Many-Body Quantum Entanglement and Topological Phases of Matter

Tarun Grover
(KITP)

Frank Zhang

Ari Turner

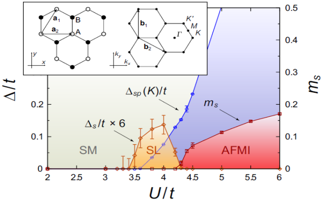
Masaki Oshikawa

Ashvin Vishwanath

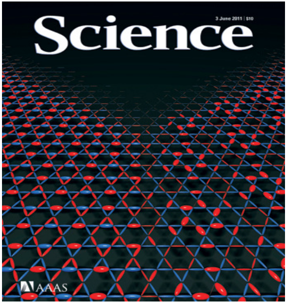
How much information about a phase of matter can be extracted from the ground state wave-function(s) alone?

Context: Quantum Spin-liquids

Quantum Spin Liquids



Honeycomb Hubbard Model (Meng et al 2010)



kagome Heisenberg (Yan et al 2011)

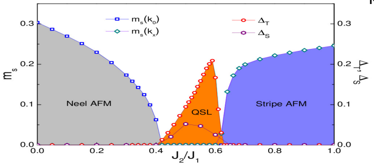


FIG. 1: **The ground state phase diagram for the spin-$\frac{1}{2}$ AFM Heisenberg J_1-J_2 model on the square lattice, as determined by accurate DMRG calculations on long cylinders with L_y up to 10.** Changing the coupling parameter J_2/J_1, three different

Square J_1-J_2 (Hongchen et al 2011)

Positive signature of gapped quantum spin-liquids =

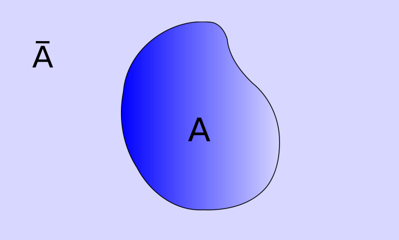
long-range quantum entanglement

(Levin, Wen 2006; Kitaev, Preskill 2006)

Entanglement Entropy

- Reduced density matrix for sub-region A:

$$\rho_A = trace_{\overline{A}}(|\psi\rangle\langle\psi|)$$



- Renyi Entropy S_n:

$$S_n = -\frac{1}{n-1}\log(\mathrm{Trace}\ \rho_A^n)$$

Topological Order ⇔ Topological Entanglement Entropy

- For topological ordered states:

$$S_A = S_{A,non-universal} - \gamma + O(\xi/L)$$

where γ is a universal number "Topological Entanglement Entropy".
(Levin, Wen 2006; Kitaev, Preskill 2006)

- γ calculable by a Variational Monte Carlo technique.

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Establishing Quantum Spin-liquid: A Simple Example

- Consider superconductor wavefunction $|BCS\rangle$
- Project wavefunction $|BCS\rangle$ down to one-particle per site:

$$|\Psi\rangle_{projected} = \prod_i (1 - n_{i\uparrow} n_{i\downarrow}) |BCS\rangle$$

- This is a putative gapped spin-liquid!
- Theoretical prediction: for $d_{xy} + id_{x^2-y^2}$ pairing, TEE $\gamma = \frac{1}{2}\log 2$ (Dong, Fradkin, Leigh, Nowling 2008).

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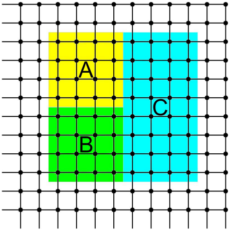
Results

State	Expected γ	$\gamma_{\text{calculated}}/\gamma_{\text{expected}}$
Chiral Spin Liquid	$\log\sqrt{2}$	0.99 ± 0.12
$\nu = 1/3$ Laughlin lattice state	$\log\sqrt{3}$	1.07 ± 0.05

(Zhang, Grover, Vishwanath 2011)



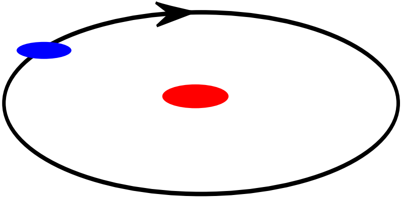
Kitaev-Preskill construction:

$$\gamma = S_A + S_B + S_C - S_{AB} - S_{BC} - S_{CA} + S_{ABC}$$

Can one do more?

Hallmark of topological order: anyonic excitations.

Mutual anyons pick up a Berry phase upon encircling.



Extract mutual fractional statistics using ground state wavefunction alone?

Entanglement to the rescue!

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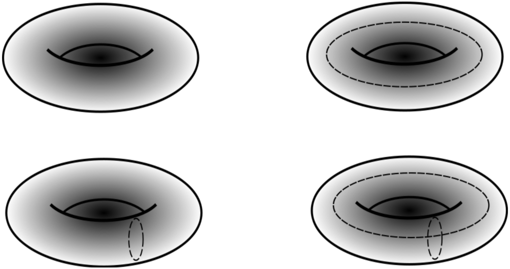
Quantum Entanglement and Mutual Statistics

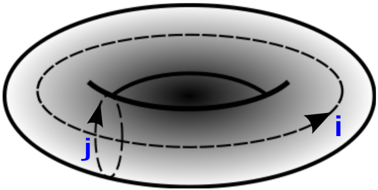
Degenerate ground states on a torus.



Threading of quasiparticles via non-contractible loops ⇒ Degenerate ground states.

Quantum Entanglement and Mutual Statistics

Mutual statistics S_{ij}: phase acquired by i'th quasiparticle when it encircles j.

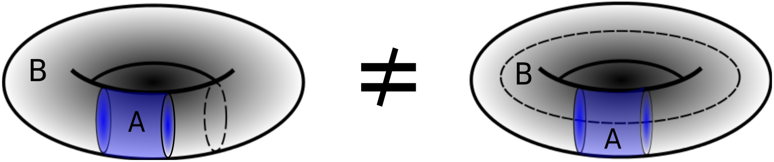


- $|\Sigma\rangle_{i,x} \equiv$ threading of i'th particle along x-direction.
- $|\Sigma\rangle_{j,y} \equiv$ threading of j'th particle along y-direction.

$$S_{ij} \propto {}_{i,x}\langle\Sigma|\Sigma\rangle_{j,y}$$

Key Observation # 1

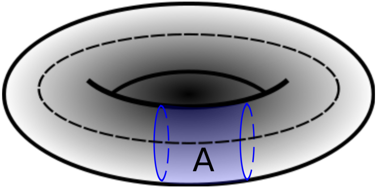
Topological Entanglement Entropy is *different for different ground states* if the entanglement cut has non-contractible boundary.



Key Observation # 2

Quasiparticle threading along $\hat{x}$ loop =

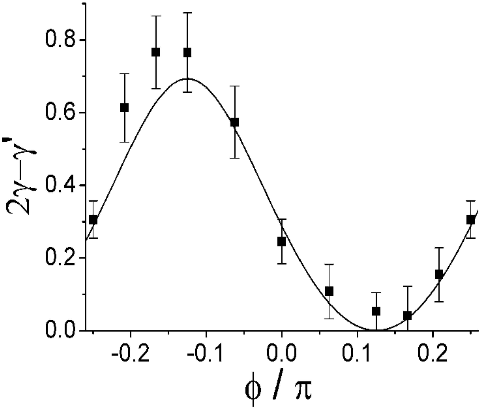
Wavefunction that minimizes entanglement entropy for a cut perpendicular to $\hat{x}$.



Intuition: *Minimum Entropy* $\Rightarrow$ *Maximum Knowledge* of quasiparticle content.

Application: Mutual Statistics in Chiral Spin Liquid

- Chiral spin-liquid: two degenerate ground states $|1\rangle$ and $|2\rangle$.
- Superpose: $|\Phi\rangle = cos(\phi)|1\rangle + sin(\phi)|2\rangle$
- Minimize entanglement entropy $S(\phi)$ numerically using Monte Carlo to get quasiparticle states.



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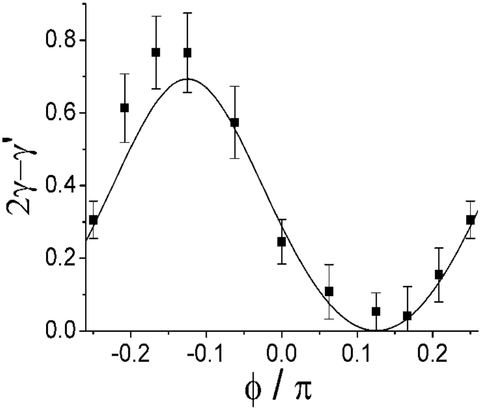

Application: Mutual Statistics in Chiral Spin Liquid

$$\sqrt{2}\mathcal{S}_{numerical} \left(\begin{array}{c|cc} & \textit{Identity} & \textit{semion} \\ \hline \textit{Identity} & 1.08 & 0.90 \\ \textit{semion} & 0.90 & -1.08 \end{array} \right)$$

- The *negative* sign on the diagonal $\Rightarrow$ semionic self-statistics!
- Consistent with the Chern-Simons effective field theory.

Zhang, Grover, Turner, Oshikawa, Vishwanath 2011

Cincio, Vidal 2012

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Summary and Open Questions

- Ground state wavefunctions "know" universal properties of excitations above the ground state, for example, entanglement Entropy can detect fractional statistics of quasiparticles ⇒ applicable to realistic spin-liquids and quantum Hall systems.
- Practical Implications for numerics such as DMRG (cf: Hong-Chen Jiang's talk).
- Interplay of global symmetries and topological order using entanglement? (cf: Ying Ran's and Mike Hermele's talk)
- Entanglement structure of interacting gapless phases?
- Braiding and statistics of extended objects (string, membranes, ...) using entanglement?