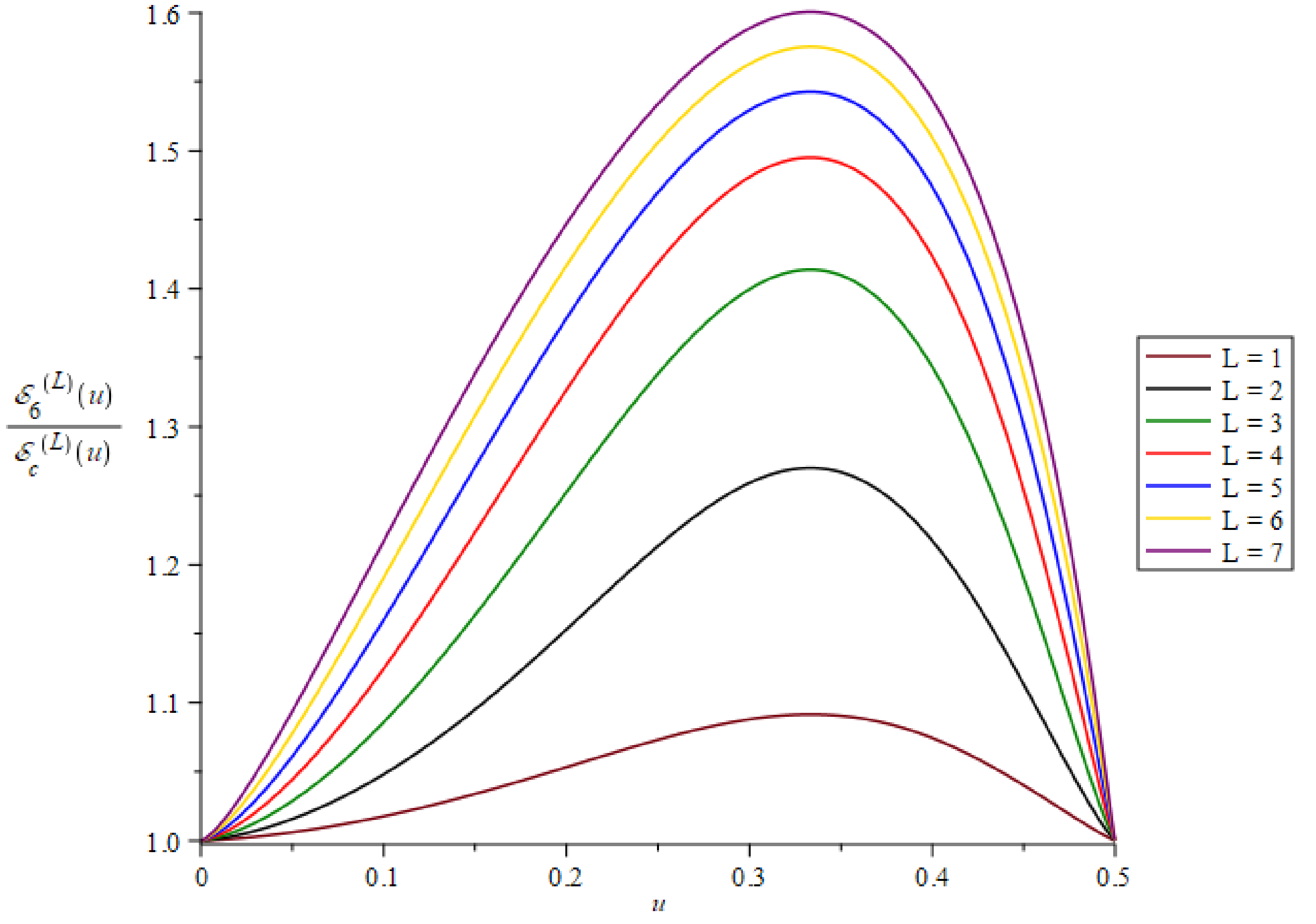
Euclidean Region



For $L > 3$, ratio at $u = \frac{1}{3}$ is within 3% of cusp anomalous dimension ratio, $\frac{\Gamma_{\rm cusp}^{(L)}}{\Gamma_{\rm cusp}^{(L-1)}}$

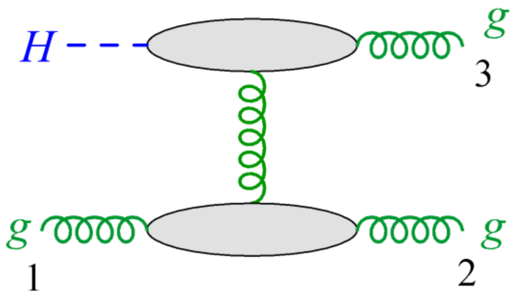
→ same finite radius of convergence?

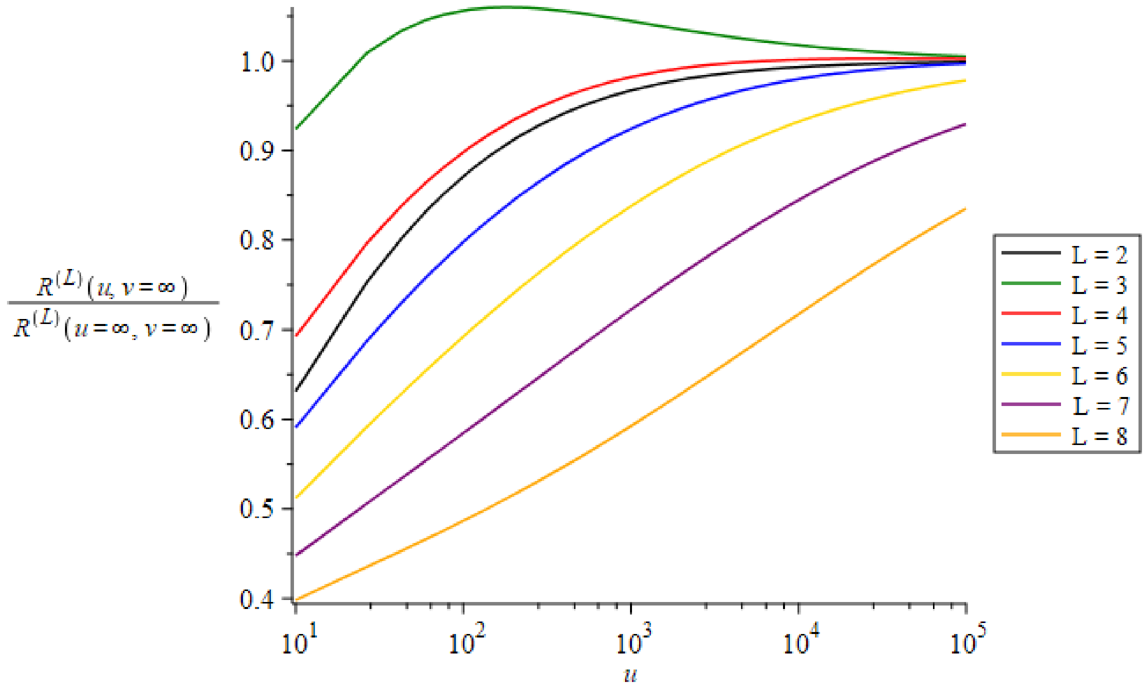
Numerical implications of antipodal duality?



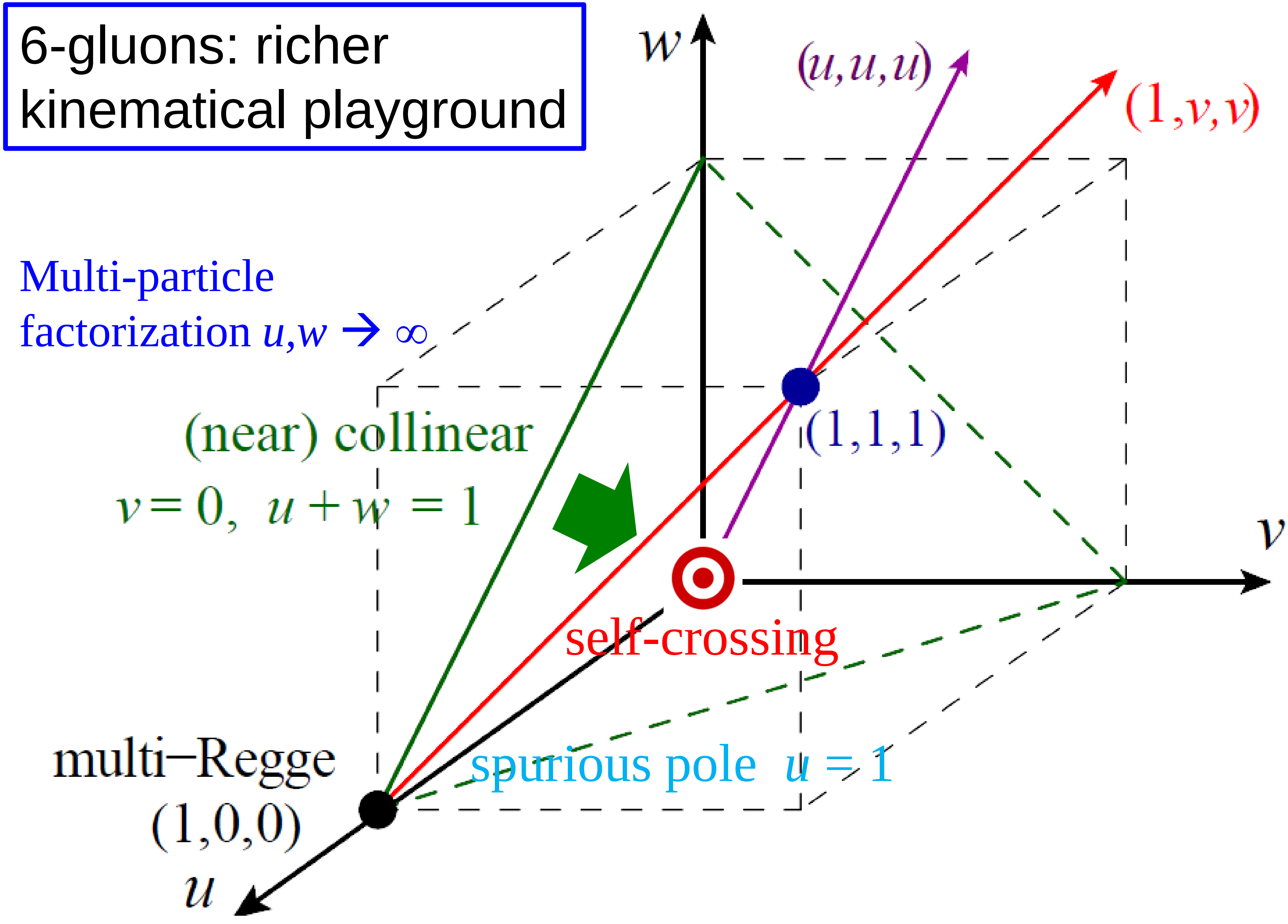
Real "impact factor" appears in space-like Regge limit, $v \to \infty$

Remainder function R is nontrivial function of $u = \frac{s_{12}}{m_H^2}$ as $s_{23} \to \infty$





6-gluons: richer kinematical playground



Number of (symbol-level) linearly independent $\{n, 1, \ldots, 1\}$ coproducts $(2L - n$ derivatives)

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$L = 1$	1	3	1														
$L = 2$	1	3	6	3	1												
$L = 3$	1	3	9	12	6	3	1										
$L = 4$	1	3	9	21	24	12	6	3	1								
$L = 5$	1	3	9	21	46	45	24	12	6	3	1						
$L = 6$	1	3	9	21	48	99	85	45	24	12	6	3	1				
$L = 7$	1	3	9	21	48	108	236	155	85	45	24	12	6	3	1		
$L = 8$	1	3	9	21	48	108	242	466	279	155	85	45	24	12	6	3	1

- Properly normalized L loop N=4 form factors $\mathcal{E}^{(L)}$ belong to a small space $\mathcal{C}$, dimension saturates on left
- $\mathcal{E}^{(L)}$ also obeys multiple-final-entry relations, saturation on right

$$
\begin{aligned}
\mathcal{E}^{(4)}(v\to\infty) = & -1920H_{0,0,0,0,0,0,0,1} - 384H_{0,0,0,0,0,1,0,1} - 192H_{0,0,0,0,0,1,1,1} - 384H_{0,0,0,0,1,0,0,1} \\
& - 96H_{0,0,0,0,1,0,1,1} - 96H_{0,0,0,0,1,1,0,1} - 384H_{0,0,0,1,0,0,0,1} - 96H_{0,0,0,1,0,0,1,1} \\
& - 144H_{0,0,0,1,0,1,0,1} - 48H_{0,0,0,1,0,1,1,1} - 64H_{0,0,0,1,1,0,0,1} - 16H_{0,0,0,1,1,0,1,1} \\
& - 16H_{0,0,0,1,1,1,0,1} - 64H_{0,0,0,1,1,1,1,1} - 384H_{0,0,1,0,0,0,0,1} - 96H_{0,0,1,0,0,0,1,1} \\
& - 144H_{0,0,1,0,0,1,0,1} - 48H_{0,0,1,0,0,1,1,1} - 128H_{0,0,1,0,1,0,0,1} - 32H_{0,0,1,0,1,0,1,1} \\
& - 32H_{0,0,1,0,1,1,0,1} - 32H_{0,0,1,0,1,1,1,1} - 48H_{0,0,1,1,0,0,0,1} - 16H_{0,0,1,1,0,1,0,1} \\
& - 16H_{0,0,1,1,0,0,1,1} - 32H_{0,0,1,1,0,1,1,1} - 16H_{0,0,1,1,1,0,0,1} - 32H_{0,0,1,1,1,0,1,1} \\
& - 32H_{0,0,1,1,1,1,0,1} - 384H_{0,1,0,0,0,0,0,1} - 96H_{0,1,0,0,0,0,1,1} - 144H_{0,1,0,0,0,1,0,1} \\
& - 48H_{0,1,0,0,0,1,1,1} - 128H_{0,1,0,0,1,0,0,1} - 32H_{0,1,0,0,1,0,1,1} - 32H_{0,1,0,0,1,1,0,1} \\
& - 32H_{0,1,0,0,1,1,1,1} - 128H_{0,1,0,1,0,0,0,1} - 32H_{0,1,0,1,0,0,1,1} - 40H_{0,1,0,1,0,1,0,1} \\
& - 40H_{0,1,0,1,0,1,1,1} - 24H_{0,1,0,1,1,0,0,1} - 32H_{0,1,0,1,1,0,1,1} - 32H_{0,1,0,1,1,1,0,1} \\
& - 40H_{0,1,0,1,1,1,1,1} - 32H_{0,1,1,0,0,0,0,1} - 16H_{0,1,1,0,0,0,1,1} - 16H_{0,1,1,0,0,1,0,1} \\
& - 32H_{0,1,1,0,0,1,1,1} - 8H_{0,1,1,0,1,0,0,1} - 32H_{0,1,1,0,1,0,1,1} - 32H_{0,1,1,0,1,1,0,1} \\
& - 40H_{0,1,1,0,1,1,1,1} - 24H_{0,1,1,1,0,0,0,1} - 32H_{0,1,1,1,0,0,1,1} - 40H_{0,1,1,1,0,1,0,1} \\
& - 40H_{0,1,1,1,0,1,1,1} - 40H_{0,1,1,1,1,0,0,1} - 48H_{0,1,1,1,1,0,1,1} - 48H_{0,1,1,1,1,1,0,1} \\
& - 240H_{0,1,1,1,1,1,1,1} \\
& + \zeta_2(96H_{0,0,0,0,1,1} + 16H_{0,0,0,1,0,1} + 112H_{0,0,0,1,1,1} + 16H_{0,0,1,0,0,1} + 80H_{0,0,1,0,1,1} \\
& \quad + 64H_{0,0,1,1,0,1} + 32H_{0,0,1,1,1,1} + 16H_{0,1,0,0,0,1} + 80H_{0,1,0,0,1,1} + 64H_{0,1,0,1,0,1} \\
& \quad + 80H_{0,1,0,1,1,1} + 64H_{0,1,1,0,0,1} + 80H_{0,1,1,0,1,1} + 80H_{0,1,1,1,0,1} + 432H_{0,1,1,1,1,1}) \\
& + \zeta_3(224H_{0,0,0,0,1} - 48H_{0,0,0,1,1} + 48H_{0,0,1,0,1} - 16H_{0,0,1,1,1} + 48H_{0,1,0,0,1} \\
& \quad - 8H_{0,1,0,1,1} + 16H_{0,1,1,0,1} + 8H_{0,1,1,1,1}) \\
& + \zeta_4(292H_{0,0,0,1} - 84H_{0,0,1,1} + 84H_{0,1,0,1} + 696H_{0,1,1,1}) \\
& + \zeta_5(264H_{0,0,1} - 72H_{0,1,1}) + 80\zeta_2\zeta_3H_{0,0,1} + \left(\frac{3782}{3}\zeta_6 - 12(\zeta_3)^2\right)H_{0,1} \\
& + \frac{49141}{36}\zeta_8 - 20\zeta_{5,3} - 352\zeta_3\zeta_5 + 8\zeta_2(\zeta_3)^2
\end{aligned}
$$

8 loop result has $\sim 2^{2\times 8-2} = 16{,}384$ terms

Values of HPLs {0,1} at $u = 1$

- Classical polylogs

evaluate to Riemann zeta values

$$\mathrm{Li}_n(u) = \int_0^u \frac{dt}{t}\mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{u^k}{k^n}$$

$$\mathrm{Li}_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta(n) \equiv \zeta_n$$

- HPL's evaluate to nested sums called multiple zeta values (MZVs):

$$\zeta_{n_1,n_2,\ldots,n_m} = \sum_{k_1>k_2>\cdots>k_m>0}^{\infty} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_m^{n_m}}$$

Weight $n = n_1 + n_1 + \ldots + n_m$

- MZV's obey many identities, e.g. stuffle

$$\zeta_{n_1}\zeta_{n_2} = \zeta_{n_1,n_2} + \zeta_{n_2,n_1} + \zeta_{n_1+n_2}$$

- All reducible to Riemann zeta values until weight 8.

  Irreducible MZVs: $\zeta_{5,3},\ \zeta_{7,3},\ \zeta_{5,3,3},\ \zeta_{9,3},\ \zeta_{6,4,1,1},\ \ldots$

Symbol is too verbose → Nested representation better

L	number of terms
1	6
2	12
3	636
4	11,208
5	263,880
6	4,916,466
7	92,954,568
8	1,671,656,292

- Define every function by its $\{n-1{,}1\}$ coproducts, i.e. its first derivatives.
- Also need to specify constants of integration at one point, e.g. $(u,v,w) = (1{,}0{,}0)$

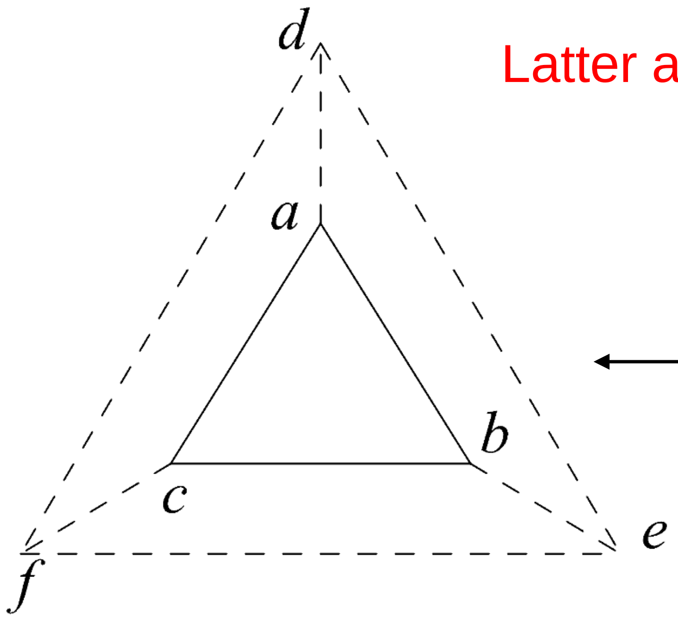
Many empirical adjacency constraints

$$F^{d,e} = F^{e,d} = F^{e,f} = F^{f,e} = F^{f,d} = F^{d,f} = 0$$

Hold for 2 loop QCD amplitudes too, planar and nonplanar!

LD, Mcleod, Wilhelm, 2012.12286

$$F^{a,d} = F^{d,a} = F^{b,e} = F^{e,b} = F^{c,f} = F^{f,c} = 0$$



Latter are NEW: Hold for planar N=4 SYM to 8 loops!

LD, Gürdoğan, Mcleod, Wilhelm, to appear

Mnemonic for dihedral symmetry;
6 dashed lines indicate 12 forbidden pairs.

Empirical multi-final entry relations

1. $\mathcal{E}^{a} = 0$ (plus dihedral images)

2. $\mathcal{E}^{a,e} = \mathcal{E}^{a,f}$ (plus …)

3. $\mathcal{E}^{a,b,d} = 0, \qquad \mathcal{E}^{a,e,e} = -\,\mathcal{E}^{a,f,f},$
$$\mathcal{E}^{e,a,f} = \mathcal{E}^{f,a,f} - \mathcal{E}^{a,f,f}$$

4. …

Symbol alphabets for n-gluon amplitudes

$n = 6$ has 9 letters: $\mathcal{S} = \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

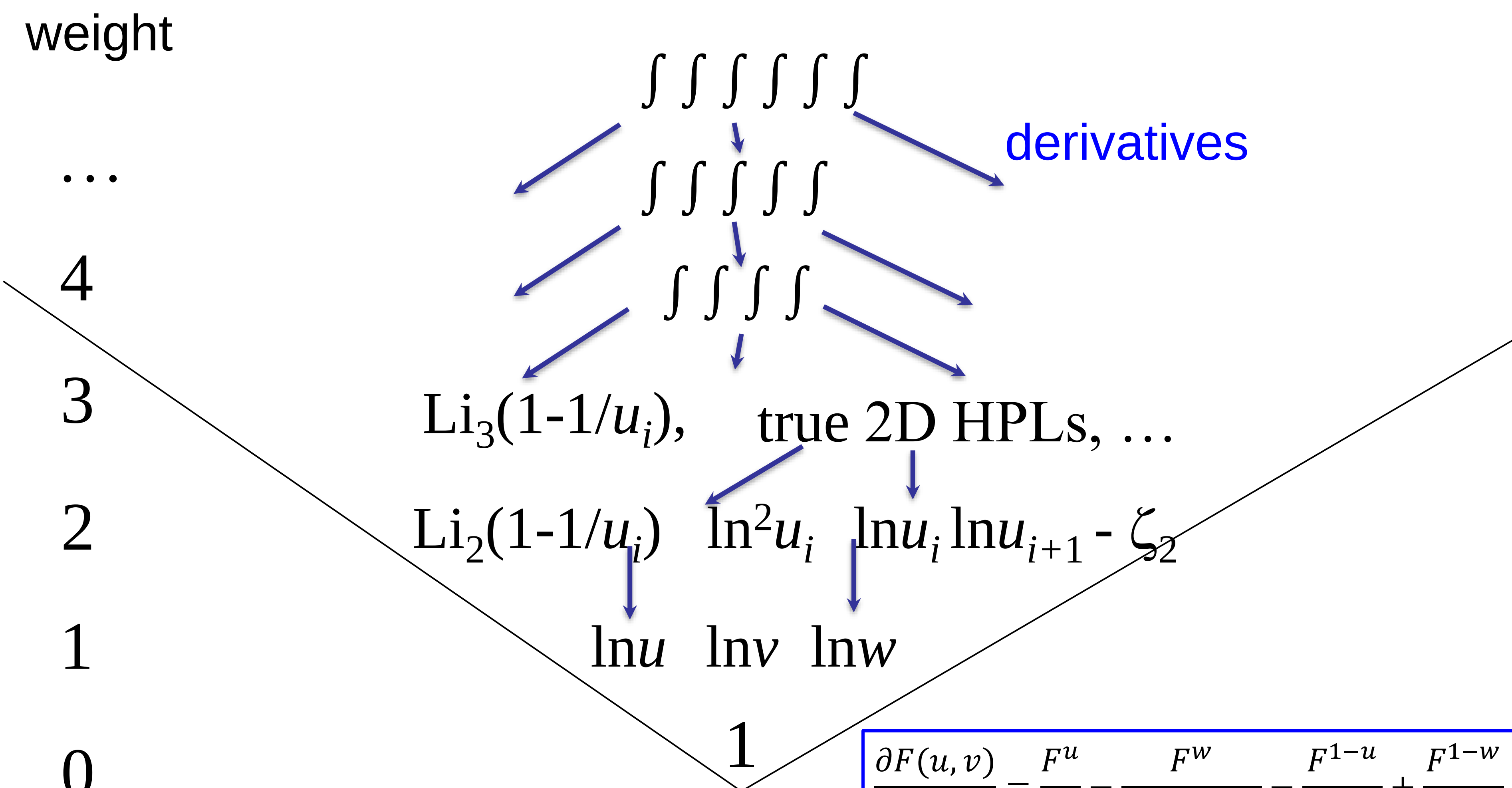
$n = 7$ has 42 letters

Golden, Goncharov, Paulos, Spradlin, Volovich, Vergu, 1305.1617, 1401.6446, 1411.3289; Drummond, Papathanasiou, Spradlin 1412.3763

$n = 8$ has at least 222 letters, could even be infinite as $L \to \infty$

Arkani-Hamed, Lam, Spradlin, 1912.08222;
Drummond, Foster, Kalousios, 1912.08217, 2002.04624;
Henke, Papathanasiou 1912.08254, 2106.01392;
Z. Li, C. Zhang, 2110.00350

Heuristic view of space



$$\frac{\partial F(u,v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{1-u-v} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{u+v}$$

Number of remaining parameters in form-factor ansatz after imposing constraints

L	2	3	4	5	6	7	8
symbols in $\mathcal{C}$	48	249	1290	6654	34219	????	????
dihedral symmetry	11	51	247	1219	????	????	????
$(L-1)$ final entries	5	9	20	44	86	???	???
L^{th} discontinuity	2	5	17	38	75	???	??
collinear limit	0	1	2	8	19	70	6
OPE $T^2 \ln^{L-1} T$	0	0	0	4	12	56	0
OPE $T^2 \ln^{L-2} T$	0	0	0	0	0	36	0
OPE $T^2 \ln^{L-3} T$	0	0	0	0	0	0	0
OPE $T^2 \ln^{L-4} T$	0	0	0	0	0	0	0
OPE $T^2 \ln^{L-5} T$	0	0	0	0	0	0	0

Table 4: Number of parameters left when bootstrapping the form factor $\mathcal{E}^{(L)}$ at L-loop order in the function space $\mathcal{C}$ at symbol level, using all the conditions on the final $(L-1)$ entries, which can be deduced at $(L-1)$ loops.

The [Dual] Conformal Group

SO(4,2) $\supset$ SO(3,1) [rotations+boosts] + translations+dilatations + special-conformal

15 = 3 + 3 + 4 + 1 + 4

- The nontrivial generators are special conformal K^μ
- Correspond to inversion · translation · inversion
- To obtain a [dual] conformally invariant function $f(x_{ij}^2)$ just have to check invariance under inversion,

$$x_i^\mu \to x_i^\mu / x_i^2$$