Entanglement Entropy in Spin Liquids, Gapless Phases and Quantum Critical Points

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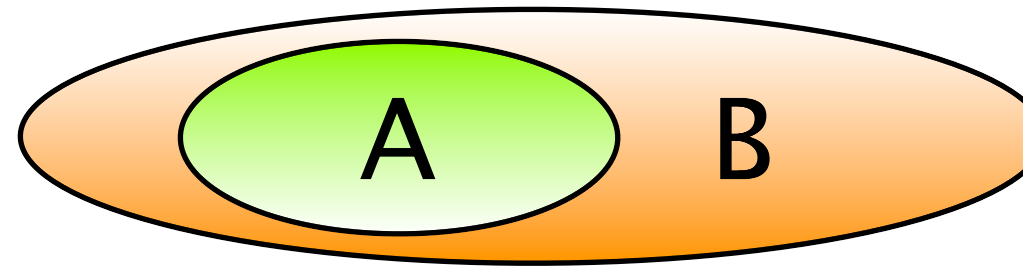
Entanglement Entropy

"The fact that information can be measured is, by now, generally accepted"
A. Renyi, 1960

Renyi Entropy:

$$S_n(\rho_A) = \frac{1}{1-n} \ln \left[\mathrm{Tr}(\rho_A^n) \right]$$



$$S_1(\rho_A) = -\mathrm{Tr}(\rho_A \ln \rho_A)$$

$$\rho = \mathrm{Tr}|\Psi\rangle\langle\Psi|$$

$$S_2(\rho_A) = -\ln \left[\mathrm{Tr}(\rho_A^2) \right]$$

$$\rho_A = \mathrm{Tr}_B(\rho)$$

- Comes for free in DMRG via the reduced density matrix
- $n \geq 2$ can be measured in Quantum Monte Carlo with a **swap** operator (T=0 projector) or a **replica trick** (T>0 world-line).

Hastings, Kallin, Gonzalez, RGM Phys. Rev. Lett. **104**, 157201 , Phys. Rev. B, **82**, 180504 (2010)

Entanglement scaling in 2D groundstates

area law $S_n = a\ell + \cdots$

topological spin liquid $S_n = a\ell - \gamma$

goldstone mode $S_n = a\ell + b\ln(\ell\ ) + \gamma(\ell_x, \ell_y)$

RVB wavefunction $S_n = a\ell + \gamma(\ell_x, \ell_y)$

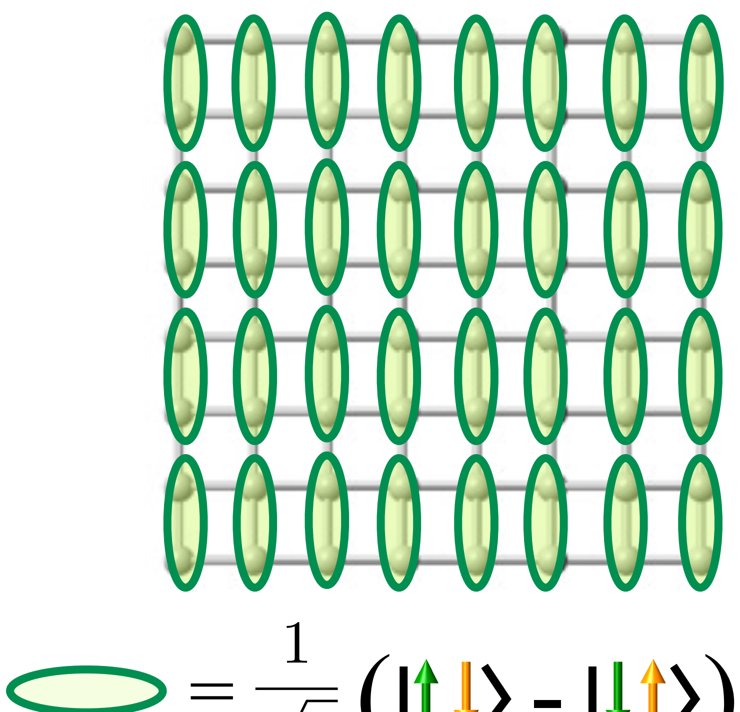
critical points $S_n = a\ell + c_n\gamma(\ell_x, \ell_y)$

Valence Bond Solid

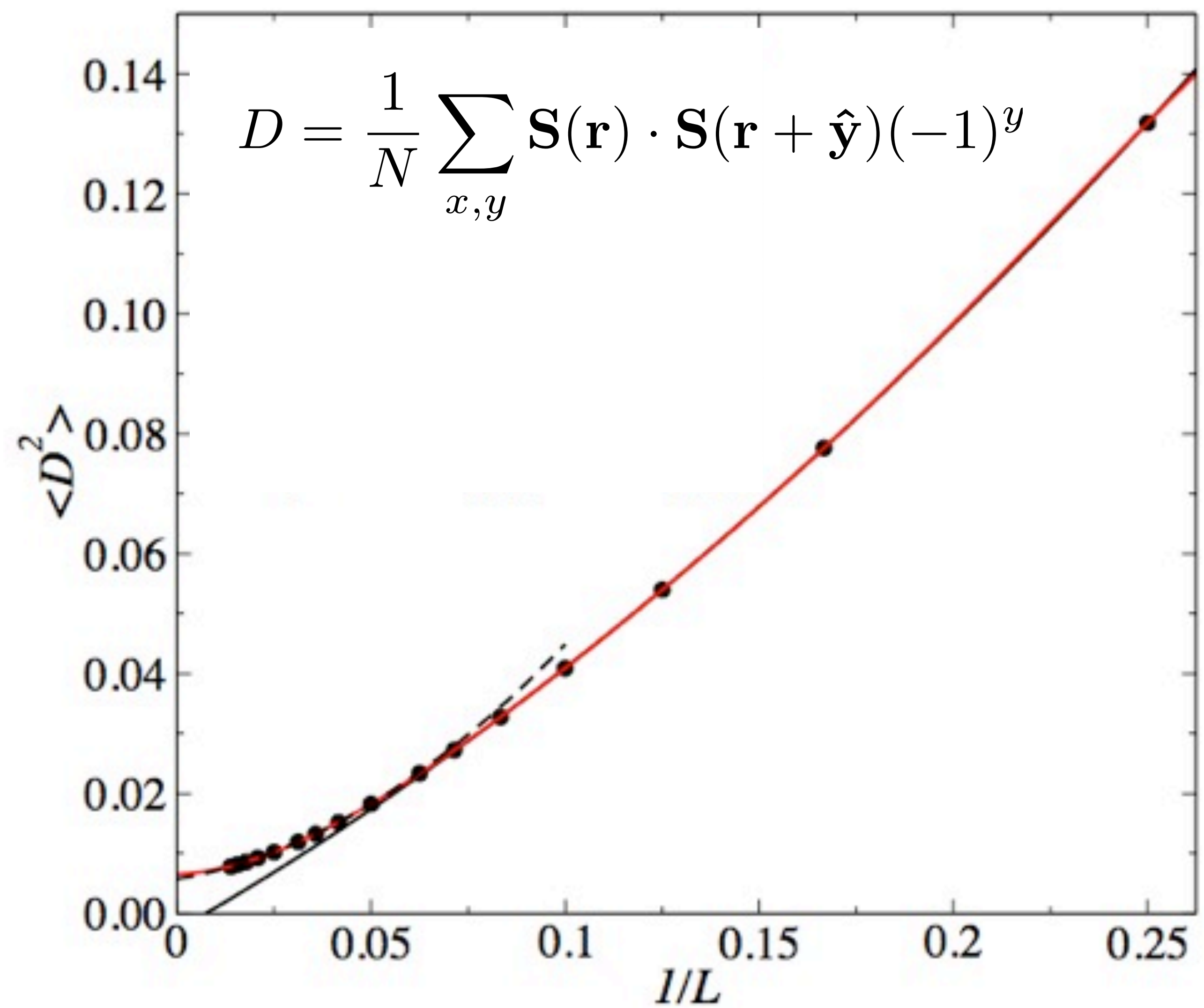
$$H = -\sum_{\langle ij \rangle} \left(\mathbf{S}_i \cdot \mathbf{S}_j - \frac{1}{4}\right)\left(\mathbf{S}_k \cdot \mathbf{S}_l - \frac{1}{4}\right)$$

Sandvik, PRB 85 134407 (2012)

- Order parameter in a columnar VBS phase:

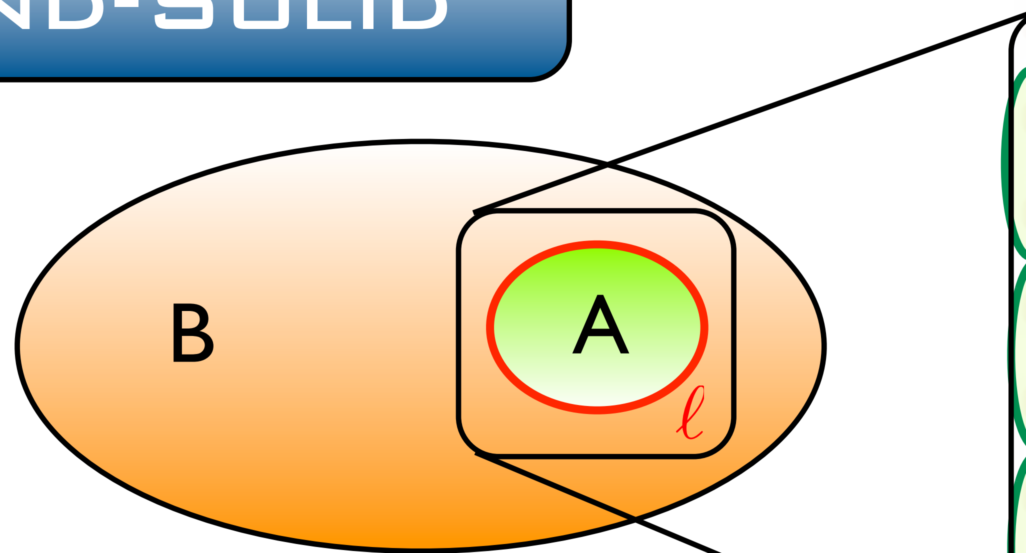


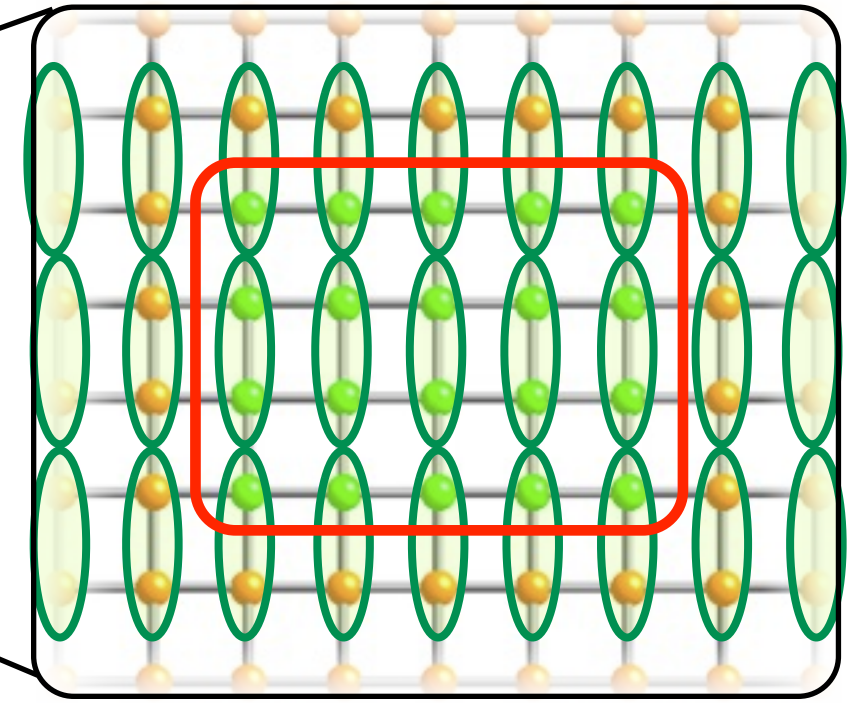
$$= \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

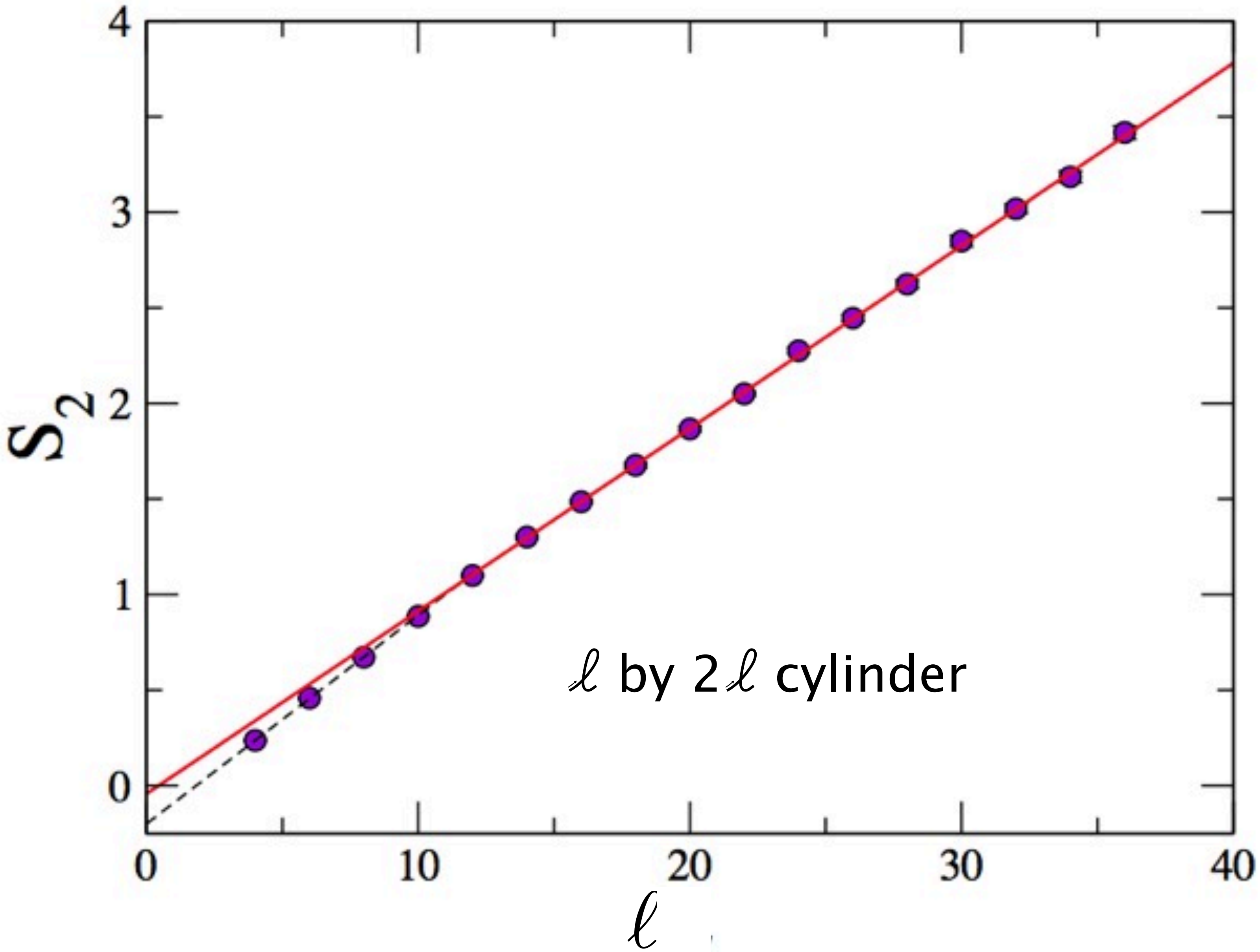


- large finite-size helps to distinguish a weak order parameter (VBS) from no order parameter (QSL)

Valence Bond-Solid







$$H = -\sum_{\langle ij \rangle} \left(\mathbf{S}_i \cdot \mathbf{S}_j - \frac{1}{4} \right) \left(\mathbf{S}_k \cdot \mathbf{S}_l - \frac{1}{4} \right)$$

need $\ell > 12$ data to see proper scaling

$$S_2 = a\ell$$

Outline

Entanglement scaling in 2D groundstates

area law $S_n = a\ell + \cdots$

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Entanglement scaling in 2D groundstates

area law $S_n = a\ell + \cdots$

topological spin liquid $S_n = a\ell - \gamma$

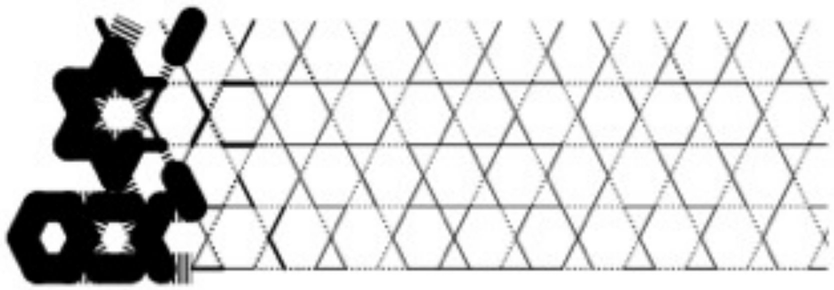
The Topological Entanglement Entropy is **independent** of Renyi index

$$S_n(\rho_A) = \frac{1}{1-n} \ln \left[\mathrm{Tr}\left(\rho_A^n \right) \right]$$

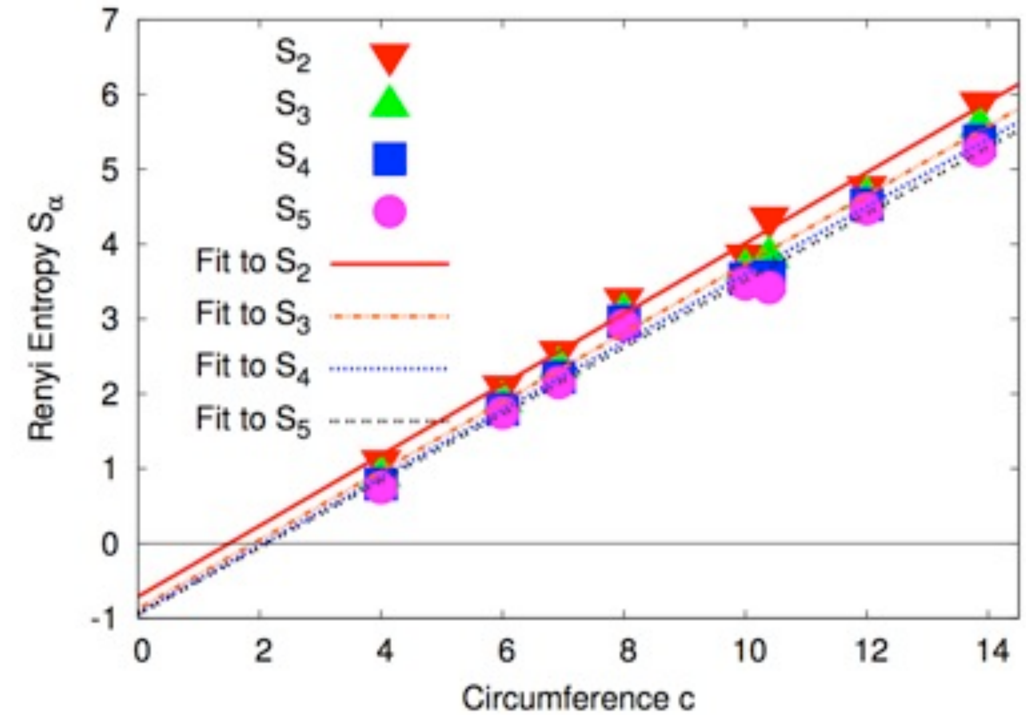
Hamma, Ionicioiu, Zanardi - Phys. Lett. A 337, 22 (2005)
- Phys. Rev. A 71, 022315 (2005)
Kitaev and Preskill - Phys. Rev. Lett. 96, 110404 (2006)
Levin and Wen, - Phys. Rev. Lett. 96, 110405 (2006)
Flammia, Hamma, Hughes, Wen, Phys. Rev. Lett 103, 261601 (2009)

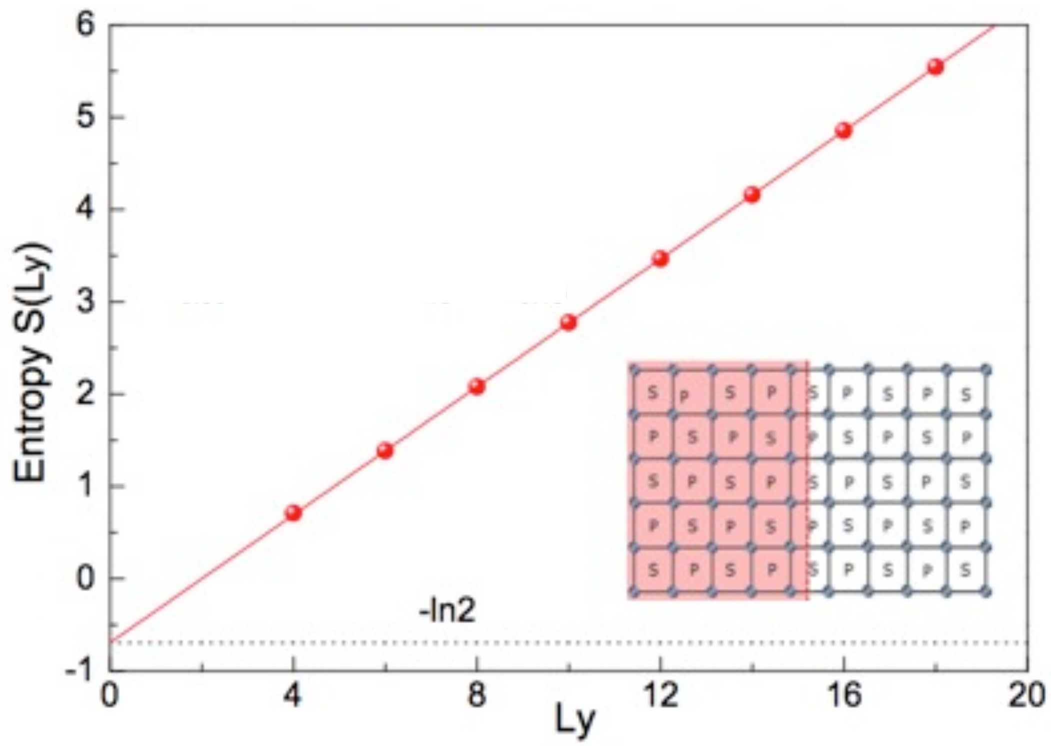
Recent T=0 approaches: DMRG

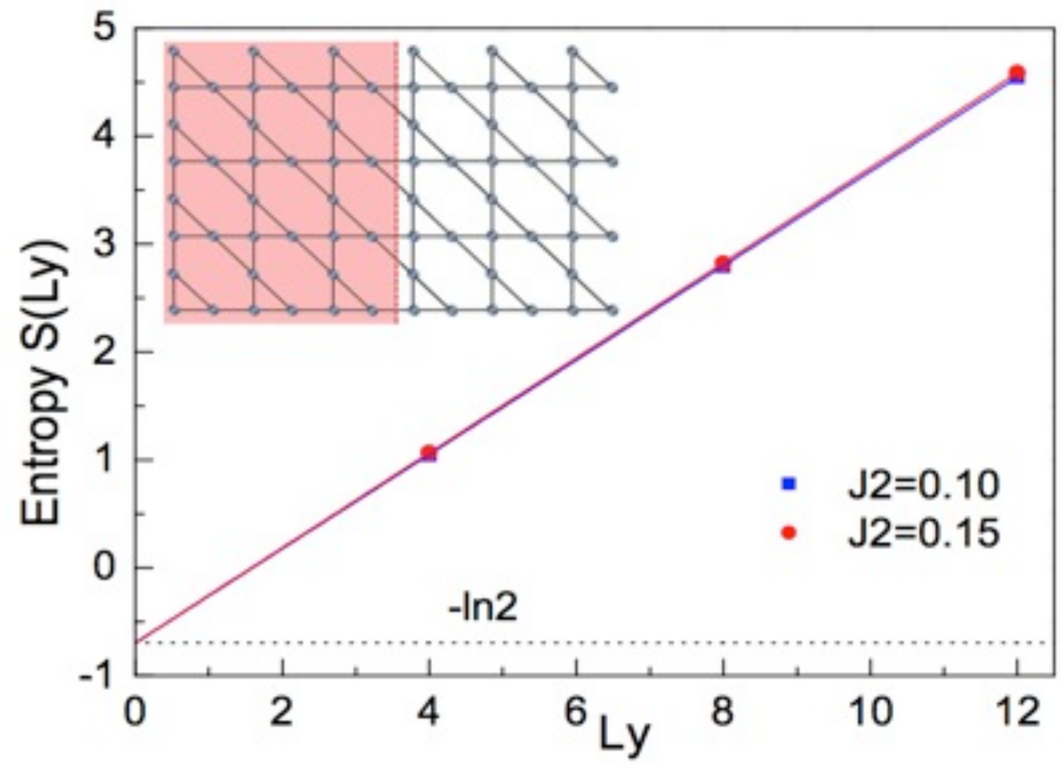
Depenbrock, McCulloch, Schollwoeck, PRL 109, 067201



$$H = J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$







Jiang, Wang, Balents, 1205:4289

Murphy's Law

In a simple model without the sign problem nothing interesting can occur

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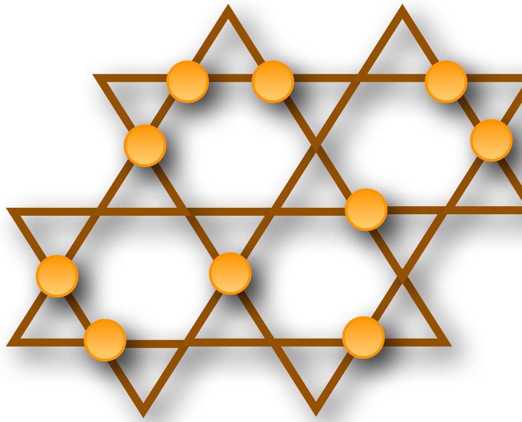
In a simple model without the sign problem nothing interesting can occur

BFG class: **Balents, Fisher, Girvin, PRB 65, 224412 (2002)**

$$H = H_0 + H_{\rm ring}$$

$$H_0 = V \sum_{\hexagon} (n_{\hexagon})^2 \qquad n_{\hexagon} = \sum_{i \in \hexagon} (n_i - 1/2)$$

$$H_{\rm ring} = -J_{\rm ring} \sum_{\langle ijkl \rangle} (b_i^\dagger b_j b_k^\dagger b_l + b_i b_j^\dagger b_k b_l^\dagger)$$



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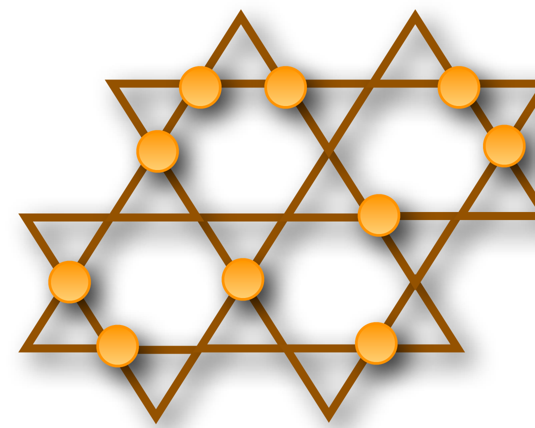
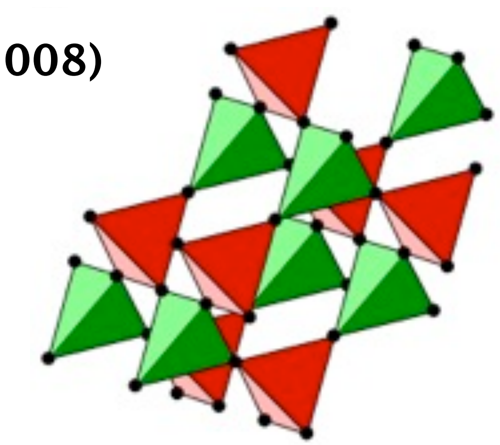


Pyrochlore U(1) spin liquid: **Banerjee, Isakov, Damle, Kim, PRL 100, 047298 (2008)**

$$H = -t \sum_{\langle ij \rangle} (b_i^\dagger b_j + b_i b_j^\dagger) + V \sum_{\langle ij \rangle} n_i n_j$$



Large-scale QMC has demonstrated several BFG models with spin liquids on the kagome lattice

$$H = -t \sum_{\langle\langle\langle ij \rangle\rangle\rangle} (b_i^\dagger b_j + b_i b_j^\dagger) + V \sum_{\hexagon} (n_{\hexagon})^2$$

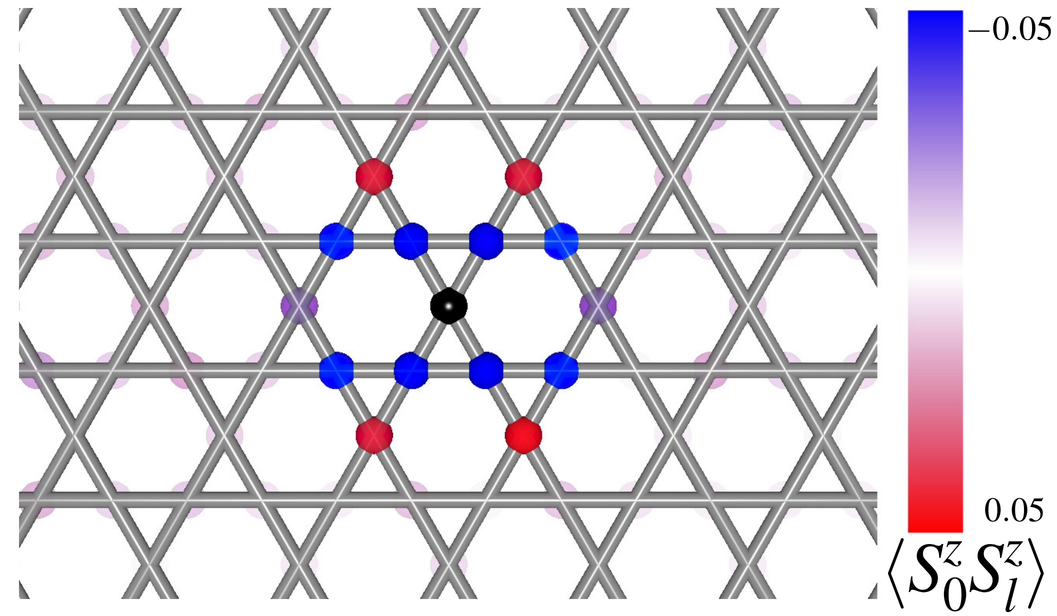
Isakov, Kim, Paramekanti   Phys. Rev. Lett. 97, 207204 (2006)

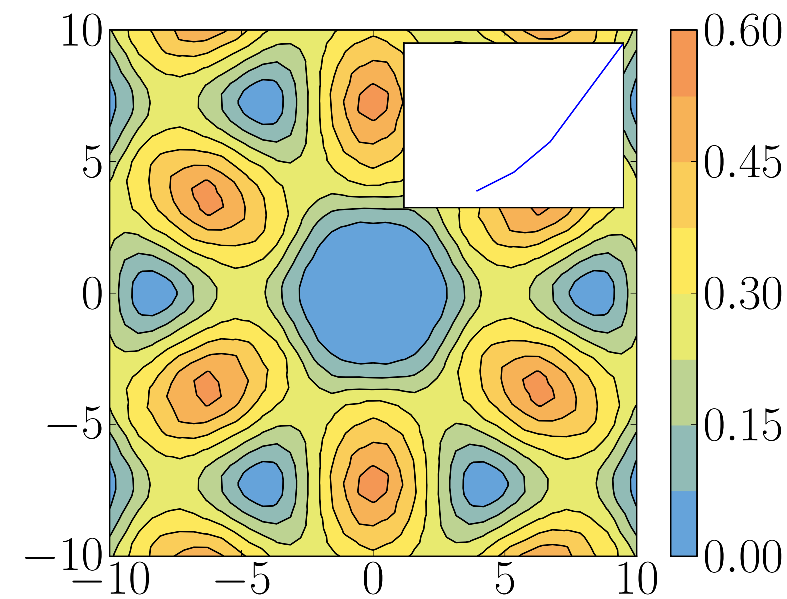
$$H = -t \sum_{\langle ij \rangle} (b_i^\dagger b_j + b_i b_j^\dagger) + V \sum_{\hexagon} (n_{\hexagon})^2$$

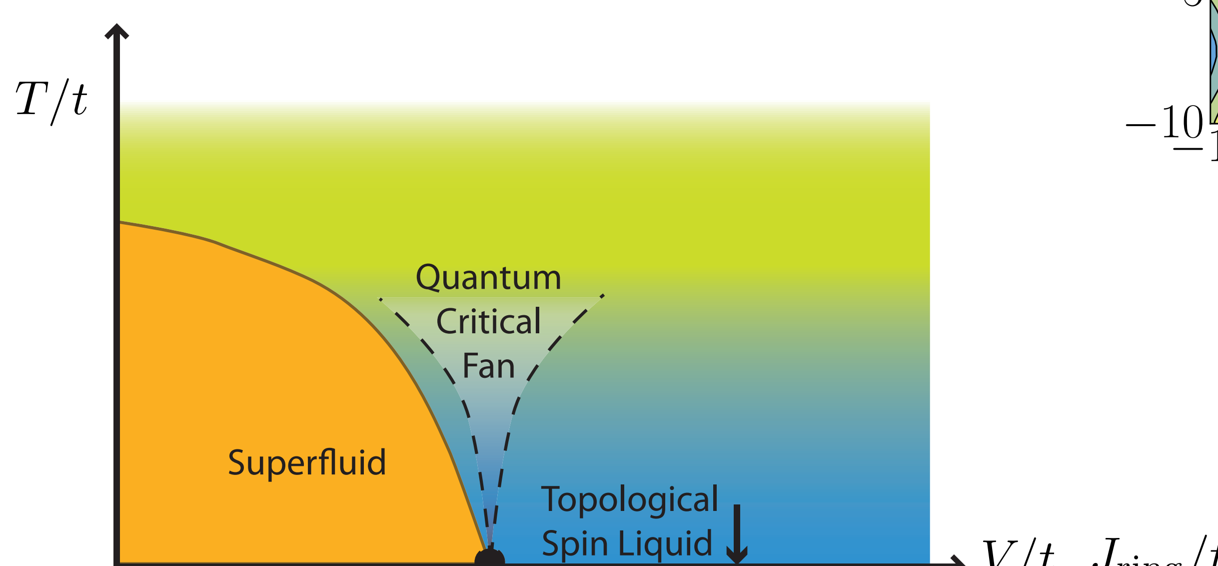
Isakov, Hastings, RGM   Nature Physics 7, 772 (2011)

$$H = -t \sum_{\langle ij \rangle} (b_i^\dagger b_j + b_i b_j^\dagger) - J_{\text{ring}} \sum_{\langle ijkl \rangle} (b_i^\dagger b_j b_k^\dagger b_l + b_i b_j^\dagger b_k b_l^\dagger)$$

Dang, Inglis, RGM   Phys. Rev. B 84, 132409 (2011)

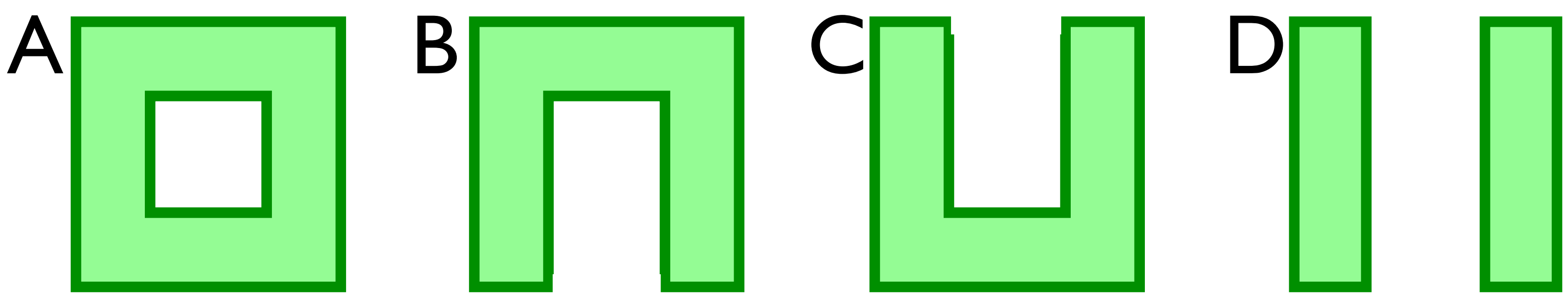




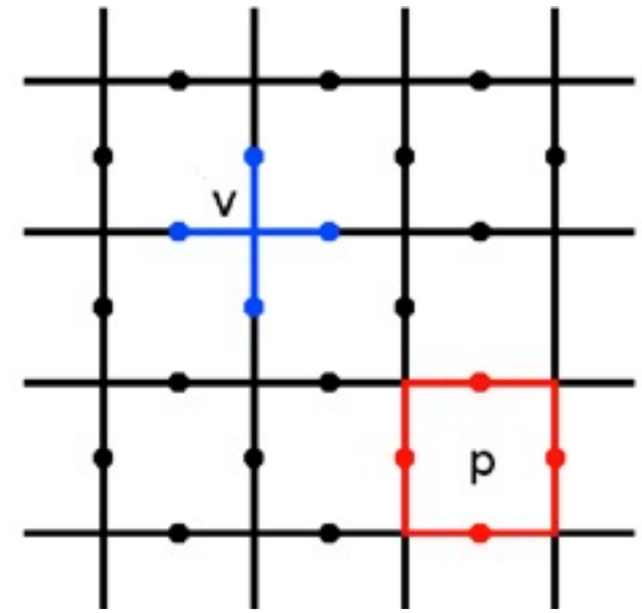


Levin and Wen, - Phys. Rev. Lett. 96, 110405 (2006)

- Boundary, corner, and bulk terms can be subtracted

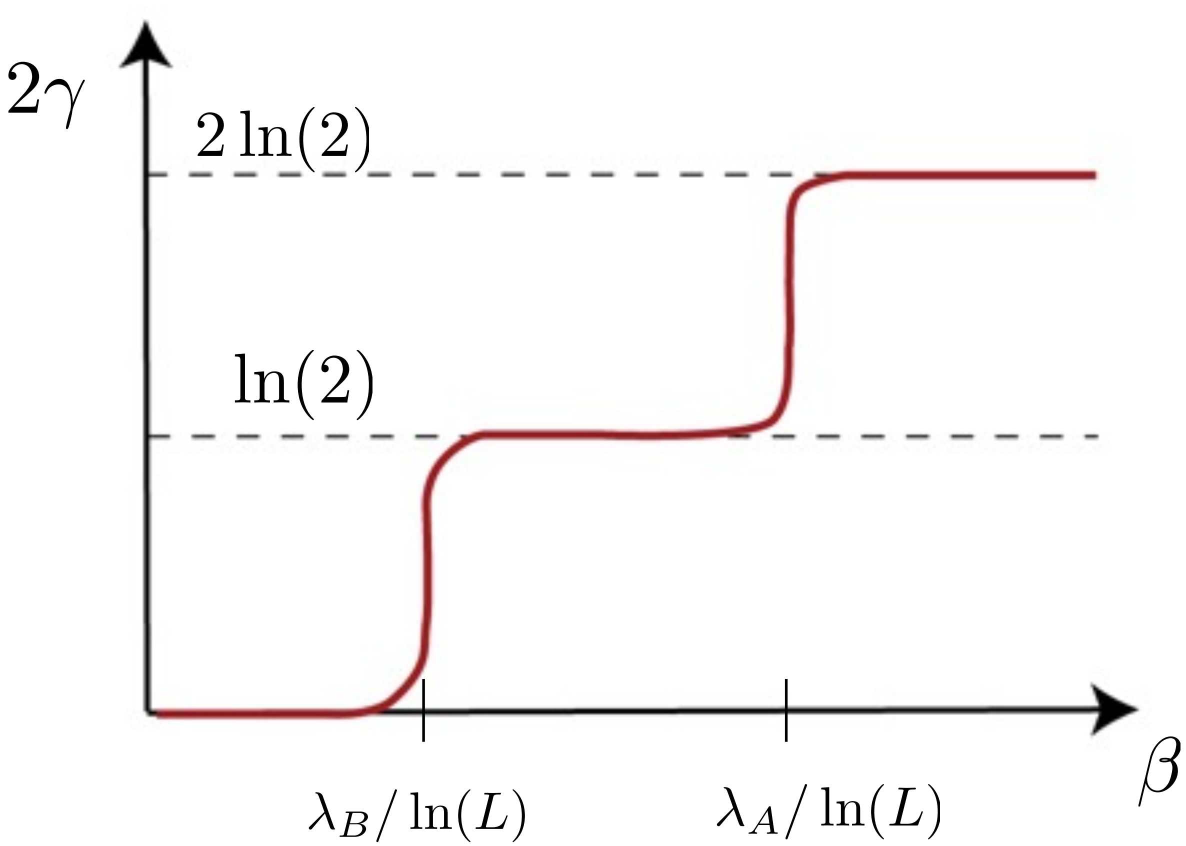


$$2\gamma = -S_n^A + S_n^B + S_n^C - S_n^D$$



$$H = -\lambda_B \sum_p B_p - \lambda_A \sum_v A_v$$

$$B_p = \prod_{i \in p} \sigma_i^z \qquad A_v = \prod_{j \in v} \sigma_j^x$$



Crossover temperatures are a result of defects in the “loop gas”

Z2 charge (spinon, e)  $\propto N e^{-\beta\lambda_A}$

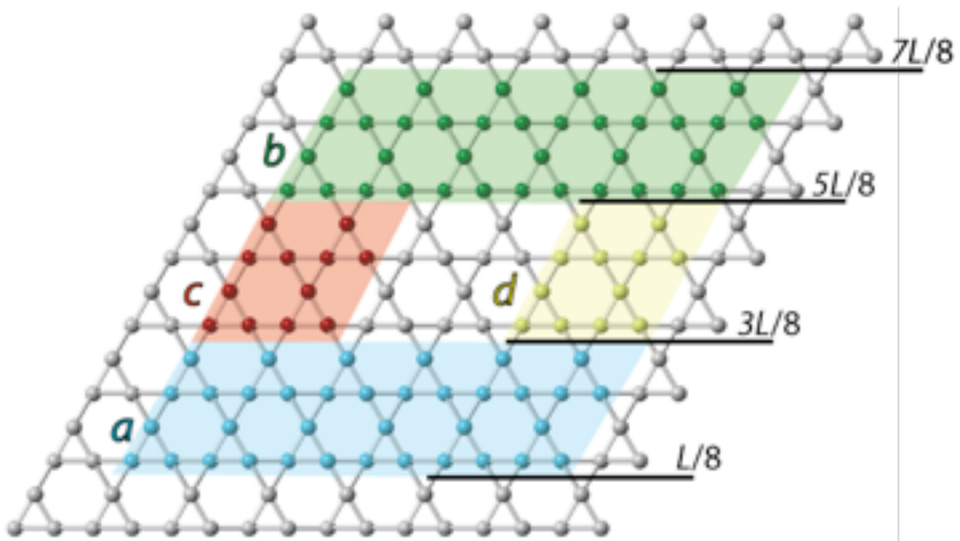
Z2 vortex (vison, m)  $\propto N e^{-\beta\lambda_B}$

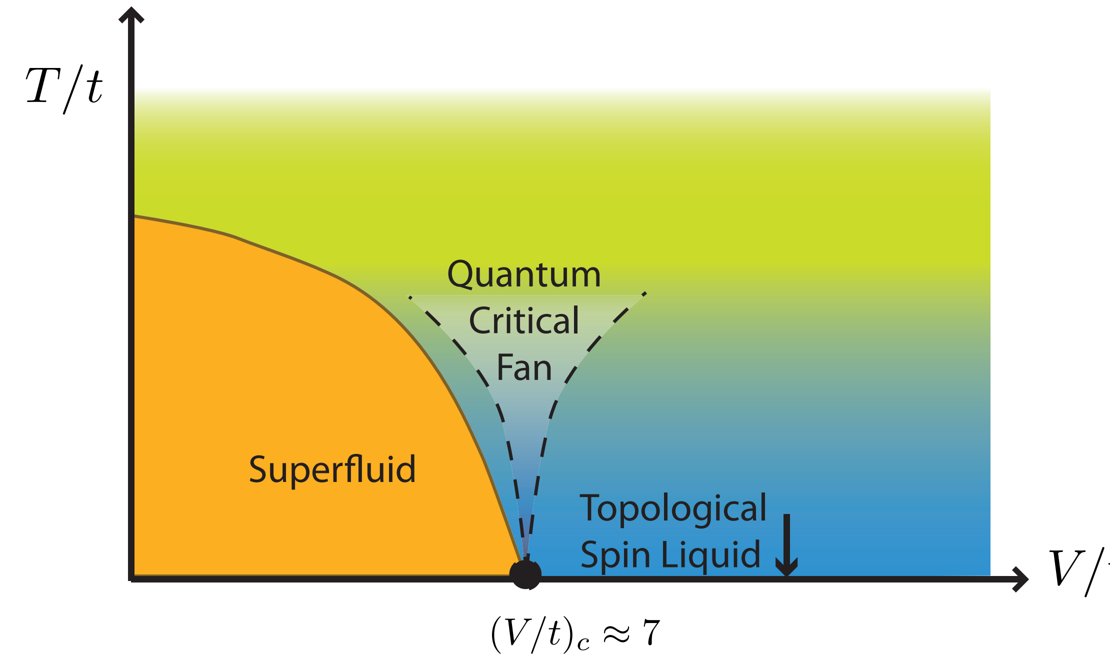
Claudio Castelnovo and Claudio Chamon, PRB 76, 184442 2007

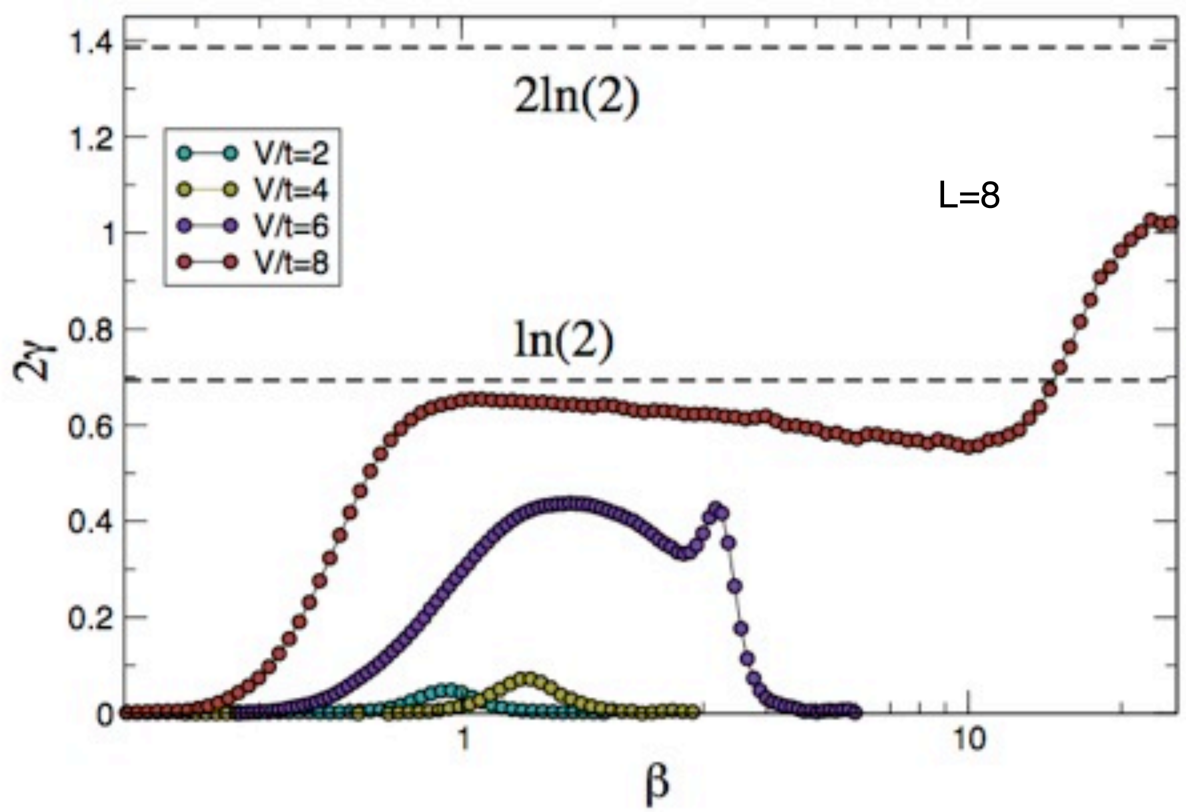
Topological EE

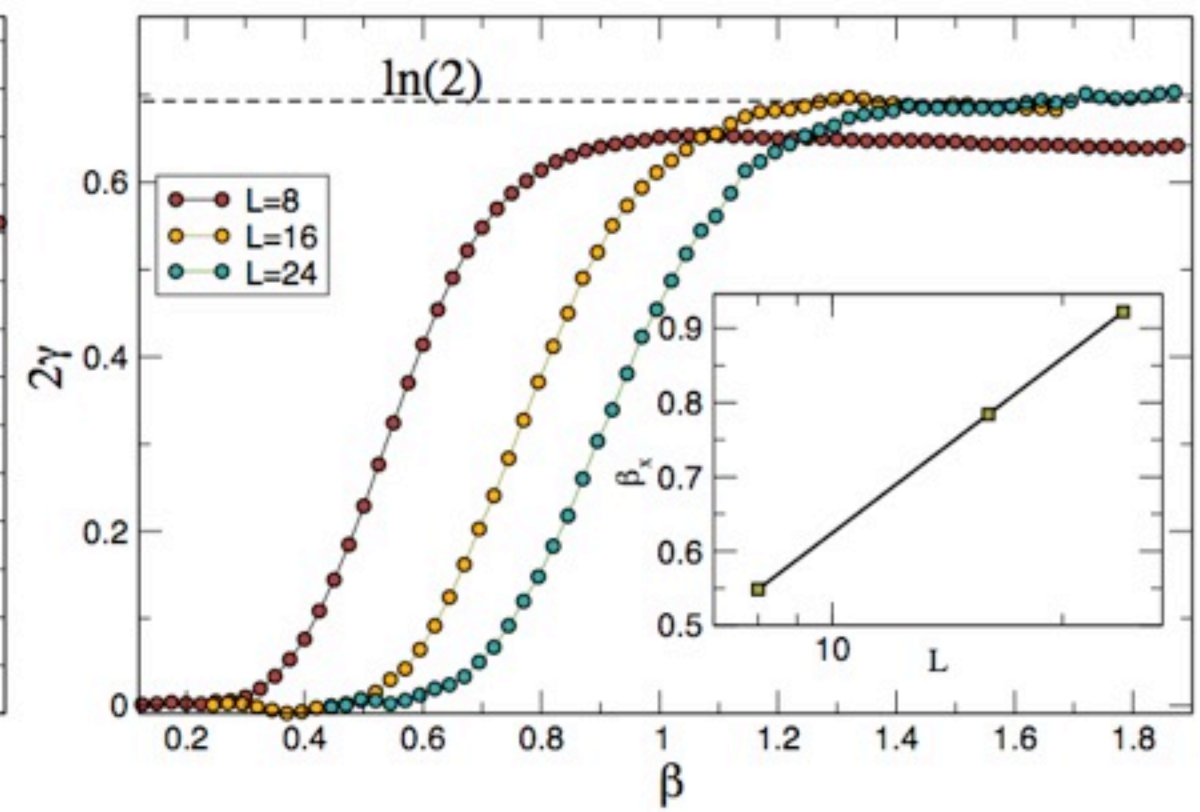
Isakov, Hastings, RGM Nature Physics 7, 772 (2011)

$$H = -t\sum_{\langle ij\rangle}(b_i^\dagger b_j + b_i b_j^\dagger) + V\sum_{\hexagon}(n_{\hexagon})^2$$



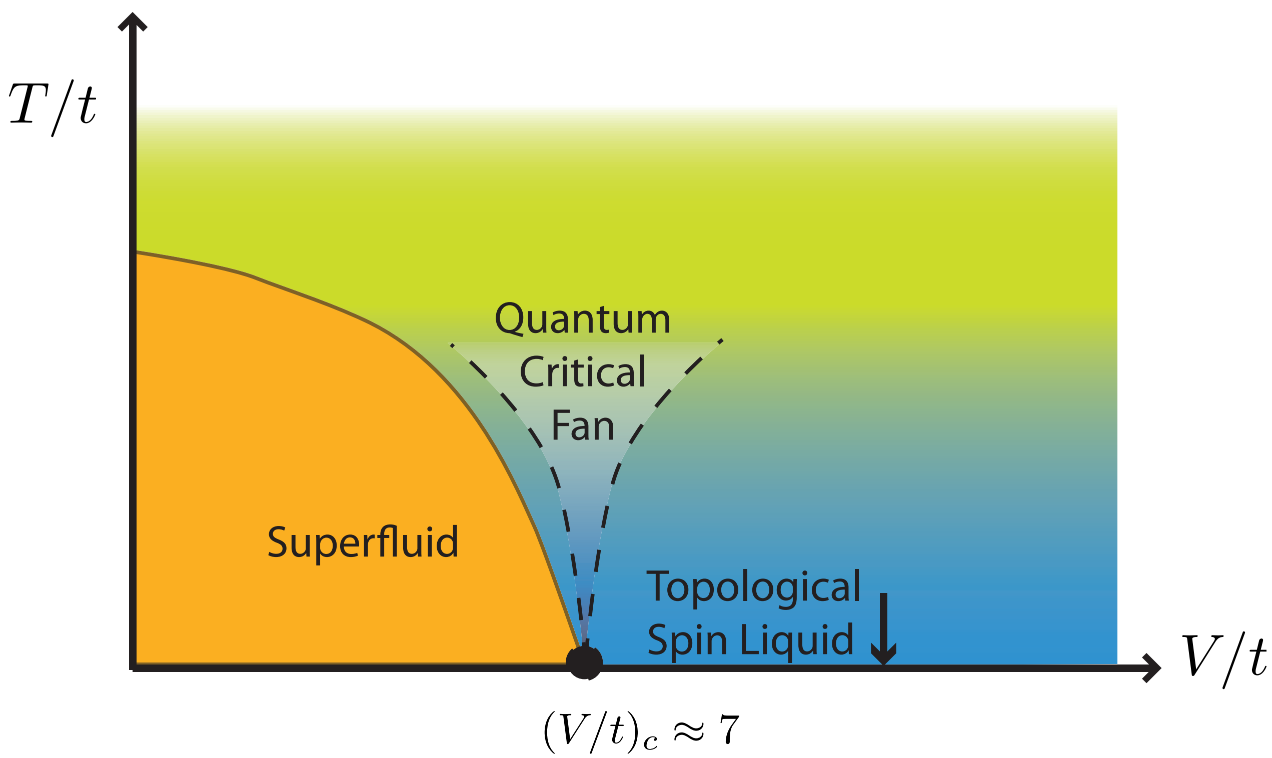


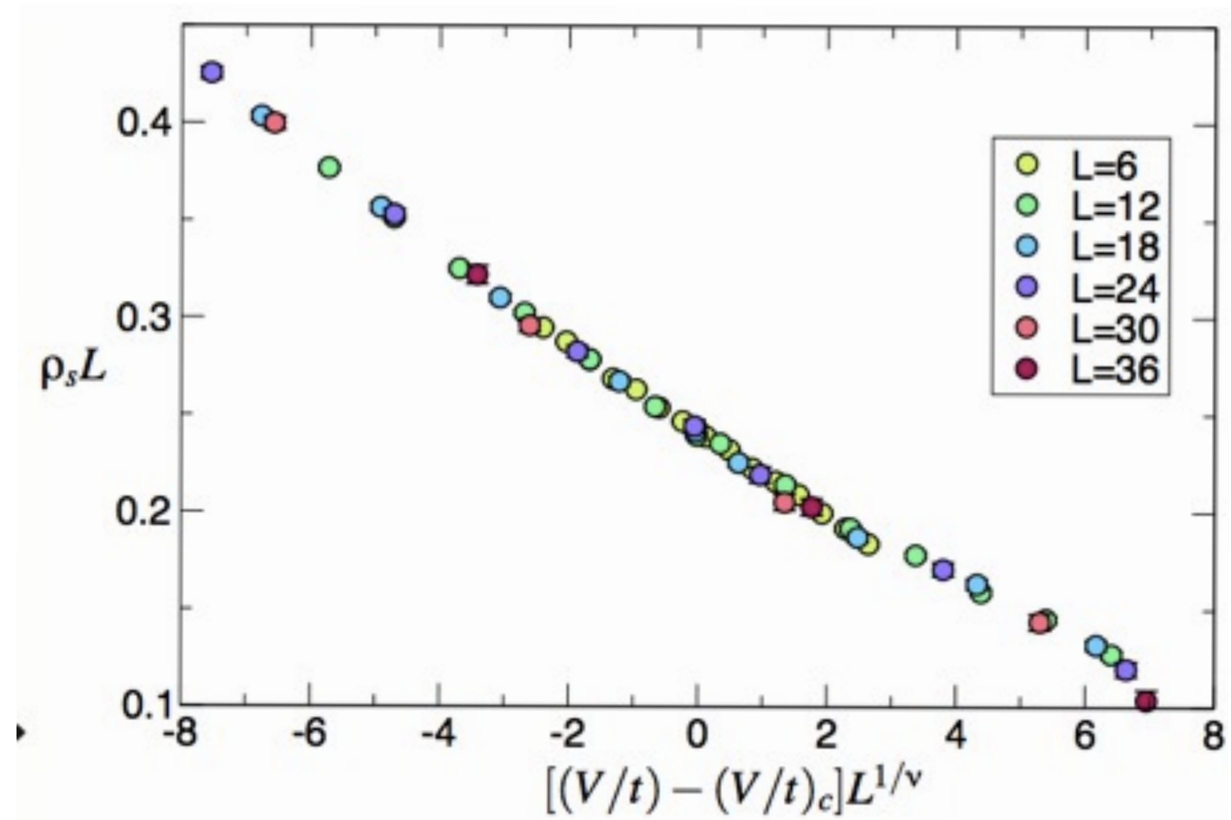




Two crossover temperature are evidence of two excitations out of the “loop gas” groundstate, in analogy with the Toric Code

SUPERFLUID/SPIN LIQUID





- QCP appears to be in the XY universality class at first glance...

$$\nu = 0.6717 \qquad z = 1$$

- But the prediction is that the **fractional charges** (spinons) undergo the XY transition, **not** the physical bosons

 $b = \phi^2$ 

A. V. Chubukov, T. Senthil, S. Sachdev, Phys. Rev. Lett. **72**, 2089 (1994)

A. V. Chubukov, S. Sachdev, T. Senthil, Nuclear Physics B **426**, 601 (1994)

S. V. Isakov, T. Senthil, Y. B. Kim, Phys. Rev. B **72**, 174417 (2005)

T. Grover, T. Senthil, Phys. Rev. B **81**, 205102 (2010)

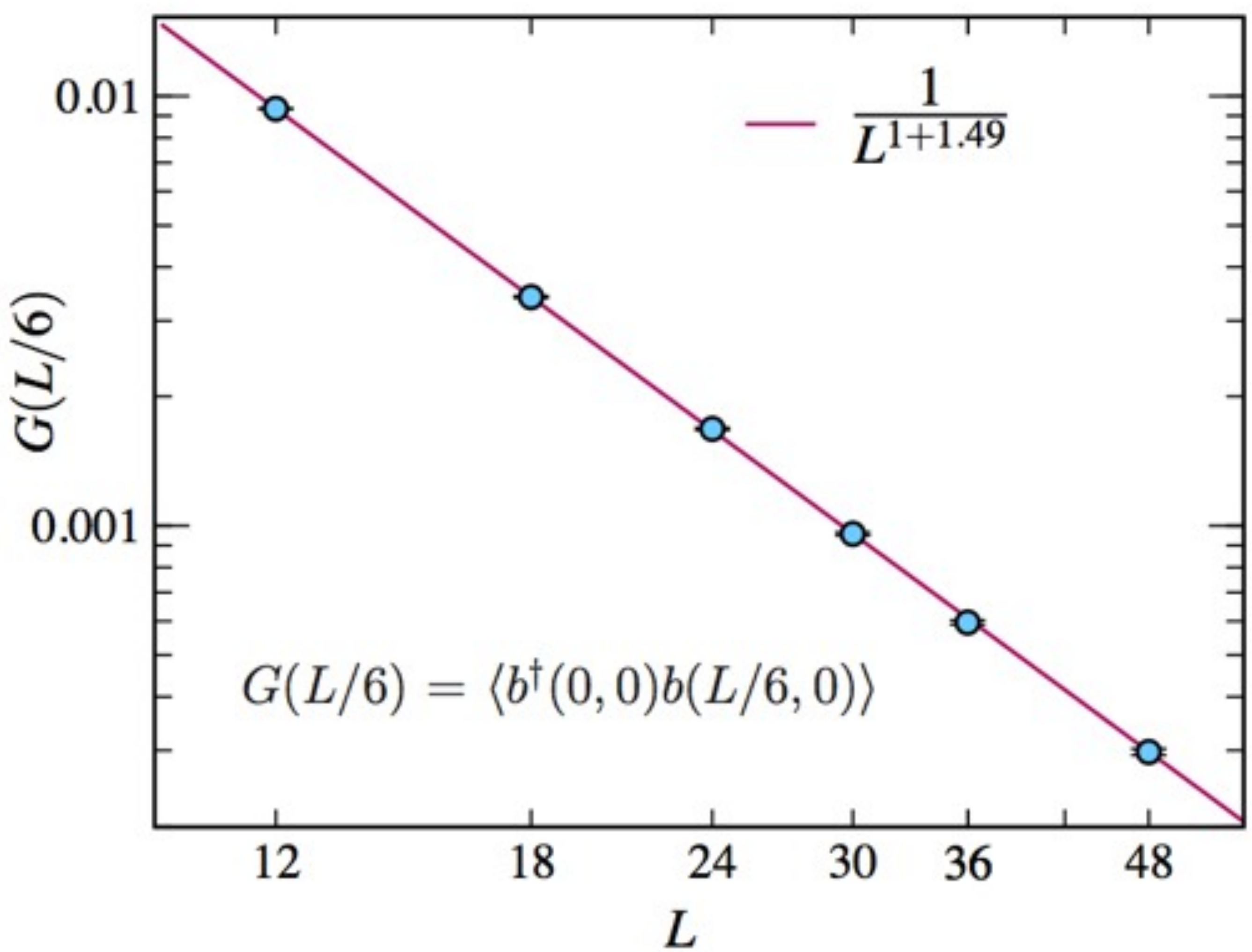
XY* TRANSITION

Critical structure factors of bilinear fields in O(N) vector models
P. Calabrese, A. Pelissetto, and E. Vicari, Phys. Rev. E 65, 046115 (2002).

M. Campostrini, M. Hasenbusch, A. Pelissetto, P. Rossi, and E. Vicari, Phys. Rev. B 63, 214503 (2001).

The most accurate field theory value: $\eta = 1.472(2)$

Isakov, RGM, Hastings Science 335, 193 (2012)



QMC: $\eta = 1.493(5)$

Compare to the 3D XY transition:

$$\eta = 0.03$$

- XY* confirms the presence of fractionalization
- Very clear cut finite-size scaling

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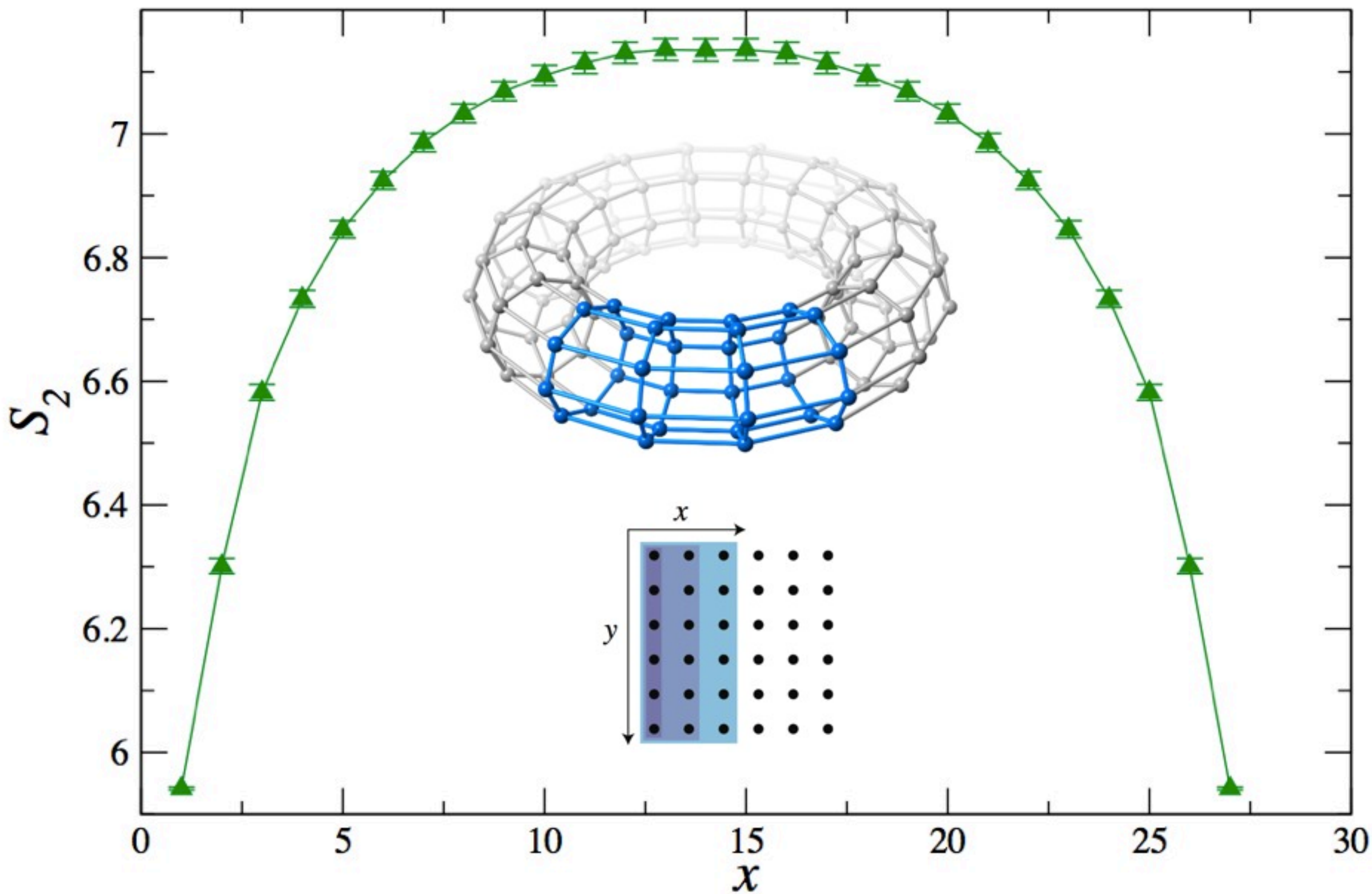
S=1/2 Heisenberg

$$H = J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$

Kallin, Hastings, RGM, Singh, PRB 84, 165134 (2011)



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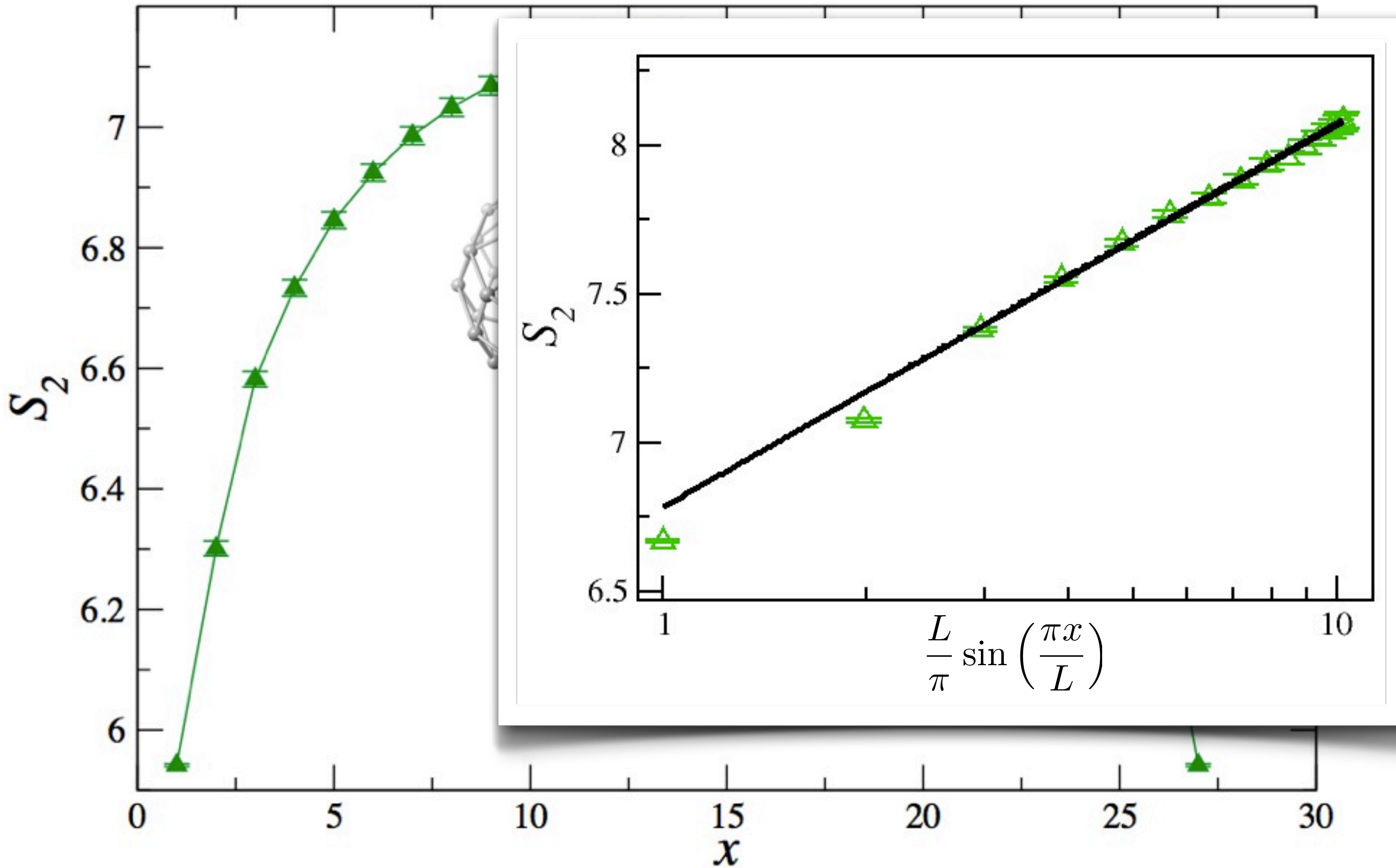
Kallin, Hastings, RGM, Singh, PRB 84, 165134 (2011)



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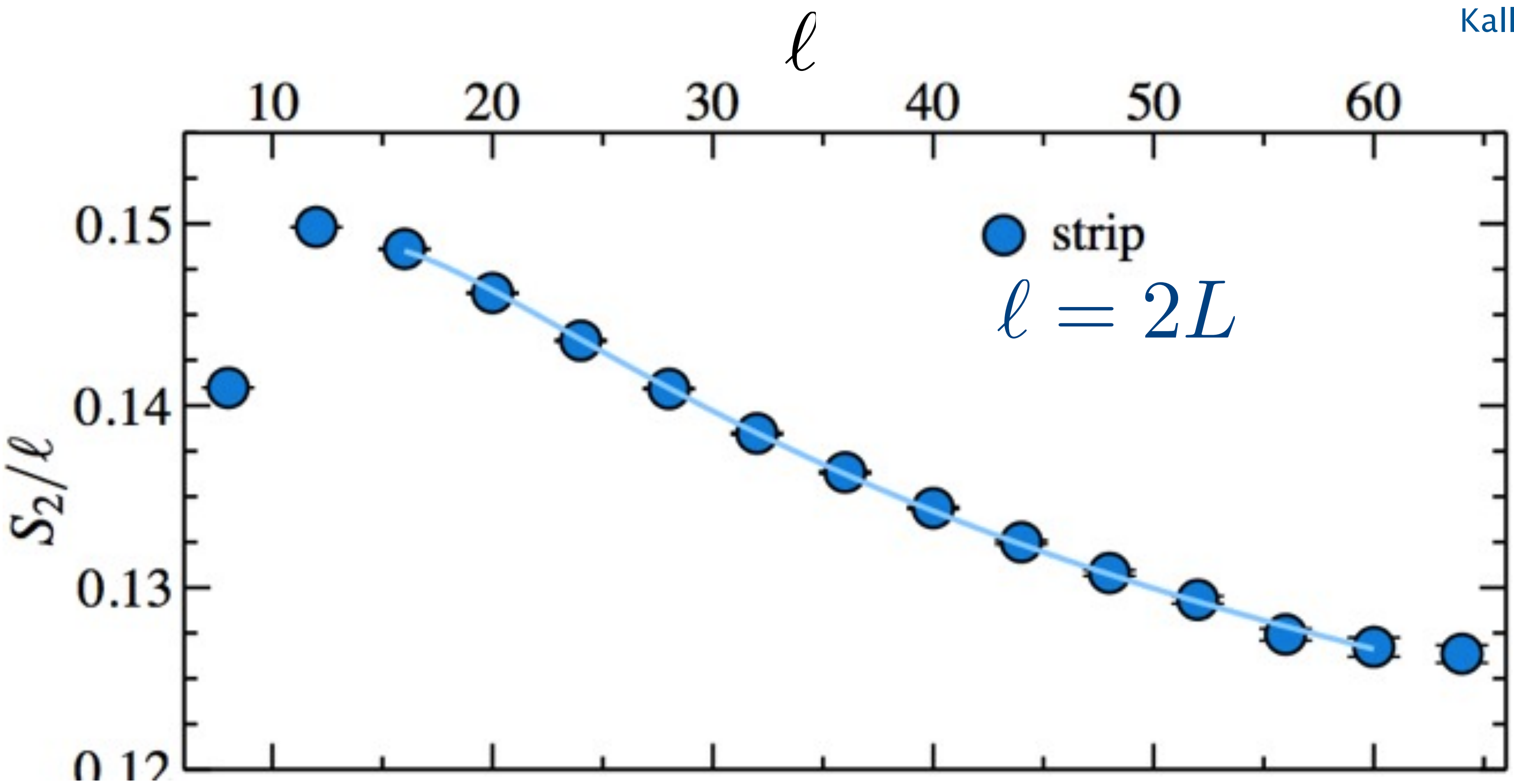
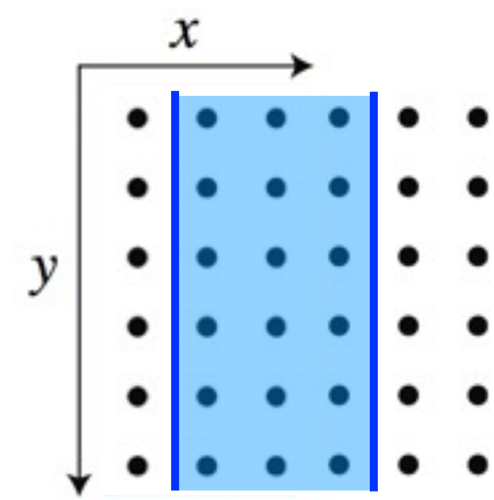
Kallin, Hastings, RGM, Singh, PRB 84, 165134 (2011)





$$a = 0.096$$
$$c = 0.74$$
$$d = -1.2$$

$$S_2 = a\ell + c\ln(\ell) + d$$

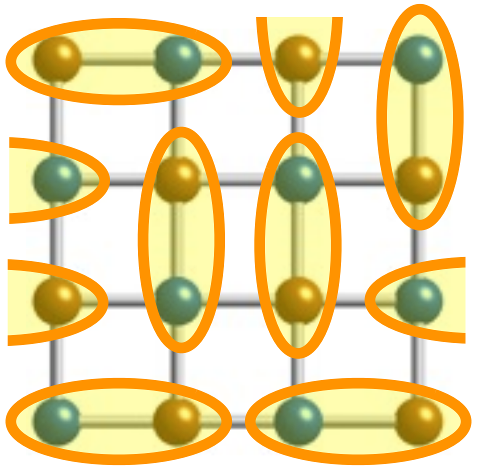
Theory prediction
(Metlitski, Grover arXiv:1112.5166)

$$c = \frac{N_g(d-1)}{2} = 1$$

Song, Laflorencie, Rachel, Le Hur Phys. Rev. B 83, 224410 (2011)

RVB WAVEFUNCTION

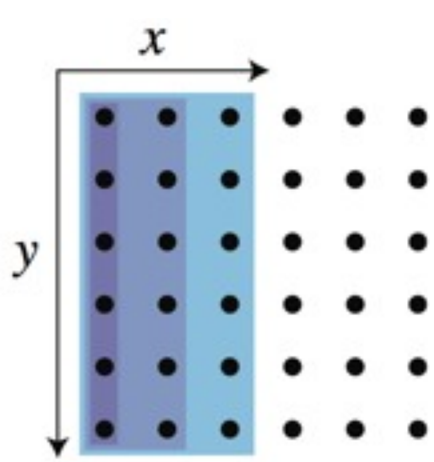
P. W. Anderson, Mat. Res. Bull, 8:153, 1973
Albuquerque, Alet PRB 82, 180408
Tang, Sandvik, Henley arXiv:1010.6146

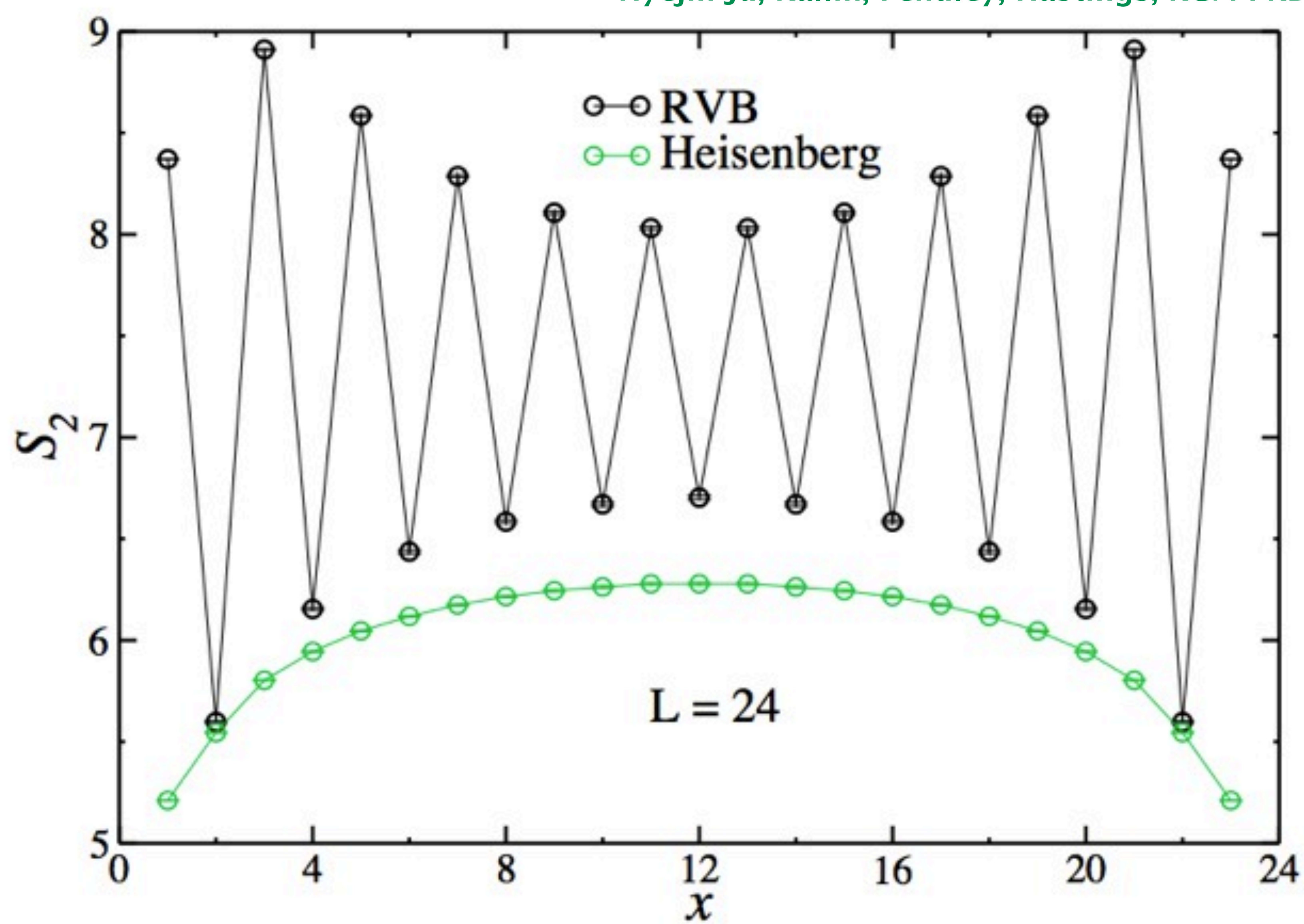


"prototype of the modern QSL"

$$|\Psi\rangle = \sum_{\mathcal{C}} |V_{\mathcal{C}}\rangle \qquad |V_{\mathcal{C}}\rangle = \frac{1}{2^{\mathcal{N}/4}} \prod_{i=1}^{\mathcal{N}/2} \left(|\uparrow_i \downarrow_{j_i}\rangle - |\downarrow_i \uparrow_{j_i}\rangle \right)$$

Hyejin Ju, Kallin, Fendley, Hastings, RGM PRB 85, 165121 (2012)

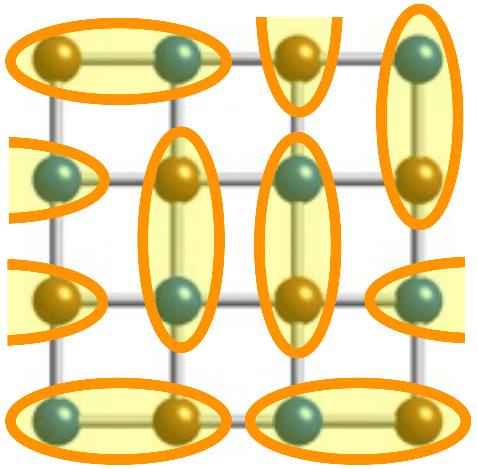




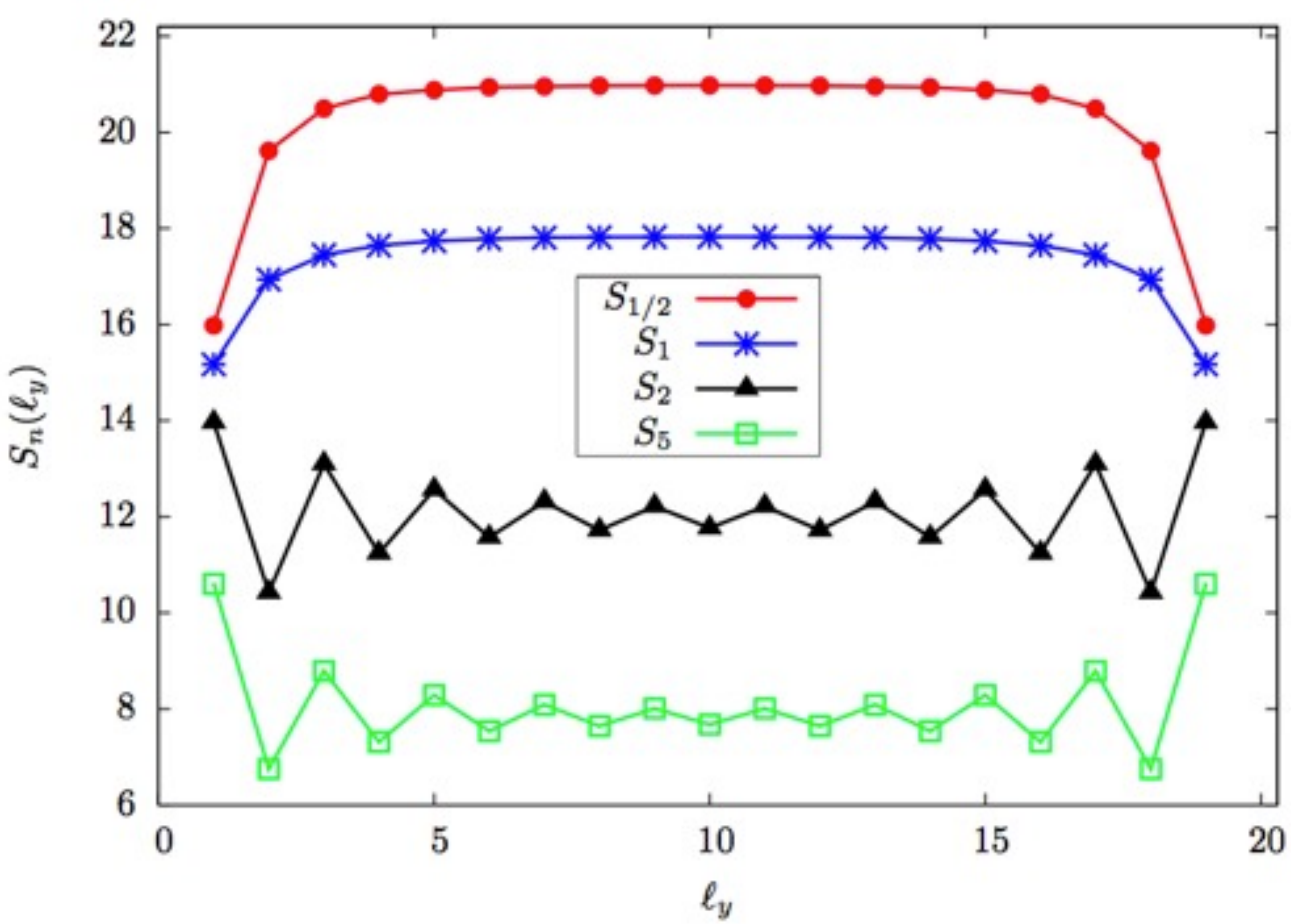
$\mathrm{SU}(N)$ versions possible – H. Ju

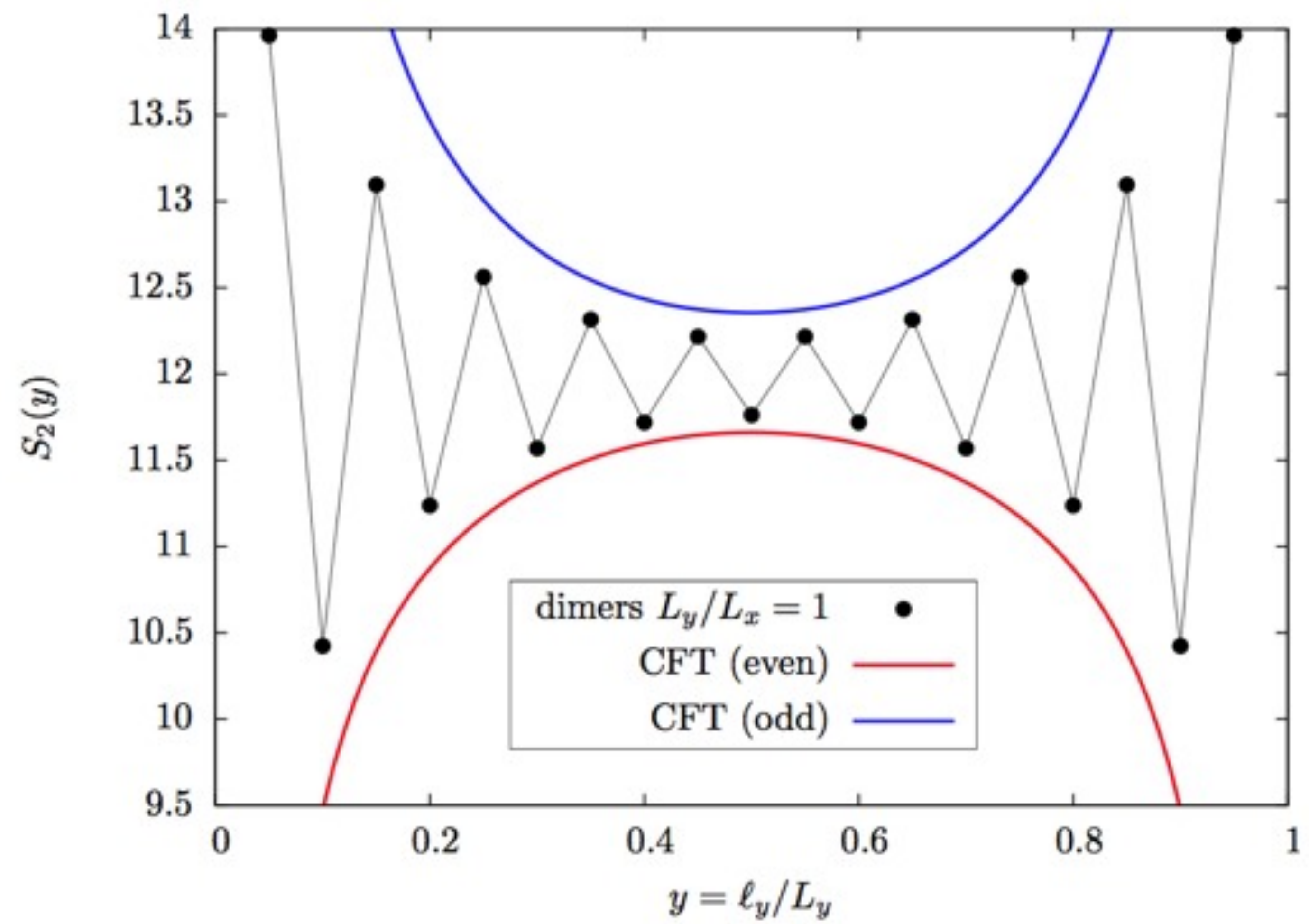
Dimer wavefunction

Stephan, Ju, Fendley, RGM arXiv:1207.3820



$$|\Psi_D\rangle = \sum_{\mathcal{C}} |D_{\mathcal{C}}\rangle \qquad \langle D_{\mathcal{C}}|D_{\mathcal{C}'}\rangle = \delta_{\mathcal{C},\mathcal{C}'}$$





- A critical Renyi index exists for the even/odd branching effect
- It persists to the infinite-size limit $s_n^{(\mathrm{even})}(y,\tau) = \frac{n}{1-n}\ln\left(\frac{\eta(\tau)^2}{\theta_3(2\tau)\theta_3(\tau/2)} \times \frac{\theta_3(2y\tau)\theta_3(2(1-y)\tau)}{\eta(2y\tau)\eta(2(1-y)\tau)}\right)$

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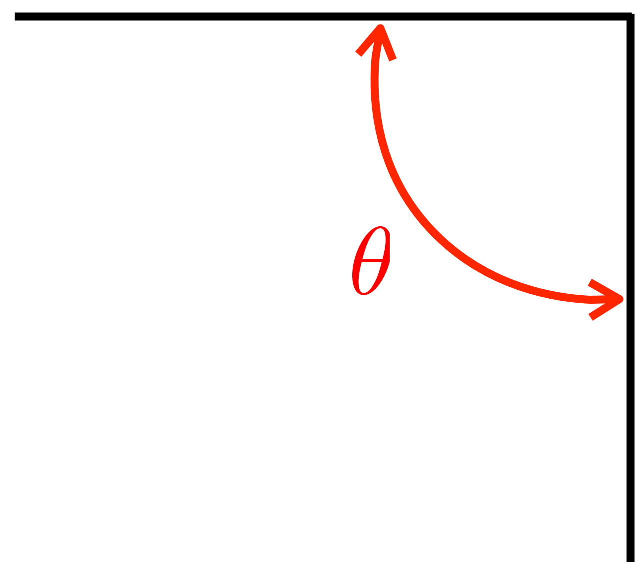
In the case of a single non-interacting corner, the shape-dependence contains a simple additive **logarithm**



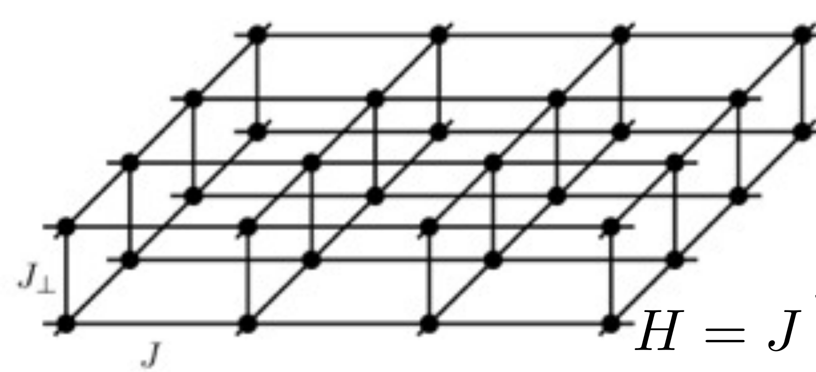
$$S_n = a\ell + c_n(\theta)\ln(\ell) + \cdots$$

"universal"

H. Casini and M. Huerta, Nucl. Phys. B 764, 183 (2007).
D. V. Fursaev, Phys. Rev. D 73, 124025 (2006).
S. Ryu and T. Takayanagi, Phys. Rev. Lett. 96, 181602 (2006).

Can compare these universal coefficients between models and field theory:

$$H = -J\sum_{\langle i,j\rangle}\sigma_i^z\sigma_j^z - h\sum_i \sigma_i^x$$

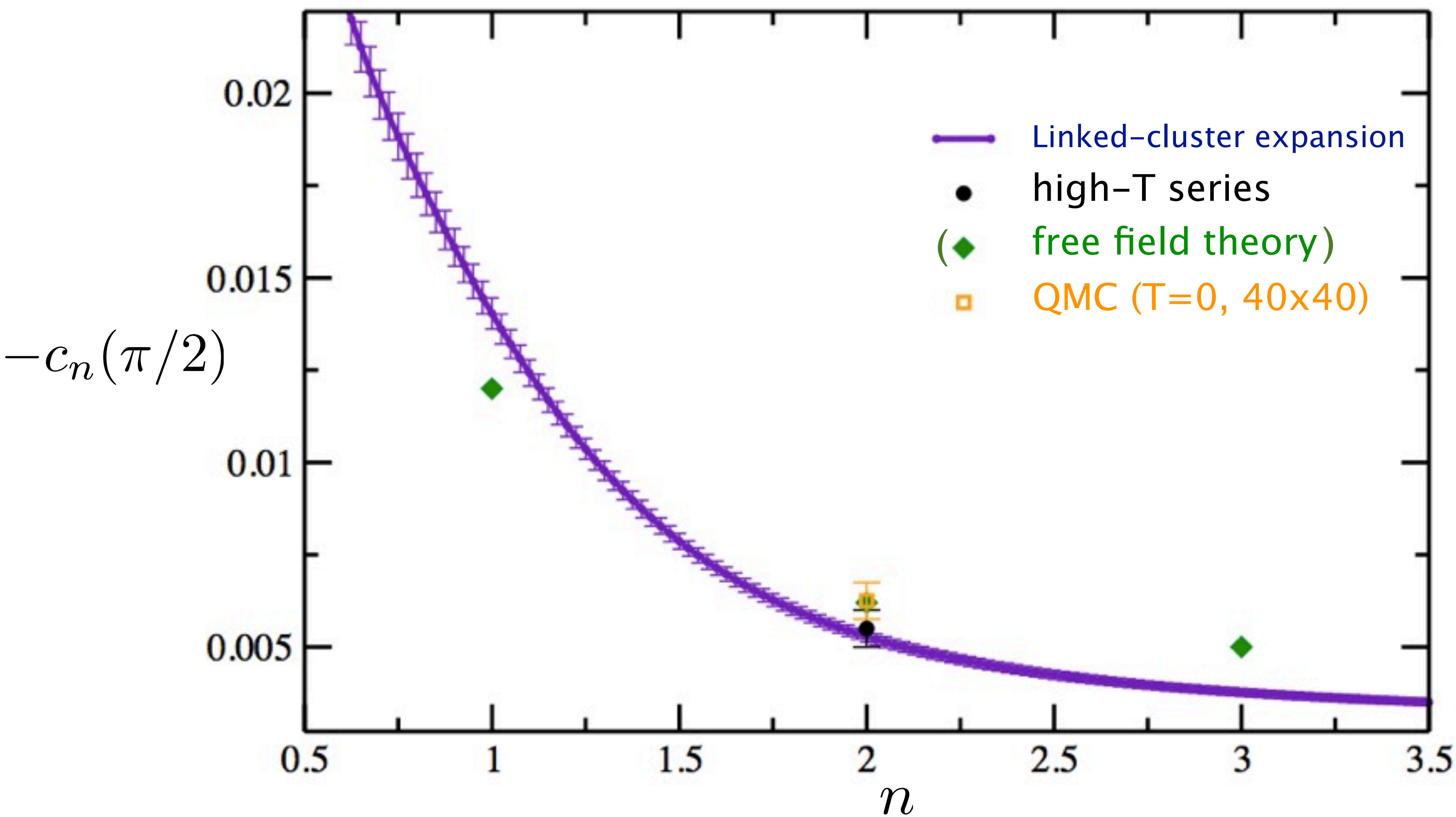


$$H = J\sum_{\langle ij\rangle}\mathbf{S}_{1i}\cdot\mathbf{S}_{1j} + J_\perp\sum_{\langle ij\rangle}\mathbf{S}_{1i}\cdot\mathbf{S}_{2j} +$$

Kallin, Hyatt, Singh, RGM (unpublished)

two-dimensional TFIM at $h/J = 3.044$

$$H = -J\sum_{\langle i,j\rangle}\sigma_i^z\sigma_j^z - h\sum_i \sigma_i^x$$



DISCUSSION:

- Murphy’s law? Many counterexamples

- Renyi entropies can be measured by QMC

- Scaling of entanglement entropy is a practical QMC tool to detect topological order in gapped spin liquids

- Finite-T behavior gives us insight into the nature of excitations

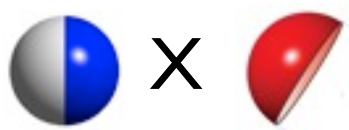
- Replica trick works in classical Monte Carlo: can measure “entanglement” (information) in loop models

To Do:

- Study BFG models in DMRG: benchmark finite-size scaling and boundary effects

- Develop entanglement entropy indicators for gapless spin liquids?

- Exotic quantum critical points driven by vison (or fermion?)



- Use universal scaling terms in entanglement entropy to categorize conventional universality classes

- Use universal scaling terms to detect fractionalization at deconfined quantum critical points

  – Swingle and Senthil, arXiv:1109.3185

$$C_{XY*} = C_{XY} + \ln(2)$$