A New Duality in Planar N=4 SYM and Possible Flux Tube Implications

LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, 2112.06243 + Y.-T. Liu, in progress

KITP Program on
Flux Tubes and Confinement
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Total cross section for producing Higgs boson at LHC via gluon fusion

Leading Order (LO)

- Higgs production at LHC is dominantly via gluon fusion, mediated by a top quark loop.
- Since \( 2m_{\text{top}} = 350 \text{ GeV} \)
  \( \gg m_{\text{Higgs}} = 125 \text{ GeV} \),
we can integrate out the top quark to get a leading operator \( HG_{\mu\nu}^{a} G_{\mu\nu}^{\alpha} \)
Leading-order (LO) predictions qualitative: poor convergence of expansion in $\alpha_s(\mu)$

Uncertainty bands from varying $\mu_R = \mu_F = \mu$

Example: Higgs gluon fusion cross section at LHC vs. CM energy $\sqrt{s}$

LO $\rightarrow$ NNNLO $\rightarrow$ factor of 2.7 increase!

Anastasiou, Duhr, Dulat, Herzog, Mistlberger, 1503.06056
NLO QCD topologies

virtual $gg \rightarrow H$

real, $gg \rightarrow Hg$
N3LO QCD topologies

$gg \to Hg$
@ 2 loops, state of art in QCD

+ $g\to Hg$
+ quarks
+ operator renormalization
+ $1/m_t^2$ corrections
+ parton distributions
Multi-loop complexities

- Multi-loop multiscale integrals typically very difficult
- All 1 loop integrals with external legs in D=4 are reducible to scalar box integrals + simpler

\[ \text{Li}_2(x) = - \int_0^x \frac{dt}{t} \ln(1 - t) \]


- At \( L \) loops, get special functions with up to \( 2L \) integrations
  Hannesdottier, McLeod, Schwartz, Vergu, 2109.09744
- Weight \( 2L \) iterated integrals, generalized polylogarithms, or worse
Planar N=4 SYM, toy model for QCD amplitudes

- QCD’s maximally supersymmetric cousin, N=4 super-Yang-Mills theory (SYM), gauge group SU($N_c$), in the large $N_c$ (planar) limit

- Structure very rigid:
  \[ \text{Amplitudes} = \sum_i \text{rational}_i \times \text{transcendental}_i \]

- For planar N=4 SYM, we understand rational structure quite well, focus on the transcendental functions.

- Furthermore, at least three dualities hold:
  1. AdS/CFT
  2. Amplitudes dual to Wilson loops
  3. New “antipodal” duality between amplitudes and form factors
N=4 SYM very special

- At one loop, cancellation of loop momenta in numerator → only scalar box integrals
  Bern, LD, Dunbar, Kosower, hep-ph/9403226

- Weight 2 functions – dilogs. E.g., \( gg \rightarrow Hg \) @ 1 loop ⊃
  \[
  = \text{Li}_2 \left( 1 - \frac{s_{123}}{s_{12}} \right) + \text{Li}_2 \left( 1 - \frac{s_{123}}{s_{23}} \right) + \frac{1}{2} \ln^2 \left( \frac{s_{12}}{s_{23}} \right) + \ldots
  \]

- QCD results also contain single log’s and rational parts from (tensor) triangle + bubble integrals
  \[
  = \frac{1}{\epsilon} - \ln(s_{123})
  \]
Higher loops

• Much evidence that N=4 SYM amplitudes have “uniform weight (transcendentality)” $2L$ at loop order $L$

• Weight $\sim$ number of integrations, e.g.

\[
\ln(s) = \int_1^s \frac{dt}{t} = \int_1^s d\ln t \tag{1}
\]

\[
\text{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1 - t) = \int_0^x d\ln t \cdot [-\ln(1 - t)] \tag{2}
\]

\[
\text{Li}_n(x) = \int_0^x \frac{dt}{t} \text{Li}_{n-1}(t) \tag{n}
\]
AdS/CFT

Conformal field theory (like N=4 SYM) is dual to a theory of gravity in anti-de Sitter space (like strings in $AdS_5 \times S^5$)

SO(4,2) isometry of 5 dimensional space-time

↔ 4d conformal symmetry

A weak-strong duality

Maldacena (1997)
Gubser, Klebanov, Polyakov; Witten (1998)
T-duality symmetry of string theory

- Exchanges string world-sheet variables $\sigma, \tau$

$$X^\mu(\tau, \sigma) = x^\mu + k^\mu \tau + \text{oscillators}$$

$\Rightarrow$ $$X^\mu(\tau, \sigma) = x^\mu + k^\mu \sigma + \text{oscillators}$$

- Strong coupling limit of planar N=4 SYM is semi-classical limit of string theory:
  world-sheet stretches tight around minimal area surface in AdS.

- Boundary determined by momenta of external states: light-like polygon with null edges = momenta $k^\mu$
Amplitudes = Wilson loops

- Polygon vertices $x_i$ are not positions but dual momenta, $x_i - x_{i+1} = k_i$
- Transform like positions under dual conformal symmetry

Duality verified to hold at weak coupling too

weak-weak duality, holds order-by-order
Dual conformal invariance

- Wilson $n$-gon invariant under inversion:
  \[ x_{ij}^2 = (k_i + k_{i+1} + \cdots + k_{j-1})^2 \equiv s_{i,i+1,\ldots,j-1} \]

- Fixed, up to functions of invariant cross ratios:
  \[ u_{ijkl} = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2} \]

- $x_{i,i+1}^2 = k_i^2 = 0 \rightarrow$ no such variables for $n = 4, 5$

\[ n = 6 \rightarrow \text{precisely 3 ratios:} \]
\[ n = 7 \rightarrow 6 \text{ ratios.} \]
In general, $3n-15$ ratios.

\[ \begin{align*}
  u &= \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}} \\
  v &= \frac{s_{23}s_{56}}{s_{234}s_{123}} \\
  w &= \frac{s_{34}s_{61}}{s_{345}s_{234}}
\end{align*} \]
Solving for Planar N=4 SYM Amplitudes

't Hooft coupling \( \lambda \)

collinear limit

minimal surface

perturbative gluons

Kinematical variables

Images: A. Sever, N. Arkani-Hamed
Flux tubes at finite coupling

Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788; Basso, Sever, Vieira, 1303.1396, 1306.2058, 1402.3307, 1407.1736, 1508.03045 BSV+Caetano+Cordova, 1412.1132, 1508.02987

- Tile $n$-gon with pentagon transitions.
- Quantum integrability $\Rightarrow$ compute pentagons exactly in 't Hooft coupling
- 4d S-matrix as expansion (OPE) in number of flux-tube excitations = expansion around near collinear limit
The new FFOPE

- Form factors are Wilson loops in a periodic space, due to injection of operator momentum
  Alday, Maldacena, 0710.1060; Maldacena, Zhiboedov, 1009.1139; Brandhuber, Spence, Travaglini, Yang, 1011.1899

- Besides pentagon transitions $\mathcal{P}$, this program needs an additional ingredient, the form factor transition $\mathcal{F}$
  Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367, 2112.10569
OPE representation

• 6-gluon amplitude:

\[ \mathcal{W}_{\text{hex}} = \sum_a \int \, du \, P_a(0|u)P_a(\bar{u}|0) \, e^{-E(u)\tau + ip(u)\sigma + im\phi} \]

\[ T = e^{-\tau}, \quad S = e^{-\sigma}, \quad F = e^{i\phi}. \quad \nu = \frac{T^2}{1+T^2} \to 0, \]

weak-coupling, \( E = k + \mathcal{O}(g^2) \to \) expansion in \( T^k \)

• 3-gluon form factor: \( \psi = \text{helicity 0 pairs of states} \)

\[ \mathcal{W}_3 = \sum \int \, e^{-E_{\psi}\tau + ip_{\psi}\sigma} \, \mathcal{P}(0|\psi) \, \mathcal{F}(\psi) \]

weak-coupling \to \) expansion in \( T^{2k} \) (no azimuthal angle \( \phi \))
“Higgs” amplitudes and N=4 SYM form factors

LD, A. McLeod, M. Wilhelm, 2012.12286
+ Ö. Gürdoğan, to appear

At leading order in $1/m_{top}$, Higgs boson couples to gluons via the operator $HG_{\mu\nu} G^{\mu\nu} a$.

3,4,5 loops
6,7,8 loops
Form factors (cont.)

- Higgs is a scalar, color singlet. In QCD its amplitudes with gluons are matrix elements of $G^a_{\mu\nu} G^{\mu\nu}_a$ with on-shell gluons: “form factors”
- In N=4, this operator is part of the (BPS-protected) stress tensor supermultiplet, which also includes for example $\phi_1^\dagger \phi_1 - \phi_2^\dagger \phi_2 (\in 20$ of $SU(4)_R$)
- $H_{gg}$ “Sudakov” form factor is “too simple”; it has no kinematic dependence beyond overall $(-s_{12})^{-L_\epsilon}$
- $H_{ggg}$ is “just right”, depends on 2 dimensionless ratios
**Hggg** kinematics is two-dimensional

\[ k_1 + k_2 + k_3 = -k_H \]

\[ s_{123} = s_{12} + s_{23} + s_{31} = m_H^2 \]

\[ s_{ij} = (k_i + k_j)^2 \quad \text{and} \quad k_i^2 = 0 \]

\[ u = \frac{s_{12}}{s_{123}} \quad v = \frac{s_{23}}{s_{123}} \quad w = \frac{s_{31}}{s_{123}} \]

\[ u + v + w = 1 \]

I = decay / Euclidean

IIa,b,c = scattering / spacelike operator

IIIa,b,c = scattering / timelike operator

**D\(_3\) \equiv S\(_3\)** dihedral symmetry generated by:

a. cycle: \( i \rightarrow i + 1 \) (mod 3), or 
\[ u \rightarrow v \rightarrow w \rightarrow u \]

b. flip: \( u \leftrightarrow v \)

N=4 amplitude is \( S_3 \) invariant

L. Dixon  Antipodal Duality
One loop integrals/amplitudes

\[ H \]

\[ g_1 \]

\[ g_2 \]

\[ g_3 \]

\[ \text{Li}_2 \left( 1 - \frac{s_{123}}{s_{12}} \right) + \text{Li}_2 \left( 1 - \frac{s_{123}}{s_{23}} \right) + \frac{1}{2} \ln^2 \left( \frac{s_{12}}{s_{23}} \right) + \cdots \]

\[ = \text{Li}_2 \left( 1 - \frac{1}{u} \right) + \text{Li}_2 \left( 1 - \frac{1}{v} \right) + \frac{1}{2} \ln^2 \left( \frac{u}{v} \right) + \cdots \]
A two-loop story

- Gehrmann et al. computed $H_{ggg}$ in QCD at 2 loops
  Gehrmann, Jaquier, Glover, Koukoutsakis, 1112.3554
- Soon after, Brandhuber et al. computed stress tensor 3-point form factor $F_3$ in N=4 SYM,
  Brandhuber, Travaglini, Yang, 1201.4170
  saw that “maximally transcendental part” of QCD result (both (+++) and (-++)) was same as N=4 result!!
- This “principle of maximal transcendentality”
  was known to work for DGLAP and BFKL anomalous dimensions, but not for generic scattering amplitudes, so this one is very special
2d HPLs

Gehrmann, Remiddi, hep-ph/0008287

Space graded by weight $n$. Every function $F$ obeys:

$$\frac{\partial F(u, v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{1 - u - v} - \frac{F^{1-u}}{1 - u} + \frac{F^{1-v}}{u + v}$$

$$\frac{\partial F(u, v)}{\partial v} = \frac{F^v}{v} - \frac{F^w}{1 - u - v} - \frac{F^{1-v}}{1 - v} + \frac{F^{1-w}}{u + v}$$

where $F^u, F^v, F^w, F^{1-u}, F^{1-v}, F^{1-w}$ are weight $n-1$ 2d HPLs.

To bootstrap $Hggg$ amplitude beyond 2 loops, find as small a subspace of 2d HPLs as possible, construct it to high weight.
Generalized polylogarithms

Chen, Goncharov, Brown,…

- Can be defined as iterated integrals, e.g.
  \[ G(a_1, a_2, \ldots, a_n, x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n, t) \]

- Or define differentially:
  \[ dF = \sum_{s_k \in S} F^{s_k} \, d \ln s_k \]

- There is a Hopf algebra that “co-acts” on the space of polylogarithms, \( \Delta: F \to F \otimes F \)
- The derivative \( dF \) is one piece of \( \Delta: \Delta_{n-1,1} F = \sum_{s_k \in S} F^{s_k} \otimes \ln s_k \)
- so we refer to \( F^{s_k} \) as a \( \{n-1,1\} \) coproduct of \( F \)
- \( s_k \) are letters in the symbol alphabet \( S \)
Generalized polylogarithms (cont.)

- The \( \{n-1,1\} \) coaction can be applied iteratively.
- Define the \( \{n-2,1,1\} \) **double** coproducts, \( F^{s_k,s_j} \), via the derivatives of the \( \{n-1,1\} \) **single** coproducts \( F^{s_j} \):

\[
dF^{s_j} \equiv \sum_{s_k \in S} F^{s_k,s_j} \, d \ln s_k
\]

- And so on for the \( \{n-m,1,\ldots,1\} \) \( m \)th coproducts of \( F \).
- The maximal iteration, \( n \) times for a weight \( n \) function, is the symbol,

\[
S[F] = \sum_{s_{i_1},\ldots,s_{i_n} \in S} F^{s_{i_1},\ldots,s_{i_n}} \, d \ln s_{i_1} \ldots d \ln s_{i_n} \equiv \sum_{s_{i_1},\ldots,s_{i_n} \in S} F^{s_{i_1},\ldots,s_{i_n}} \, s_{i_1} \otimes \ldots \otimes s_{i_n}
\]

where now \( F^{s_{i_1},\ldots,s_{i_n}} \) are just rational numbers

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

L. Dixon  Antipodal Duality  KITP - 2022/01/26
Example: The classical polylogarithms

\[
\text{Li}_1(x) = -\ln(1 - x) = \sum_{k=1}^{\infty} \frac{x^k}{k}
\]

\[
\text{Li}_n(x) = \int_0^x \frac{dt}{t} \text{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{x^k}{kn}
\]

- Regular at \( x = 0 \), branch cut starts at \( x = 1 \).
- Iterated differentiation gives the symbol:
  \[
  S[\text{Li}_n(x)] = S[\text{Li}_{n-1}(x)] \otimes x = \cdots = -(1 - x) \otimes x \otimes \cdots \otimes x
  \]
- Branch cut discontinuities displayed in first entry of symbol, e.g. clip off leading \((1 - x)\) to compute discontinuity at \( x = 1 \).
- Derivatives computed from symbol by clipping last entry, multiplying by that \( d \ln(\ldots) \).
**Example:** Harmonic Polylogarithms in one variable (HPLs \{0,1\})

Remiddi, Vermaseren, hep-ph/9905237

- Generalize the classical polylogs:
  \[
  \text{Li}_n(u) = \int_0^u \frac{dt}{t} \text{Li}_{n-1}(t), \quad \text{Li}_1(t) = -\ln(1 - t)
  \]

- Define HPLs by iterated integration:
  \[
  H_{0,\vec{w}}(u) = \int_0^u \frac{dt}{t} H_{\vec{w}}(t), \quad H_{1,\vec{w}}(u) = \int_0^u \frac{dt}{1 - t} H_{\vec{w}}(t)
  \]

- Or by derivatives:
  \[
  dH_{0,\vec{w}}(u) = H_{\vec{w}}(u) \ d\ln u \quad dH_{1,\vec{w}}(u) = -H_{\vec{w}}(u) d\ln(1 - u)
  \]

- Symbol letters: \( S = \{u, 1 - u\} \)

- Weight \( n = \) length of binary string \( \vec{w} \)

- Number of functions at weight \( n = 2L : 2^{2L} \)

- Branch cuts dictated by **first** integration/entry in symbol

- **Derivatives** dictated by **last** integration/entry in symbol
Symbol alphabet for $H_{ggg}$

Gehrmann, Remiddi, hep-ph/0008287

- Comparing
  \[
  \frac{\partial F(u, v)}{\partial u} = \frac{F_u}{u} - \frac{F_w}{1 - u - v} - \frac{F^{1-u}}{1 - u} + \frac{F^{1-w}}{u + v}
  \]
  \[
  \frac{\partial F(u, v)}{\partial v} = \frac{F_v}{v} - \frac{F_w}{1 - u - v} - \frac{F^{1-v}}{1 - v} + \frac{F^{1-w}}{u + v}
  \]

with
\[
dF = \sum_{S_k \in S} F^{S_k} \, d \ln s_k
\]

we see that
\[
S = \{ u, v, w, 1 - u, 1 - v, 1 - w \} \quad w = 1 - u - v
\]

$\exists$ dihedral symmetry $D_3 \equiv S_3$, permutations of $\{ u, v, w \}$

For example, all permutations of (finite part of) box integral are in this space.

For example, all permutations of (finite part of) box integral are in this space.

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
= \text{Li}_2 \left( 1 - \frac{1}{u} \right) + \text{Li}_2 \left( 1 - \frac{1}{v} \right) + \frac{1}{2} \ln^2 \left( \frac{u}{v} \right) + \ldots
\]
A better alphabet

• Motivated by a similar change of variables in the 6 gluon case Caron-Huot, LD, von Hippel, McLeod, 1609.00669 (which exposes the Steinmann relations there), we also switch to the alphabet

\[ S' = \{ a = \frac{u}{vw}, b = \frac{v}{wu}, c = \frac{w}{uv}, d = \frac{1-u}{u}, e = \frac{1-v}{v}, f = \frac{1-w}{w} \} \]

• We find that the symbols of the (suitably normalized) form factor \( F_3^{(L)} \) at one and two loops simplify remarkably, down to just 1 and 2 terms, plus dihedral images(!!!):

\[
S \left[ F_3^{(1)} \right] = (-1) \ b \otimes d + \text{dihedral}
\]

\[
S \left[ F_3^{(2)} \right] = 4 \ b \otimes d \otimes d \otimes d + 2 \ b \otimes b \otimes b \otimes d + \text{dihedral}
\]
Simplest analytic form is for $v \to \infty$

$\to$ Harmonic polylogarithms $H_{\overline{w}} \equiv H_{\overline{w}}(1 - \frac{1}{u})$

\[
F_3^{(1)}(v \to \infty) = 2H_{0,1} + 6\zeta_2
\]
\[
F_3^{(2)}(v \to \infty) = -8H_{0,0,0,1} - 4H_{0,1,1,1} + 12\zeta_2 H_{0,1} + 13\zeta_4
\]
\[
F_3^{(3)}(v \to \infty) = 96H_{0,0,0,0,0,1} + 16H_{0,0,0,1,0,1} + 16H_{0,0,0,1,1,1} + 16H_{0,0,1,0,0,1} + 8H_{0,0,1,0,1,1,1}
\]
\[
+ 8H_{0,0,1,1,0,1} + 16H_{0,1,0,0,0,1} + 8H_{0,1,0,0,1,1} + 12H_{0,1,0,1,0,1} + 4H_{0,1,0,1,1,1}
\]
\[
+ 8H_{0,1,1,0,0,1} + 4H_{0,1,1,0,1,1} + 4H_{0,1,1,1,0,1} + 24H_{0,1,1,1,1,1}
\]
\[
- \zeta_2(32H_{0,0,0,1,1} + 8H_{0,0,1,1} + 4H_{0,1,0,1} + 52H_{0,1,1,1})
\]
\[
- \zeta_3(8H_{0,0,1} - 4H_{0,1,1}) - 53\zeta_4 H_{0,1} - \frac{167}{4}\zeta_6 + 2(\zeta_3)^2
\]

8 loop result has $\sim 2^{2\times8-2} = 16,384$ terms
6-gluon amplitude is simplest for $(\hat{u}, \hat{v}, \hat{w}) = (1, \hat{v}, \hat{v})$

- Let $H_{\overrightarrow{w}} \equiv H_{\overrightarrow{w}}(1 - \frac{1}{\hat{v}})$

\[
A_6^{(1)}(1, \hat{v}, \hat{v}) = 2H_{0,1} \\
A_6^{(2)}(1, \hat{v}, \hat{v}) = -8H_{0,1,1,1,1} - 4H_{0,0,0,1} - 4\zeta_2 H_{0,1} - 9\zeta_4 \\
A_6^{(3)}(1, \hat{v}, \hat{v}) = 96H_{0,1,1,1,1,1} + 16H_{0,1,0,1,1,1} + 16H_{0,0,0,1,1,1} + 16H_{0,1,1,0,1,1} + 8H_{0,0,1,0,1,1} + 8H_{0,1,0,0,1,1} + 16H_{0,1,1,1,0,1} + 8H_{0,0,1,1,0,1} + 12H_{0,1,0,1,0,1} + 4H_{0,0,0,1,0,1} + 8H_{0,1,1,0,0,1} + 4H_{0,0,1,0,0,1} + 4H_{0,1,0,0,0,1} + 24H_{0,0,0,0,0,1} + \zeta_2 (8H_{0,0,0,1} + 8H_{0,1,0,1} + 48H_{0,1,1,1}) + 42\zeta_4 H_{0,1} + 121\zeta_6
\]

There is an exact map at symbol level, with $\frac{1}{\hat{v}} = 1 - \frac{1}{u}$, $0 \leftrightarrow 1$, if you also reverse the order of the symbol entries!!! It works to 7 loops, where $\sim 2^{2 \times 7 - 2} = 4,096$ terms agree.
Antipodal duality in 2d

weak-weak duality

\[ F_3^{(L)} (u, v, w) = S \left( A_6^{(L)} (\hat{u}, \hat{v}, \hat{w}) \right) \]

where the antipode \( S \), at symbol level, reverses the order of all letters:

\[ S(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = (-1)^m x_m \otimes \cdots \otimes x_2 \otimes x_1 \]

and the kinematic map is

\[ \hat{u} = \frac{vw}{(1 - v)(1 - w)}, \quad \hat{v} = \frac{wu}{(1 - w)(1 - u)}, \quad \hat{w} = \frac{uv}{(1 - u)(1 - v)} \]

which maps \( u + v + w = 1 \) to the parity-preserving surface

\[ \Delta \equiv (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0 \]

corresponding to \( \hat{k}_{i+n}^\mu = -\hat{k}_i^\mu, i = 1,2,\ldots,n \quad (n = 3 \text{ here}) \)
6-gluon alphabet and symbol map

Goncharov, Spradlin, Vergu, Volovich, 1006.5703; LD, Drummond, Henn, 1108.4461; Caron-Huot, LD, von Hippel, McLeod, 1609.00669

- $S_6 = \{ \hat{u}, \hat{v}, \hat{w}, 1 - \hat{u}, 1 - \hat{v}, 1 - \hat{w}, \hat{u}, \hat{v}, \hat{w} \}$

$\rightarrow S'_6 = \{ \hat{a} = \frac{\hat{u}}{\hat{v}\hat{w}}, \hat{b} = \frac{\hat{v}}{\hat{w}u}, \hat{c} = \frac{\hat{w}}{\hat{u}\hat{v}}, \hat{d} = \frac{1-\hat{u}}{\hat{u}}, \hat{e} = \frac{1-\hat{v}}{\hat{v}}, \hat{f} = \frac{1-\hat{w}}{\hat{w}} \}$

- Kinematic map on letters:
  \[ \sqrt{\hat{a}} = d, \quad \hat{d} = a, \quad \text{plus cyclic relations} \]

\[
S \left[ A_6^{(1)} \right] = (-\frac{1}{2})\hat{b} \otimes \hat{d} + \text{dihedral}
\]

\[
S \left[ A_6^{(2)} \right] = b \otimes d \otimes d \otimes d + \frac{1}{2} b \otimes b \otimes b \otimes d + \text{dihedral}
\]

- Works through 7 loops!

## L loop symbol

<table>
<thead>
<tr>
<th>$L$</th>
<th>number of terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
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<tr>
<td>2</td>
<td>12</td>
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<tr>
<td>3</td>
<td>636</td>
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<td>7</td>
<td>92,954,568</td>
</tr>
<tr>
<td>8</td>
<td>1,671,656,292</td>
</tr>
</tbody>
</table>
Map covers entire phase space for 3-gluon form factor

- Soft is dual to collinear; collinear is dual to soft
- White regions in \((u, v)\) map to some of \(\hat{u}, \hat{v}, \hat{w} > 1\)
Many special dual points

There is an “$f$” alphabet at all of these points, which is a way of writing multiple zeta values (MZV’s) so that the coaction is manifest.

F. Brown, 1102.1310; O. Schnetz, HyperlogProcedures

<table>
<thead>
<tr>
<th></th>
<th>$(\hat{u}, \hat{v}, \hat{w})$</th>
<th>$(u, v, w)$</th>
<th>functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>▽</td>
<td>$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$</td>
<td>$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$</td>
<td>$\sqrt[6]{1}$</td>
</tr>
<tr>
<td>□</td>
<td>$(\frac{1}{2}, \frac{1}{2}, 0)$</td>
<td>$(0, 0, 1)$</td>
<td>$\text{Li}_2(\frac{1}{2}) + \text{logs}$</td>
</tr>
<tr>
<td>•</td>
<td>$(1, 1, 1)$</td>
<td>$\lim_{u \to \infty} (u, u, 1-2u)$</td>
<td>MZVs</td>
</tr>
<tr>
<td>○</td>
<td>$(0, 0, 1)$</td>
<td>$\lim_{u \to \infty} (\frac{1}{2}, \frac{1}{2}, 0)$</td>
<td>MZVs + logs</td>
</tr>
<tr>
<td>△</td>
<td>$(\frac{3}{4}, \frac{3}{4}, \frac{1}{4})$</td>
<td>$(-1, -1, 3)$</td>
<td>$\sqrt[6]{1}$</td>
</tr>
<tr>
<td>□□</td>
<td>$(\infty, \infty, \infty)$</td>
<td>$(1, 1, -1)$</td>
<td>alternating sums</td>
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<tr>
<td>⊗</td>
<td>$\lim_{\hat{v} \to \infty} (1, \hat{v}, \hat{v})$</td>
<td>$\lim_{v \to \infty} (1, v, -v)$</td>
<td>MZVs</td>
</tr>
<tr>
<td></td>
<td>$(1, \hat{v}, \hat{v})$</td>
<td>$\lim_{v \to \infty} (u, v, 1-u-v)$</td>
<td>HPL${0, 1}$</td>
</tr>
<tr>
<td></td>
<td>$(\hat{u}, \hat{u}, (1-2\hat{u})^2)$</td>
<td>$(u, u, 1-2u)$</td>
<td>HPL${-1, 0, 1}$</td>
</tr>
</tbody>
</table>
The simplest point

• \((\hat{u}, \hat{v}, \hat{w}) = (1,1,1) \iff u, v \to \infty\)

• At this point,

\[
A_6^{(1)}(\cdot) = 0 \quad F_3^{(1)}(\cdot) = 8\zeta_2
\]
\[
A_6^{(2)}(\cdot) = -9\zeta_4 \quad F_3^{(2)}(\cdot) = 31\zeta_4
\]
\[
A_6^{(3)}(\cdot) = 121\zeta_6 \quad F_3^{(3)}(\cdot) = -145\zeta_6
\]
\[
A_6^{(4)}(\cdot) = 120f_{3,5} - 48\zeta_2f_{3,3} - \frac{6381}{4}\zeta_8
\]
\[
A_6^{(5)}(\cdot) = -2688f_{3,7} - 1560f_{5,5} + \mathcal{O}(\pi^2)
\]
\[
A_6^{(6)}(\cdot) = 48528f_{3,9} + 37296f_{5,7} + 21120f_{7,5} + \mathcal{O}(\pi^2)
\]

• Reversing ordering of words in \(f\)-alphabet, the blue values show that antipodal duality holds at these points beyond symbol level, modulo \(i\pi\)

• modulo \(i\pi\) seems to be the best we can get from the antipode
OPE parametrizations

• Amplitude:

\[ \hat{u} = \frac{1}{1 + (\hat{T} + \hat{S}\hat{F})(\hat{T} + \hat{S}/\hat{F})}, \]
\[ \hat{v} = \hat{u}\hat{w}\hat{S}^2/\hat{T}^2, \quad \hat{w} = \frac{\hat{T}^2}{1 + \hat{T}^2} \]

\[ u = \frac{1}{1 + S^2 + T^2}, \quad v = \frac{T^2}{1 + T^2}, \]
\[ w = \frac{1}{(1 + T^2)(1 + S^{-2}(1 + T^2))}, \]

• Form factor:

\( \hat{F} = 1 \) for \( \Delta = 0 \)

• Apply the kinematic map \( \hat{T} = \frac{T}{S}, \quad \hat{S} = \frac{1}{TS} \)

• There is apparently a correspondence between single flux tube excitations for the amplitude \( (T^1) \) and double (or bound state) excitations for the form factor \( (T^2) \)
8-gluon Amp $\leftrightarrow$ 4-gluon FF

- We have a candidate kinematic map for a 4-dimensional surface (4-gluon FF is 5d).
- $S[R_8^{(2)}]$ is known  
  S. Caron-Huot, 1105.5606
- The kinematic+antipodal maps take it to a symbol with 40 letters, the first 8 of which are “right”:  
  $u_i = \frac{s_{i,i+1}}{s_{1234}}$,  
  $v_i = \frac{s_{i,i+1,i+2}}{s_{1234}}$
- But we still have to run more checks on this candidate 2-loop 4-gluon form factor
8-4 Kinematic Map in OPE Parametrization

- 8-point amplitude has $D_8$ dihedral symmetry; change it to $D_4$ of the form factor by requiring
  \[ \hat{T}_3 = \hat{T}_1, \quad \hat{S}_3 = \hat{S}_1, \quad \hat{F}_3 = \hat{F}_1 \]
- To get $S[R_8^{(2)}]$ to have only 8 final entries, we also fix $\hat{F}_1 = \hat{F}_2 = 1$.
- The kinematic map becomes
  \[
  \hat{T}_1 = \frac{T}{S}, \quad \hat{S}_1 = \frac{1}{TS}, \\
  \hat{T}_2 = \frac{T_2}{S_2}, \quad \hat{S}_2 = \frac{1}{T_2S_2}
  \]
  and requires $F_2 = i$
Beyond 8-4

• The map \( \hat{T}_1 = \frac{T}{S}, \hat{S}_1 = \frac{1}{TS}, \hat{T}_2 = \frac{T_2}{S_2}, \hat{S}_2 = \frac{1}{T_2S_2} \)

seems likely to generalize to give rise to a \( 2(n - 2) \)
parameter subspace of the full \( 3n - 7 \) dimensional \( n \)-point form factor kinematics, presumably from setting \( F_2 = \cdots = F_{n-2} = i \)

• We can conjecture that antipodal duality applies on this subspace

• But there is still a lot to be checked!
Summary & Outlook

- Form factors as well as scattering amplitudes in planar N=4 SYM can now be bootstrapped to high loop order.
- By comparing the 3-gluon form factor to the 6-gluon amplitude, we found a strange new antipodal duality, which swaps the role of branch cuts and derivatives, and seems to map single flux-tube excitations (amplitude) to doubles (form factor).
- What is the underlying physical reason for this duality?
- (How) does it hold at strong coupling?
- (How much) can we verify of it at the 8-4 level, and beyond?
- How much can we exploit it to learn more about both amplitudes and form factors?
Extra Slides
Removing Amplitude (or Form Factor) Infrared Divergences

- On-shell amplitudes **IR divergent** due to long-range gluons

- Polygonal Wilson loops **UV divergent** at cusps, anomalous dimension $\Gamma_{\text{cusp}}$
  - known to all orders in planar N=4 SYM: Beisert, Eden, Staudacher, hep-th/0610251

- Both removed by dividing by **BDS-like ansatz**
  Bern, LD, Smirnov, hep-th/0505205, Alday, Gaiotto, Maldacena, 0911.4708

- Normalized [MHV] amplitude is finite, dual conformal invariant, also uniquely (up to constant) maintains important symbol adjacency relations due to causality (Steinmann relations for 3-particle invariants):

\[
\mathcal{E}(u_i) = \lim_{\epsilon \to 0} \frac{\mathcal{A}_6(s_{i,i+1}, \epsilon)}{\mathcal{A}_{6\text{-like}}(s_{i,i+1}, \epsilon)} = \exp\left[\frac{\Gamma_{\text{cusp}}}{4} \mathcal{E}^{(1)} + R_6\right]
\]

\(\mathcal{E}(u_i)\) is the remainder function.
BDS & BDS-like normalization for $\mathcal{F}_3$

\[
\frac{\mathcal{F}_3}{\mathcal{F}_{3,\text{MHV, tree}}} = \exp\left\{ \sum_{L=1}^{\infty} g^{2L} \left[ \left( \frac{\Gamma_{\text{cusp}}^{(L)}}{4} + \mathcal{O}(\epsilon) \right) M^{1-\text{loop}}(L\epsilon) + C^{(L)} + R^{(L)}(u, v, w) \right] \right\}
\]

BDS ansatz

split 1-loop amplitude judiciously:

\[
\frac{\mathcal{F}_{3,1-\text{loop}}}{\mathcal{F}_{3,\text{MHV, tree}}} \equiv M^{1-\text{loop}}(\epsilon) = M(\epsilon) + \mathcal{E}^{(1)}(u, v, w)
\]

\[
M(\epsilon) = -\frac{1}{\epsilon^2} \sum_{i=1}^{3} \left( \frac{\mu^2}{-s_{i,i+1}} \right)^{\epsilon} - \frac{7}{2} \zeta_2 + \sum_{i=1}^{3}
\]

\[
\mathcal{E}^{(1)}(u, v, w) = \left( 1 - \frac{1}{u} \right) + \text{Li}_2 \left( 1 - \frac{1}{w} \right)
\]

\[
\mathcal{E}^{(1), u} + \mathcal{E}^{(1), 1-u} = 0
\]

Now divide by $\mathcal{E}$ obeys “adjacency constraints”

\[
\frac{\mathcal{F}_{3,\text{BDS-like}}}{\mathcal{F}_{3,\text{MHV, tree}}} = \exp\left\{ \sum_{L=1}^{\infty} g^{2L} \left[ \left( \frac{\Gamma_{\text{cusp}}}{4} + \mathcal{O}(\epsilon) \right) M(L\epsilon) + C^{(L)} \right] \right\} \Rightarrow \mathcal{E} = \exp\left[ \frac{\Gamma_{\text{cusp}}}{4} \mathcal{E}^{(1)} + R \right]
\]
Different routes to perturbative amplitudes

Draw all Feynman graphs $G_i$

Evaluate all Feynman rules: $I_i$

Perform all loop integrations: $A_i$

$$ A = \sum_i A_i $$

Evaluate all unitarity cuts $C_\alpha$

Construct local: integrand $I$

Perform all loop integrations: $A_\alpha$

$$ A = \sum_\alpha A_\alpha $$

Bootstrap: Guess

$$ A = \sum_m c_m F_m $$

$F_m$ known functions

$c_m \in \mathbb{Q}$ unknown constants

Solve constraints, linear equations for $c_m \rightarrow r_m$

$$ A = \sum_m r_m F_m $$
Some numerics

\[ w = 0 \]

\[ v = 1 \]

\[ w = 1 \]

I = decay / Euclidean

IIa,b,c = scattering / spacelike operator

IIIa,b,c = scattering / timelike operator
Euclidean Region

For $L > 3$, ratio at $u = \frac{1}{3}$ is within 3% of cusp anomalous dimension ratio, $\frac{\Gamma^{(L)}_{\text{cusp}}}{\Gamma^{(L-1)}_{\text{cusp}}}$

$\rightarrow$ same finite radius of convergence?
Numerical implications of antipodal duality?
Real “impact factor” appears in space-like Regge limit, $\nu \to \infty$

Remainder function $R$ is nontrivial function of $u = \frac{s_{12}}{m_H^2}$ as $s_{23} \to \infty$
6-gluons: richer kinematical playground

Multi-particle factorization $u, w \to \infty$

(near) collinear
$v = 0, u + w = 1$

multi-Regge
$(1,0,0)$

self-crossing

spurious pole $u = 1$
Number of (symbol-level) linearly independent \( \{n, 1, \ldots, 1\} \) coproducts \((2L – n \text{ derivatives})\)

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<td>24</td>
<td>12</td>
<td>6</td>
<td>3</td>
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</table>

- Properly normalized \(L\) loop N=4 form factors \(\mathcal{E}^{(L)}\) belong to a small space \(\mathcal{C}\), dimension saturates on left
- \(\mathcal{E}^{(L)}\) also obeys multiple-final-entry relations, saturation on right
\[ \mathcal{E}^{(4)}(v \to \infty) = -1920H_{0,0,0,0,0,0,1,0,1} - 384H_{0,0,0,0,0,0,1,1,1} - 192H_{0,0,0,0,0,0,1,1,1} - 384H_{0,0,0,0,0,0,1,0,0,1} \\
- 96H_{0,0,0,0,0,1,0,0,1} - 96H_{0,0,0,0,0,1,1,0,1} - 384H_{0,0,0,0,0,1,0,0,0,1} - 96H_{0,0,0,0,0,1,0,0,0,1} \\
- 144H_{0,0,0,0,1,0,0,1,0,1} - 48H_{0,0,0,0,1,0,1,1,1} - 64H_{0,0,0,0,1,1,0,1,0,1} - 16H_{0,0,0,0,1,1,1,0,1,1} \\
- 16H_{0,0,0,0,1,1,1,0,1,1} - 64H_{0,0,0,0,1,1,1,0,0,0,1} - 384H_{0,0,1,0,0,0,0,0,1} - 96H_{0,0,1,0,0,0,0,0,1} \\
- 144H_{0,0,1,0,0,1,0,0,1} - 48H_{0,0,1,0,1,0,1,1,1} - 128H_{0,0,1,0,1,0,0,1,0,1} - 32H_{0,0,1,0,1,0,1,0,1,1} \\
- 32H_{0,0,1,0,1,1,0,1,0,1} - 32H_{0,0,1,0,1,1,0,0,0,1} - 48H_{0,0,1,0,1,0,0,1,0,1} - 16H_{0,0,1,0,1,1,1,0,1,1} \\
- 32H_{0,0,1,0,1,1,0,0,0,1} - 384H_{0,0,1,0,0,0,0,0,1} - 96H_{0,0,1,0,0,0,0,0,1} - 144H_{0,0,1,0,0,0,0,0,1} \\
- 48H_{0,1,0,0,0,0,0,1,1} - 128H_{0,1,0,0,0,1,0,0,1} - 32H_{0,1,0,0,1,0,1,1,1} - 32H_{0,1,0,0,1,0,0,1,0,1} \\
- 32H_{0,1,0,0,1,1,0,1,0,1} - 128H_{0,1,0,0,1,0,0,0,1} - 32H_{0,1,0,0,1,0,0,0,1} - 40H_{0,1,0,0,1,0,0,0,1} \\
- 40H_{0,1,0,0,1,1,0,1,0,1} - 24H_{0,1,0,0,1,1,0,0,1} - 32H_{0,1,0,0,1,1,0,0,1,0,1} - 32H_{0,1,0,0,1,1,1,0,1,1} \\
- 40H_{0,1,0,0,1,1,1,0,0,1} - 32H_{0,1,0,1,1,0,0,0,1} - 16H_{0,1,1,1,0,0,0,1} - 16H_{0,1,1,1,0,0,0,1} \\
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- 240H_{0,1,1,1,1,1,1,1,1} \\
+ \zeta_2(96H_{0,0,0,0,0,1,1,1} + 16H_{0,0,0,0,1,0,1,0,1} + 112H_{0,0,0,0,1,1,0,1,1} + 16H_{0,0,0,0,1,0,0,1,0,1} + 80H_{0,0,0,0,1,0,0,0,1,1} \\
+ 64H_{0,0,1,1,0,0,0,1} + 32H_{0,0,1,1,0,0,1,0,1} + 16H_{0,0,1,1,0,0,0,1,1} + 80H_{0,0,1,1,0,0,0,0,1} + 64H_{0,0,1,0,1,0,0,1,1} \\
+ 80H_{0,0,1,0,1,1,0,0,1} + 64H_{0,0,1,1,0,0,0,1} + 80H_{0,0,1,1,0,0,0,0,1} + 80H_{0,0,1,1,0,0,0,1} + 432H_{0,0,1,1,1,1,1}) \\
+ \zeta_3(224H_{0,0,0,0,0,0,0,1} - 48H_{0,0,0,0,0,0,1,0,1} - 48H_{0,0,0,0,0,0,1,1,1} - 48H_{0,0,0,0,0,0,1,0,0,1} \\
- 8H_{0,0,0,0,0,1,0,1,1} + 16H_{0,0,0,0,0,1,1,0,1,1} + 16H_{0,0,0,0,0,1,0,1,1,1} + 16H_{0,0,0,0,0,1,0,0,1,1,1} \\
+ \zeta_4(292H_{0,0,0,0,0,0,0,1} - 84H_{0,0,0,0,0,0,1,0,1} + 84H_{0,0,0,0,0,0,1,1,1} + 696H_{0,0,0,0,0,0,1,1,1}) \\
+ \zeta_5(264H_{0,0,0,0,0,0,0,0,1} - 72H_{0,0,0,0,0,0,0,1,1} + 80\zeta_2\zeta_3H_{0,0,0,0,0,0,1} + \left(\frac{3782}{3}\zeta_6 - 12(\zeta_3)^2\right)H_{0,1} \\
+ \frac{49141}{360}\zeta_8 - 20\zeta_{5,3} - 352\zeta_3\zeta_5 + 8\zeta_2(\zeta_3)^2)
\]

8 loop result has $\sim 2^{2\times 8-2} = 16,384$ terms
Values of HPLs \{0,1\} at \( u = 1 \)

- Classical polylogs evaluate to Riemann zeta values

\[
\text{Li}_n(u) = \int_0^u \frac{dt}{t} \text{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{u^k}{k^n}
\]

\[
\text{Li}_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta(n) \equiv \zeta_n
\]

- HPL’s evaluate to nested sums called multiple zeta values (MZVs):

\[
\zeta_{n_1,n_2,\ldots,n_m} = \sum_{k_1>k_2>\ldots>k_m>0}^{\infty} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_m^{n_m}}
\]

Weight \( n = n_1 + n_1 + \ldots + n_m \)

- MZV’s obey many identities, e.g. stuffle

\[
\zeta_{n_1} \zeta_{n_2} = \zeta_{n_1,n_2} + \zeta_{n_2,n_1} + \zeta_{n_1+n_2}
\]

- All reducible to Riemann zeta values until weight 8.

Irreducible MZVs: \( \zeta_{5,3}, \zeta_{7,3}, \zeta_{5,3,3}, \zeta_{9,3}, \zeta_{6,4,1,1}, \ldots \)
Symbol is too verbose
→ Nested representation better

- Define every function by its \( \{n-1, 1\} \) coproducts, i.e. its first derivatives.
- Also need to specify constants of integration at one point, e.g. \((u, v, w) = (1, 0, 0)\)

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Many empirical adjacency constraints

\[ F^{d,e} = F^{e,d} = F^{e,f} = F^{f,e} = F^{f,d} = F^{d,f} = 0 \]

Hold for 2 loop QCD amplitudes too, planar and nonplanar!
LD, Mcleod, Wilhelm, 2012.12286

\[ F^{a,d} = F^{d,a} = F^{b,e} = F^{e,b} = F^{c,f} = F^{f,c} = 0 \]

Latter are NEW: Hold for planar N=4 SYM to 8 loops!
LD, Gürdoğan, Mcleod, Wilhelm, to appear

Mnemonic for dihedral symmetry;
6 dashed lines indicate 12 forbidden pairs.
Empirical multi-final entry relations

1. $\mathcal{E}^a = 0$ (plus dihedral images)

2. $\mathcal{E}^{a,e} = \mathcal{E}^{a,f}$ (plus ...)

3. $\mathcal{E}^{a,b,d} = 0$, $\mathcal{E}^{a,e,e} = - \mathcal{E}^{a,f,f}$, 
   $\mathcal{E}^{e,a,f} = \mathcal{E}^{f,a,f} - \mathcal{E}^{a,f,f}$

4. ...

L. Dixon   Antipodal Duality
KITP - 2022/01/26  56
Symbol alphabets for $n$-gluon amplitudes

$n = 6$ has 9 letters: $S = \{u, v, w, 1 - u, 1 - v, 1 - w, y_u, y_v, y_w\}$

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

$n = 7$ has 42 letters

Golden, Goncharov, Paulos, Spradlin, Volovich, Vergu, 1305.1617, 1401.6446, 1411.3289; Drummond, Papathanasiou, Spradlin 1412.3763

$n = 8$ has at least 222 letters, could even be infinite as $L \to \infty$

Arkani-Hamed, Lam, Spradlin, 1912.08222; Drummond, Foster, Kalousios, 1912.08217, 2002.04624; Henke, Papathanasiou 1912.08254, 2106.01392; Z. Li, C. Zhang, 2110.00350
Heuristic view of space

weight

\[
\begin{align*}
\int & \int \int \int \int \\
\int & \int \int \int \int \\
\int & \int \int \int \\
\int & \int \int \\
\int & \int \\
\int & \\
1 & \\
0 &
\end{align*}
\]

definitions:

- \( \text{Li}_3(1-1/u_i) \), true 2D HPLs, ...
- \( \text{Li}_2(1-1/u_i) \)
- \( \ln^2 u_i \)
- \( \ln u_i \ln u_{i+1} \) - \( \zeta_2 \)
- \( \ln u \), \( \ln v \), \( \ln w \)

1

\[
\frac{\partial F(u, v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{1-u-v} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{u+v}
\]
Number of remaining parameters in form-factor ansatz after imposing constraints

<table>
<thead>
<tr>
<th>$L$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>symbols in $\mathcal{C}$</td>
<td>48</td>
<td>249</td>
<td>1290</td>
<td>6654</td>
<td>34219</td>
<td>????</td>
<td>????</td>
</tr>
<tr>
<td>dihedral symmetry</td>
<td>11</td>
<td>51</td>
<td>247</td>
<td>1219</td>
<td>????</td>
<td>????</td>
<td>????</td>
</tr>
<tr>
<td>$(L - 1)$ final entries</td>
<td>5</td>
<td>9</td>
<td>20</td>
<td>44</td>
<td>86</td>
<td>???</td>
<td>???</td>
</tr>
<tr>
<td>$L^{th}$ discontinuity</td>
<td>2</td>
<td>5</td>
<td>17</td>
<td>38</td>
<td>75</td>
<td>???</td>
<td>??</td>
</tr>
<tr>
<td>collinear limit</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>8</td>
<td>19</td>
<td>70</td>
<td>6</td>
</tr>
<tr>
<td>OPE $T^2 \ln^{L-1} T$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>12</td>
<td>56</td>
<td>0</td>
</tr>
<tr>
<td>OPE $T^2 \ln^{L-2} T$</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>36</td>
<td>0</td>
</tr>
<tr>
<td>OPE $T^2 \ln^{L-3} T$</td>
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<td>0</td>
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<td>0</td>
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<tr>
<td>OPE $T^2 \ln^{L-4} T$</td>
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<td>0</td>
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<tr>
<td>OPE $T^2 \ln^{L-5} T$</td>
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<td>0</td>
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</tr>
</tbody>
</table>

Table 4: Number of parameters left when bootstrapping the form factor $\mathcal{E}^{(L)}$ at $L$-loop order in the function space $\mathcal{C}$ at symbol level, using all the conditions on the final $(L - 1)$ entries, which can be deduced at $(L - 1)$ loops.
The [Dual] Conformal Group

SO(4,2) \supset SO(3,1) \ [rotations+boosts] + translations+dilatations + special-conformal

15 = 3 + 3 + 4 + 1 + 4

- The nontrivial generators are special conformal $K^\mu$
- Correspond to inversion \cdot translation \cdot inversion
- To obtain a [dual] conformally invariant function $f(x_{ij}^2)$
  just have to check invariance under inversion,
  \[ x_i^\mu \rightarrow x_i^\mu / x_i^2 \]