Quantum loops and topological order

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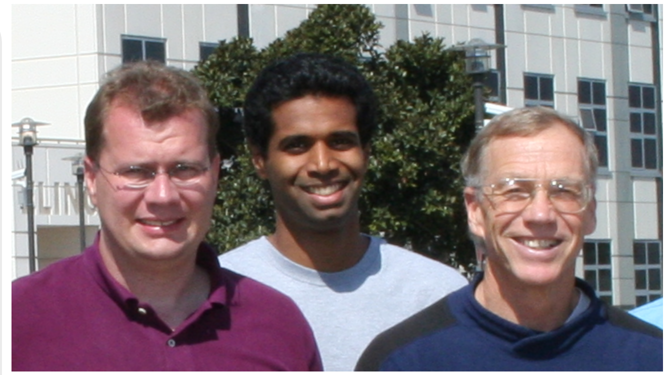
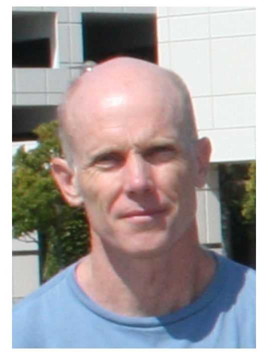
Collaborators

- Michael Freedman — Station Q
- Chetan Nayak — Station Q
- Zhenghan Wang — Station Q
- Kevin Walker — Station Q
- Simon Trebst* — Station Q
- Matthias Troyer — ETH Zürich

* Special thanks for many of these slides





Freedman, Nayak, KS, Walker, Wang, Ann. Phys. 2003
Freedman, Nayak & KS, PRL 2005; PRL 2006; PRB 2008
Troyer, Trebst, KS & Nayak, PRL 2008

Some history: Early 2000s...

Dr. Michael Freedman
Microsoft Research
One Microsoft Way
Redmond WA 98052
8 November 2000

Dear Dr. Freedman
Thank you for submitting your paper 1056860 ***"Topological Quantum Computation"*** to **SCIENCE**. The submission was considered by the Board of Reviewing Editors and by several members of the editorial staff. ***I am sorry to say that the manuscript has not been assigned sufficient priority during this initial screening to warrant sending it out for in depth peer review.***
As you are aware, we receive many more papers that we can accept, and many of them are in competitive fields. We therefore send for in-depth review only those papers most likely to be ultimately published in SCIENCE. Therefore, our decision is not necessarily a reflection of the quality of your research, but rather of our stringent space limitations.
Because most decisions about manuscripts submitted to SCIENCE are based on relative quality rather than absolute merit, we do not consider resubmitted manuscripts.
We wish you every success when you submit the paper elsewhere.

Yours sincerely
Dr. Ian Osborne
Associate Editor

Some history: Early 2000s...

PHYSICAL REVIEW B, VOLUME 64, 064422

Microscopic models of two-dimensional magnets with fractionalized excitations

Chetan Nayak and Kirill Shtengel*

Department of Physics and Astronomy, University of California at Los Angeles, Los Angeles, California 90095-1547

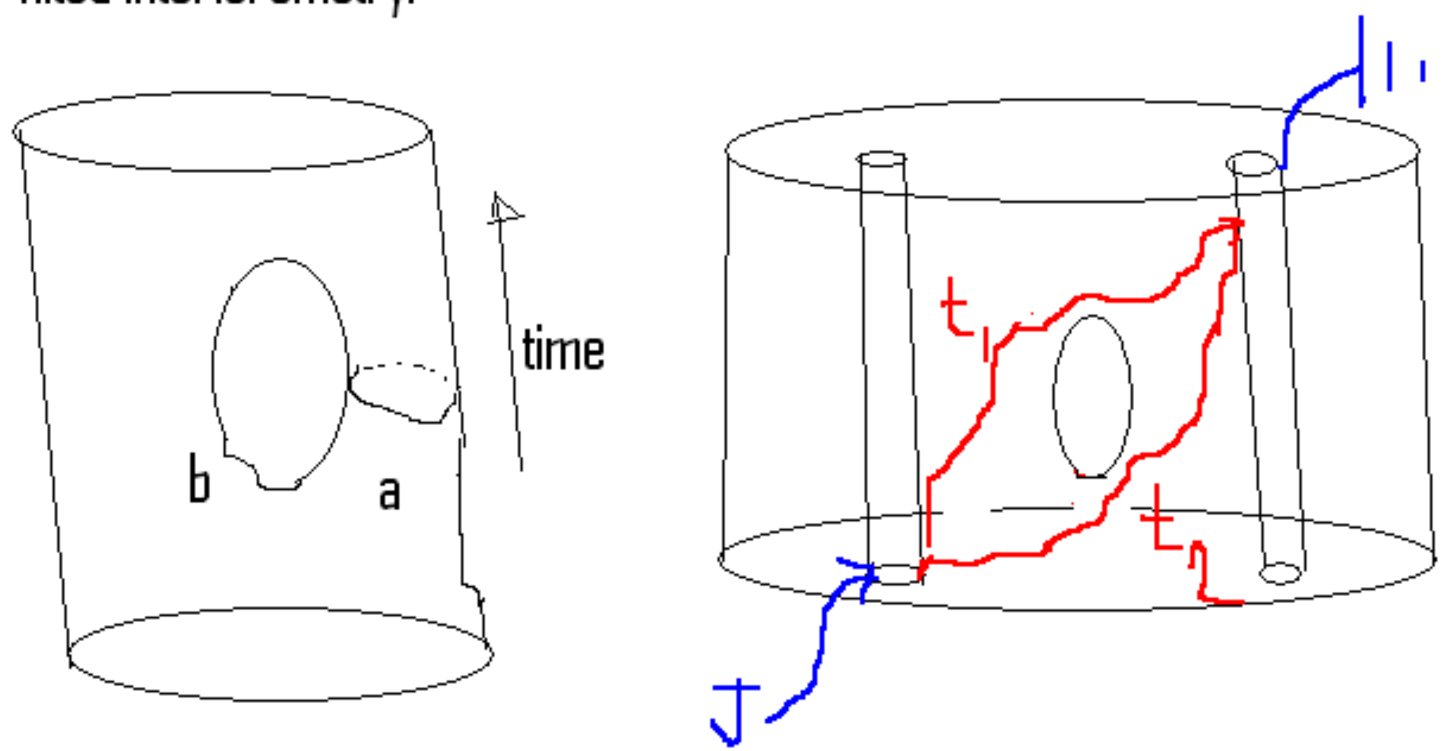
(Received 21 December 2000; published 24 July 2001)

We demonstrate that spin-charge separation can occur in two dimensions and note its confluence with superconductivity, topology, gauge theory, and fault-tolerant quantum computation. We construct a microscopic Ising-like model and, at a special coupling constant value, find its exact ground state as well as neutral spin-$\frac{1}{2}$ (spinon), spinless charge e (holon), and Z_2 vortex (vison) states and energies. The fractionalized excitations reflect the topological order of the ground state which is evinced by its fourfold degeneracy on the torus—a degeneracy which is unrelated to translational or rotational symmetry—and is described by a Z_2 gauge theory. A magnetic moment coexists with the topological order. Our model is a member of a family of topologically ordered models, one of which is integrable and realizes the toric quantum error correction code but does not conserve any component of the spin. We relate our model to a dimer model which could be a spin SU(2) symmetric realization of topological order and its concomitant quantum number fractionalization.

Some history: 2002...

10

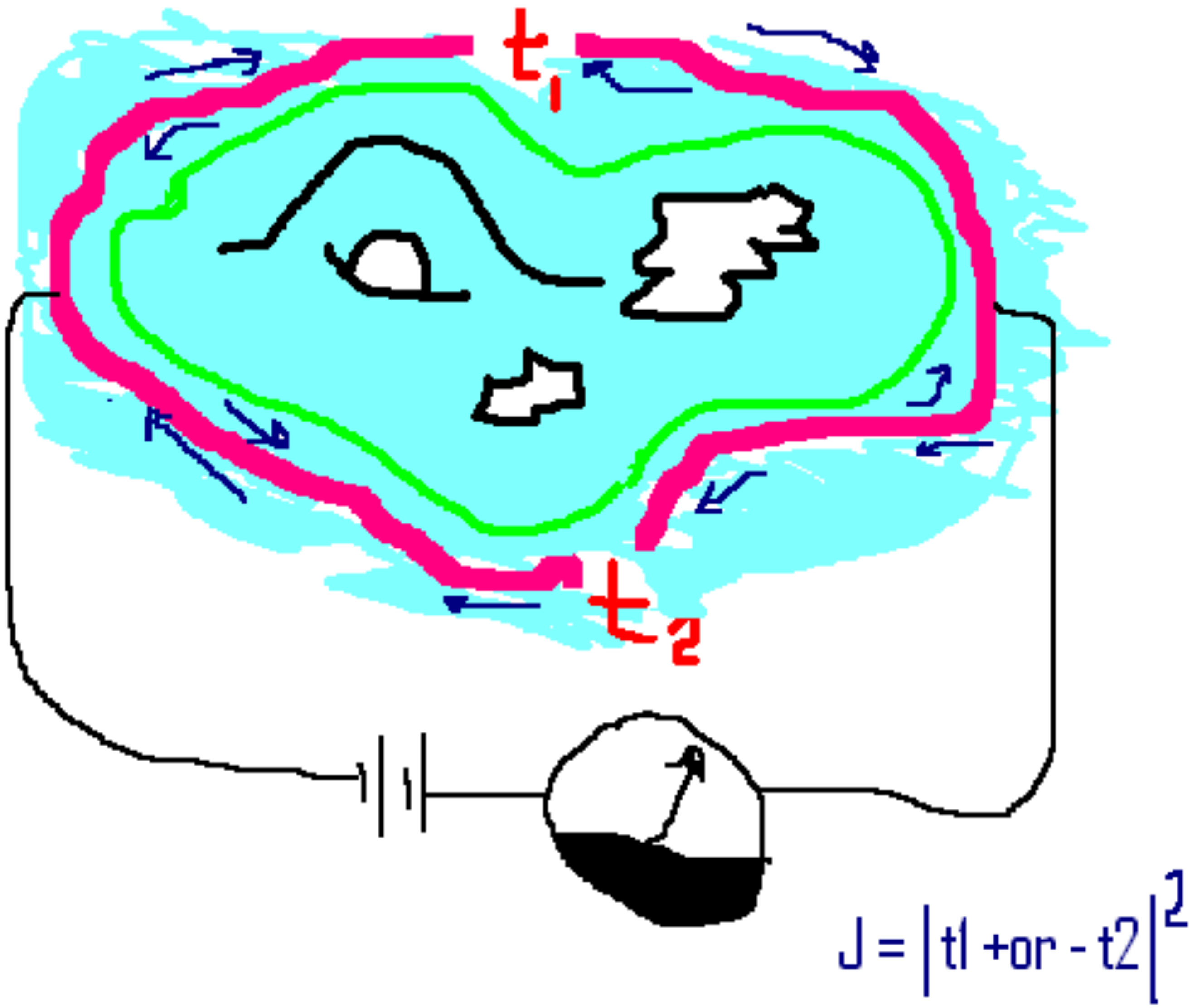
Tilted interferometry:



Find charge on b

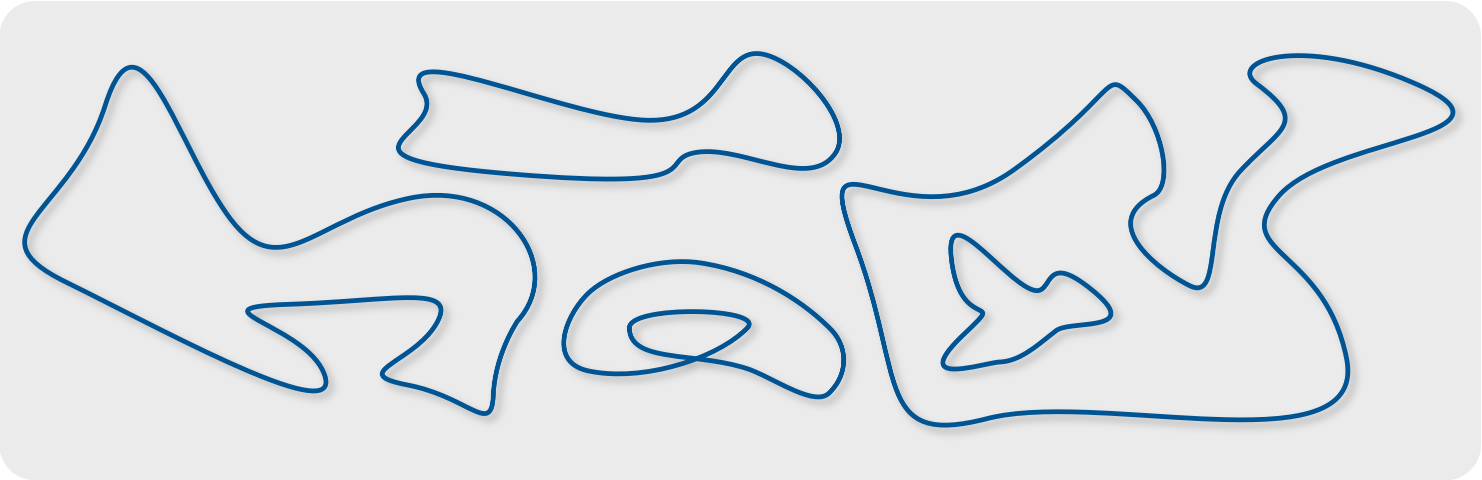
An actual slide from Mike's talk at "Emergent Phenomena in Quantum Hall Systems", Taos 2005

Or topologically the same thing



Quantum loop gases

The “Ising models” of topological phases

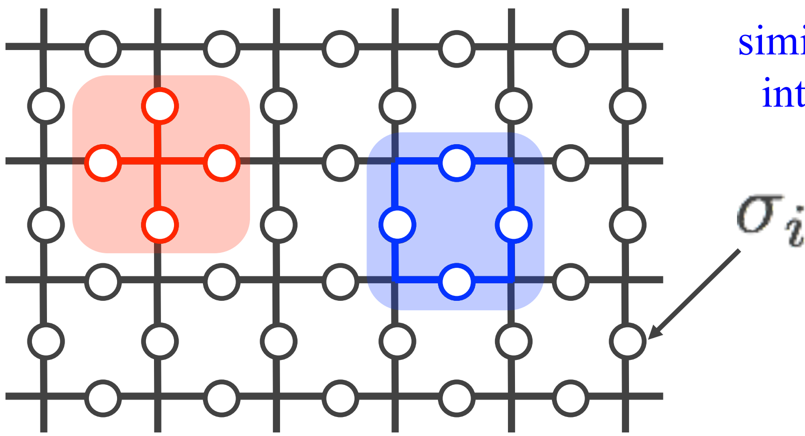


The toric code: the simplest loop gas

A. Kitaev, Ann. Phys. **303**, 2 (2003).

$$H_{\mathrm{TC}} = -A \sum_{v} \prod_{j \in \mathrm{vertex}(v)} \sigma_j^z - B \sum_{p} \prod_{j \in \mathrm{plaquette}(p)} \sigma_j^x$$

similar to ring exchange introduces frustration



Hamiltonian has only local terms.

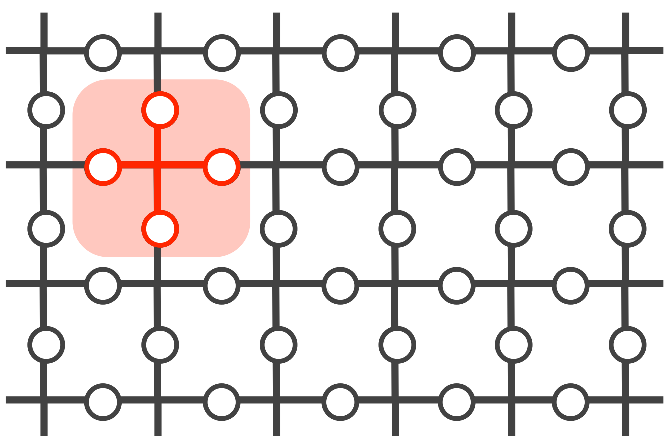
All terms commute → **exact solution!**

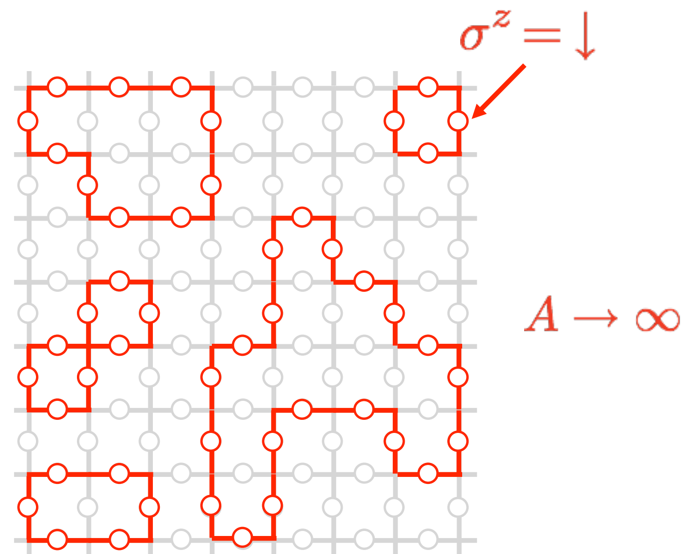
The vertex term

A. Kitaev, Ann. Phys. **303**, 2 (2003).

$$H_{\mathrm{TC}} = -A\sum_v \prod_{j\in\mathrm{vertex}(v)} \sigma_j^z - B\sum_p \prod_{j\in\mathrm{plaquette}(p)} \sigma_j^x$$

- is minimized by an **even** number of down-spins around a vertex.
- Replacing down-spins by line segments maps ground state to closed loops.
- Open ends are (charge) excitations costing energy $2A$.



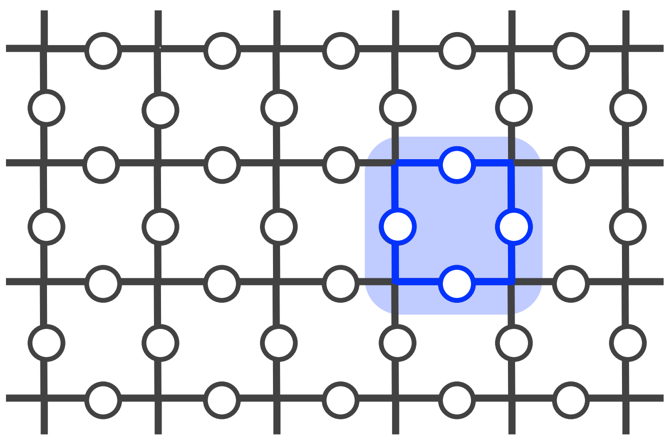


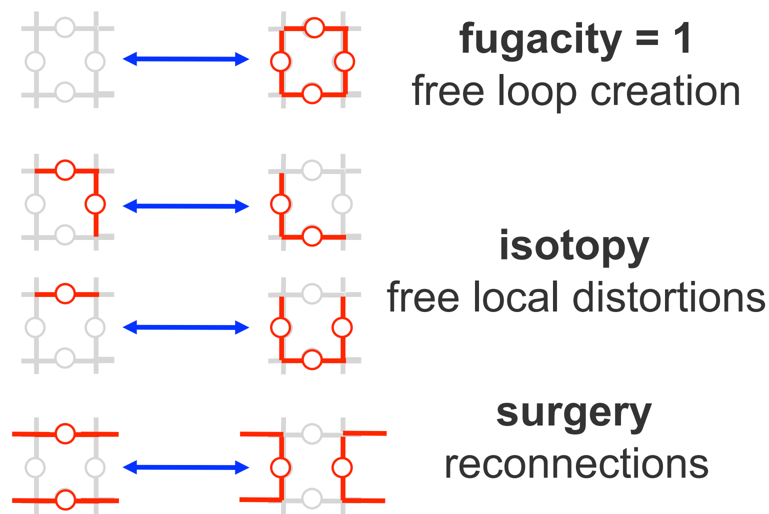
The plaquette term

A. Kitaev, Ann. Phys. **303**, 2 (2003).

$$H_{\mathrm{TC}} = -A \sum_{v} \prod_{j \in \mathrm{vertex}(v)} \sigma_j^z - B \sum_{p} \prod_{j \in \mathrm{plaquette}(p)} \sigma_j^x$$

- flips all spins on a plaquette.
- favors equal amplitude superposition of all loop configurations.
- Sign changes upon flip (vortices) cost energy $2B$.

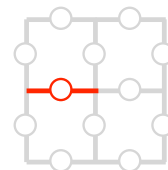




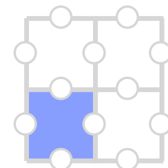
Anyons in the toric code

A. Kitaev, Ann. Phys. **303**, 2 (2003).

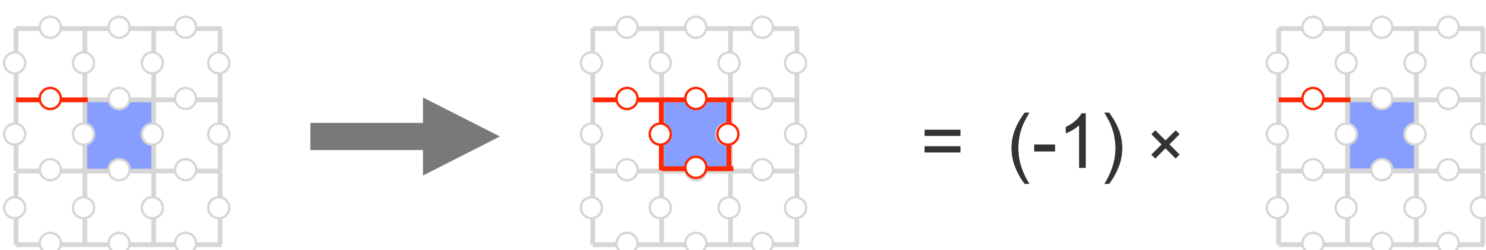
- Two types of excitations:
  - **electric charges**: open loop ends violate vertex constraint



  - **magnetic vortices**: plaquettes giving -1 when flipped



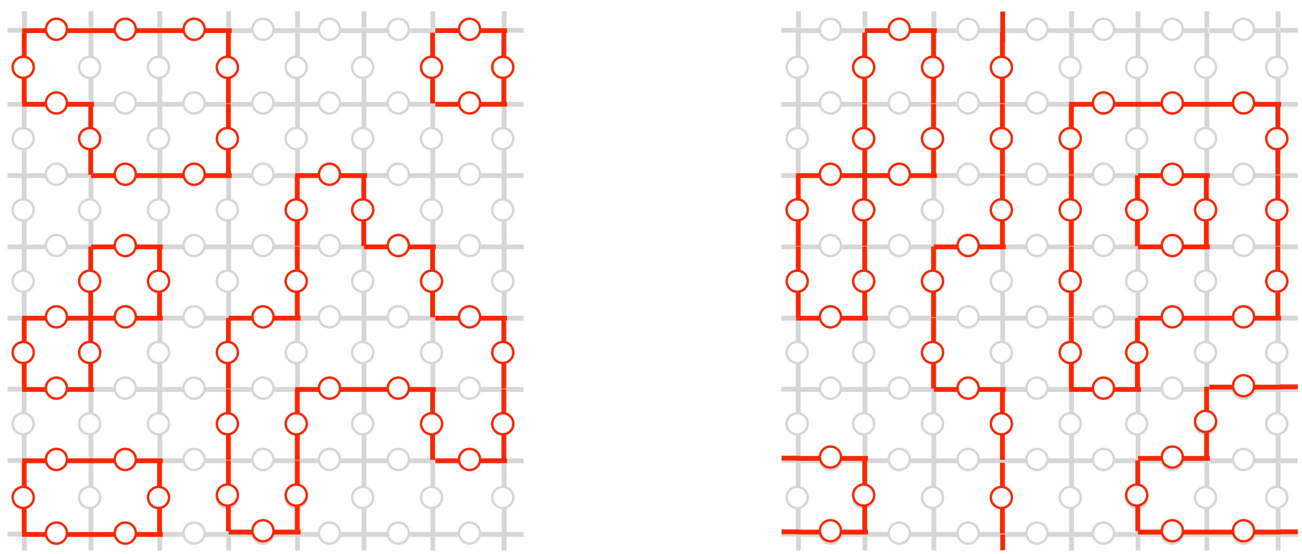
We get a minus sign taking one around the other:
electric charges and magnetic vortices are **mutual anyons**



Toric code & quantum loop gases

A. Kitaev, Ann. Phys. **303**, 2 (2003).

Ground-state manifold is a **quantum loop gas**.

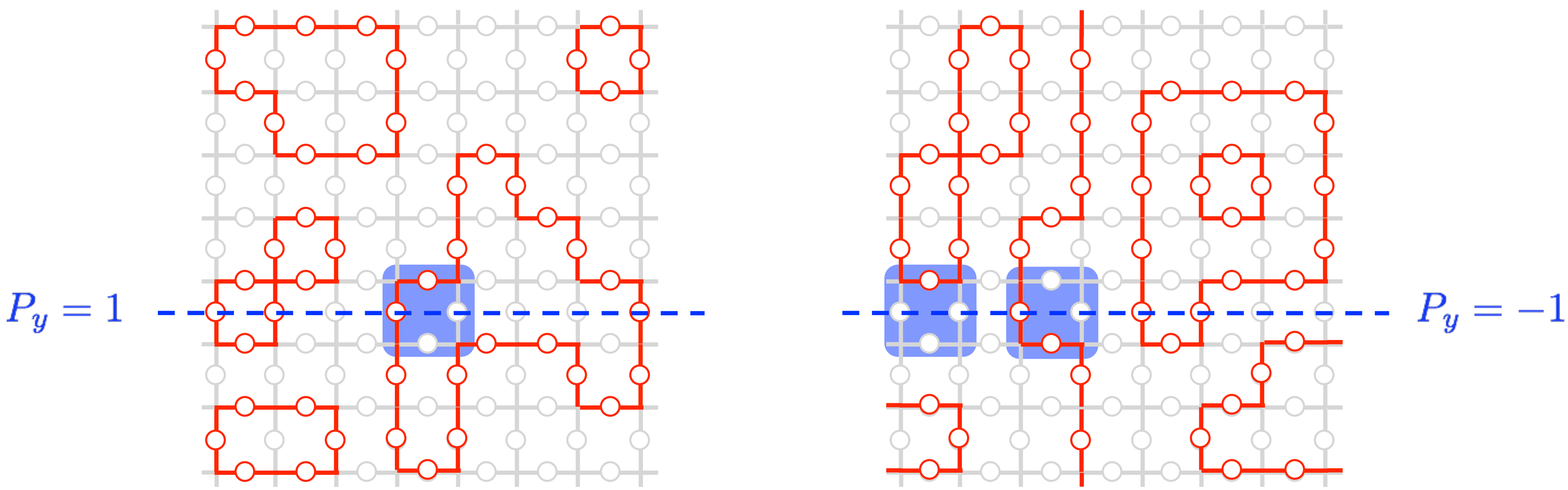


Ground-state wavefunction is **equal superposition** of loop configurations.

The toric code

A. Kitaev, Ann. Phys. **303**, 2 (2003).

Ground-state manifold is a **quantum loop gas**.

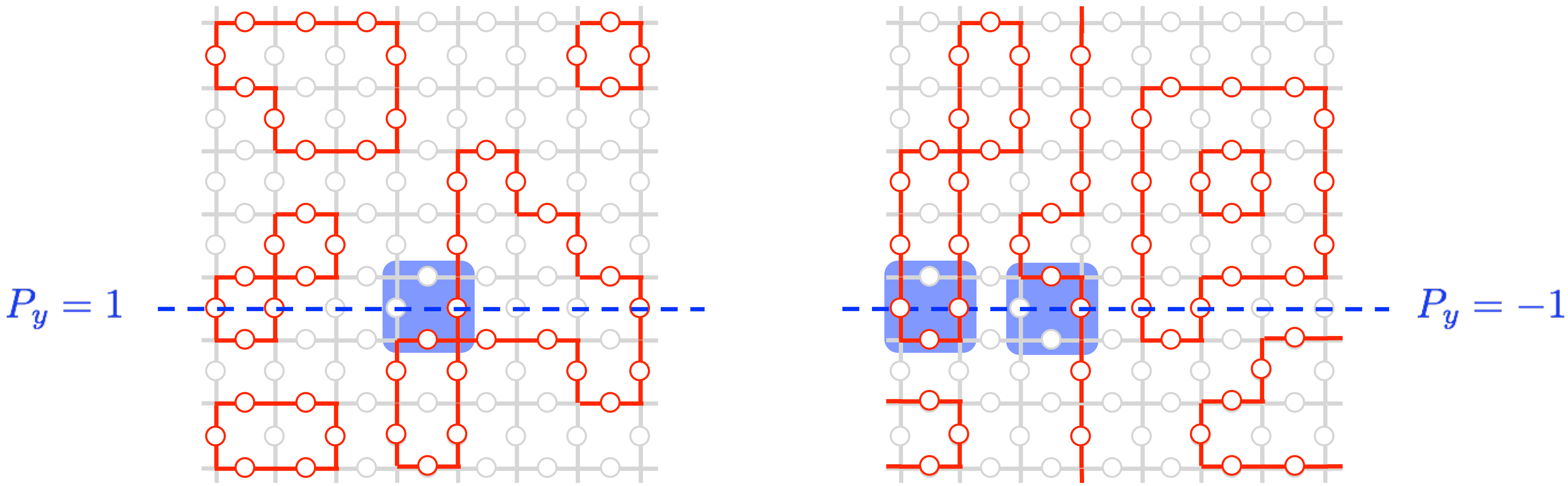


Topological sectors defined by winding number parity $P_{y/x} = \prod_{i \in c_{x/y}} \sigma_i^z$

The toric code

A. Kitaev, Ann. Phys. **303**, 2 (2003).

Ground-state manifold is a **quantum loop gas**.



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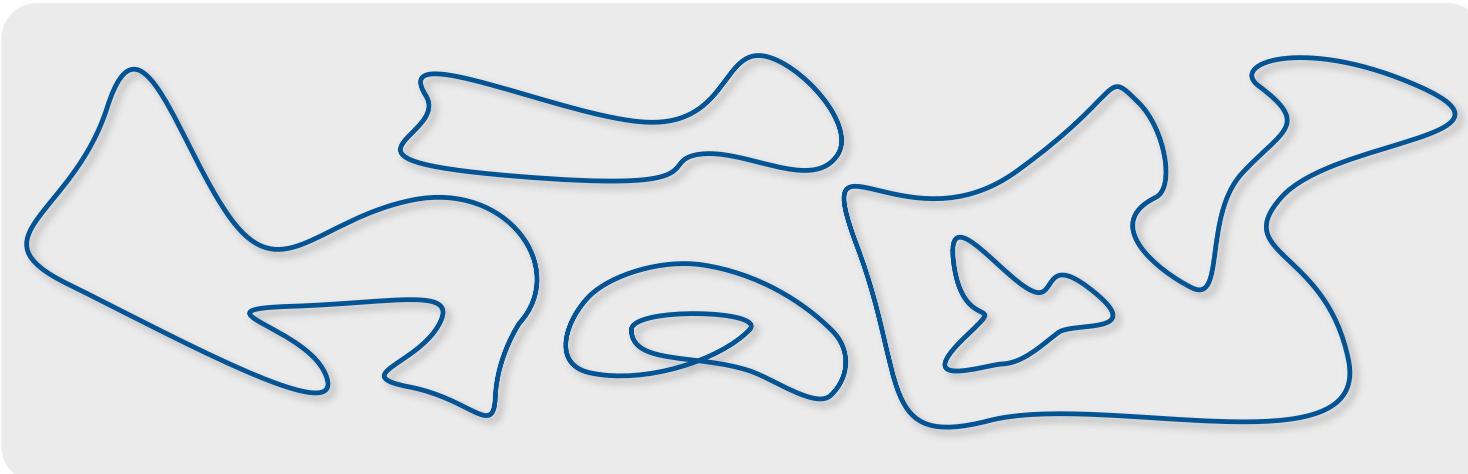
Quantum loop gases

The "Ising models" of topological phases

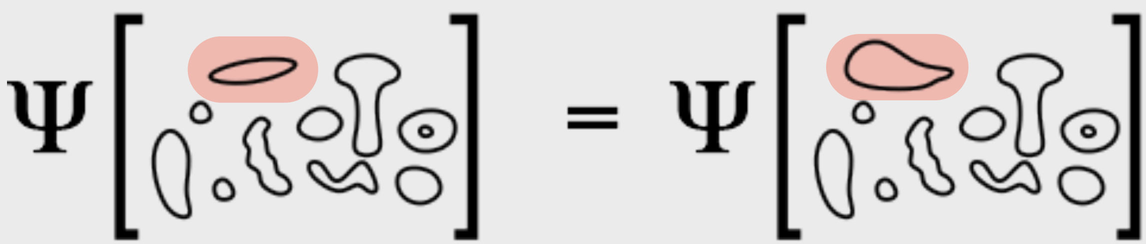


Three conditions:

1. *Isotopy*
2. *Loop fugacity*
3. *Surgery*

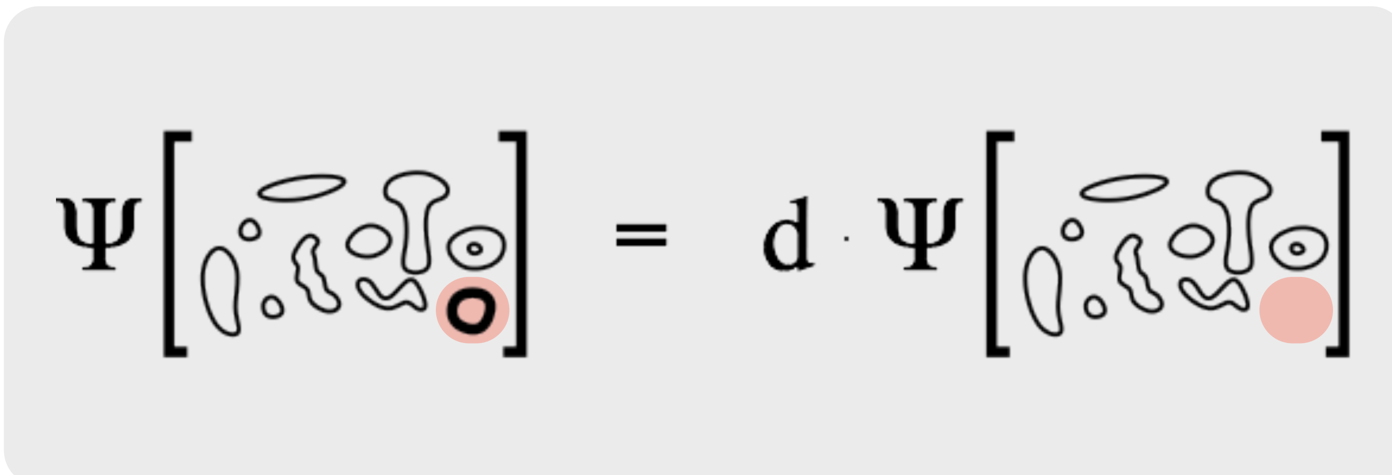
Isotopy

- Invariance under smooth deformations (*isotopy*).



Loop fugacity

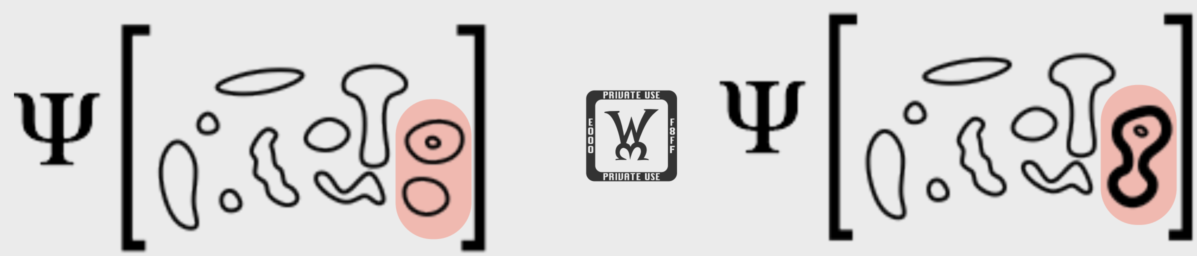
- A ‘fugacity’ for small, contractible loops



$$\left|\Psi_0^{(d)}\right\rangle \propto \sum_{\{\mathcal{L}\}} d^{\ell(\mathcal{L})} \left|\mathcal{L}\right\rangle$$

Surgery

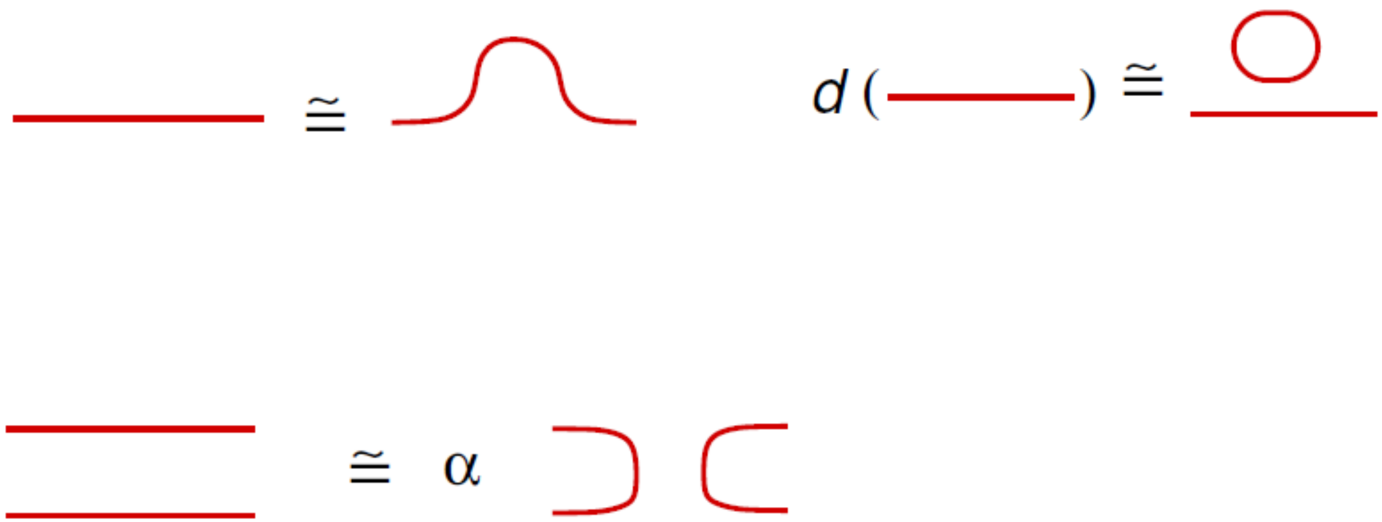
- ‘Surgery’ relation:



Without a surgery relation we have an infinite ground-state degeneracy (all winding numbers) on the torus.

Surgery

- Consistency between d-isotopy & ‘surgery’:



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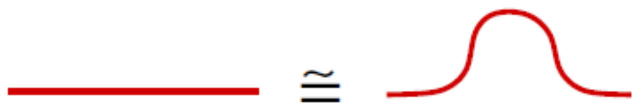

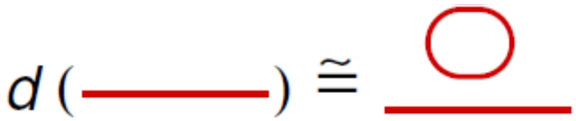

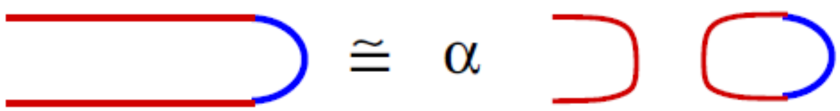

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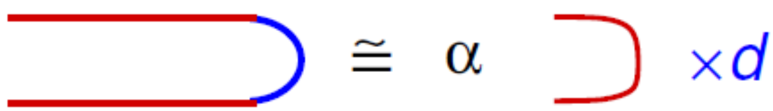
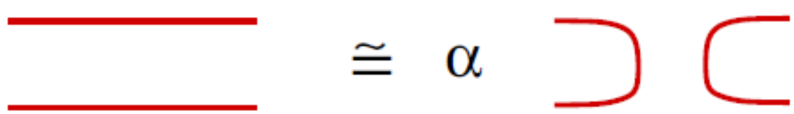


$$1 = \alpha d$$

Surgery

- Consistency between d-isotopy & ‘surgery’:



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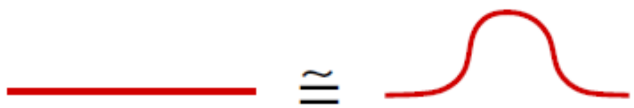
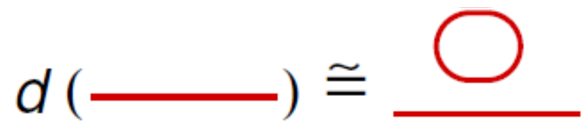
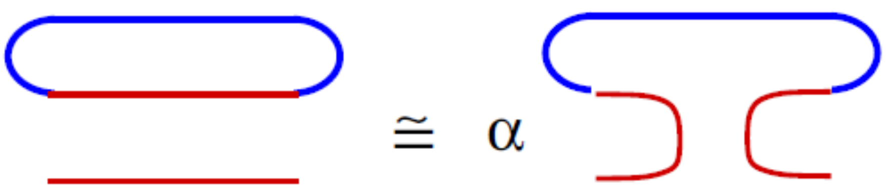


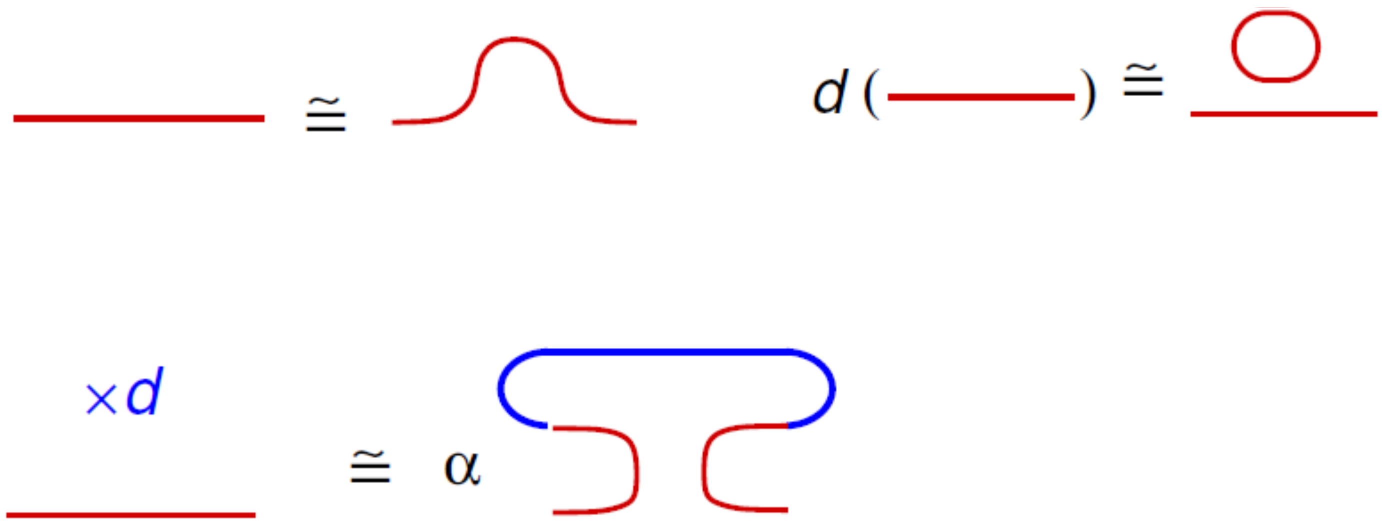




$$\mathbf{1 = \alpha\, d}$$

Surgery

- Consistency between d-isotopy & ‘surgery’:

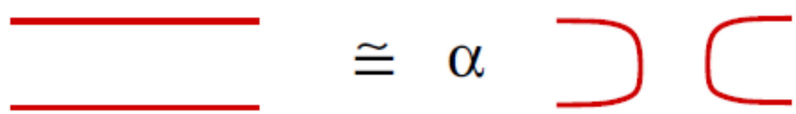


$$\mathbf{1 = \alpha\, d}$$

$$\mathbf{d = \alpha}$$

Surgery

- Consistency between d-isotopy & ‘surgery’:



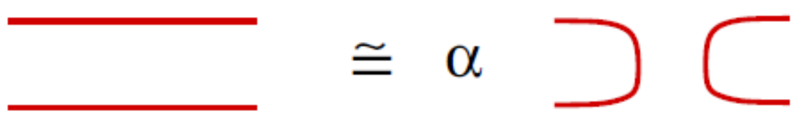
$$1 = \alpha d$$

$$d = \alpha$$

$$\alpha = d = \pm 1$$

Surgery

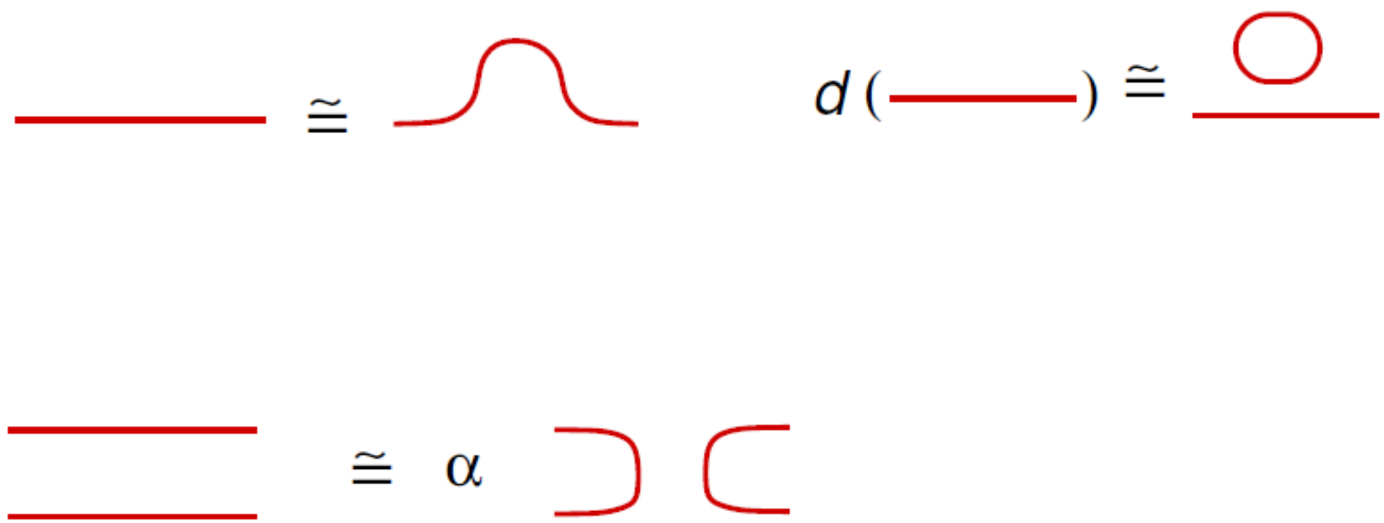
- Consistency between d-isotopy & ‘surgery’:



$\boldsymbol{\alpha = d = +1}$ - Z_2 gauge theory

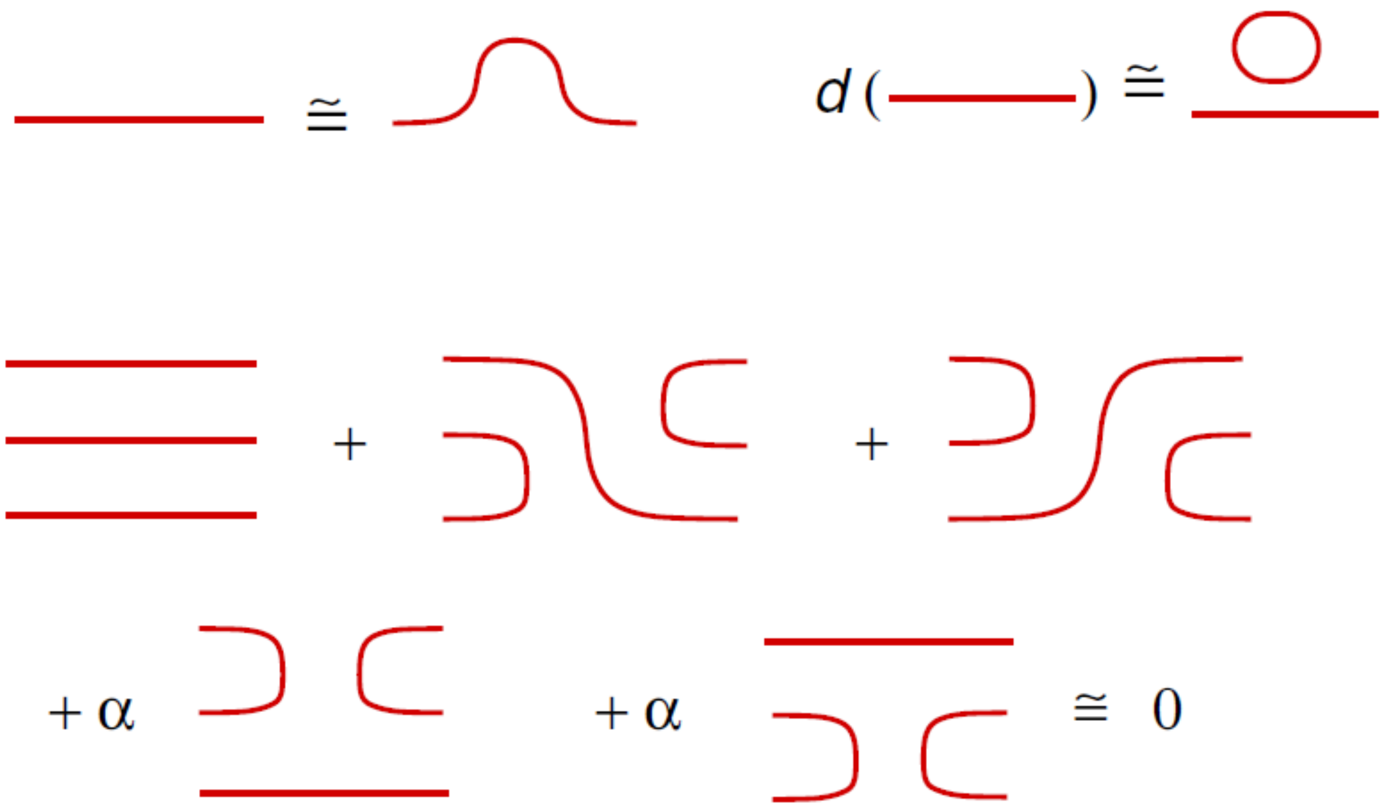
Surgery

- Consistency between d-isotopy & ‘surgery’:



$\boldsymbol{\alpha = d = -1}$ - doubled U(1) Chern-Simons theory with semions

Can we go beyond Abelian theories?



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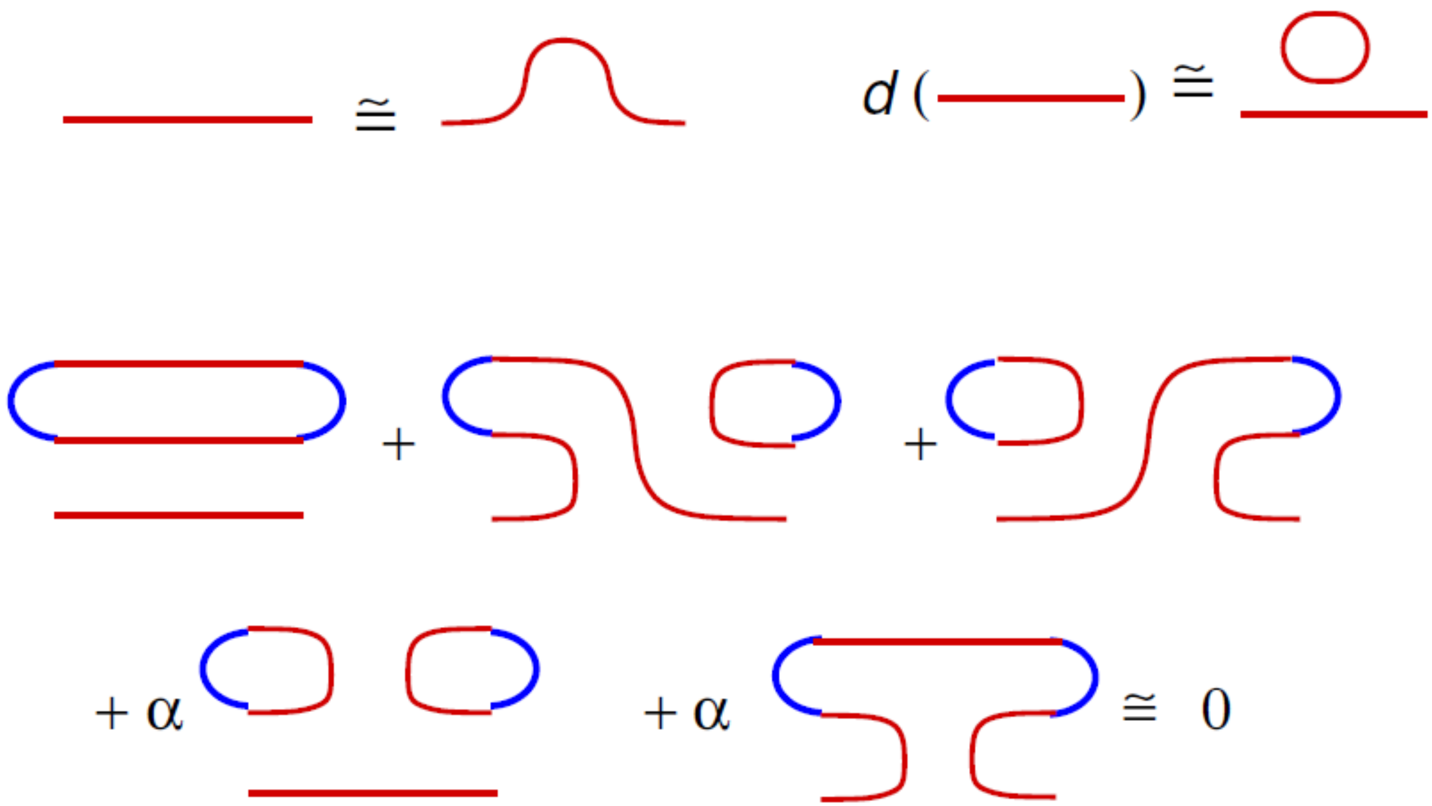

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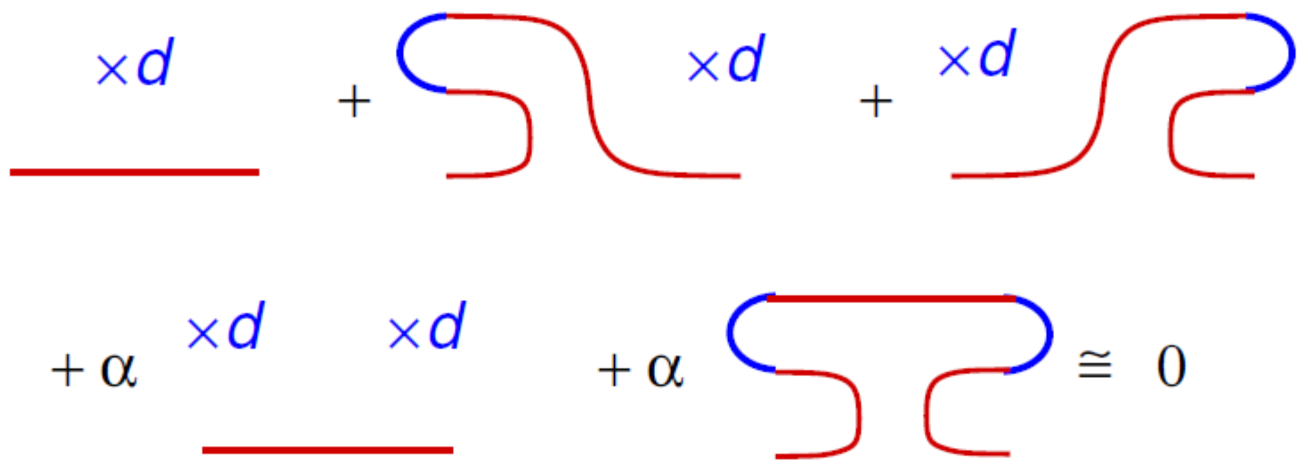
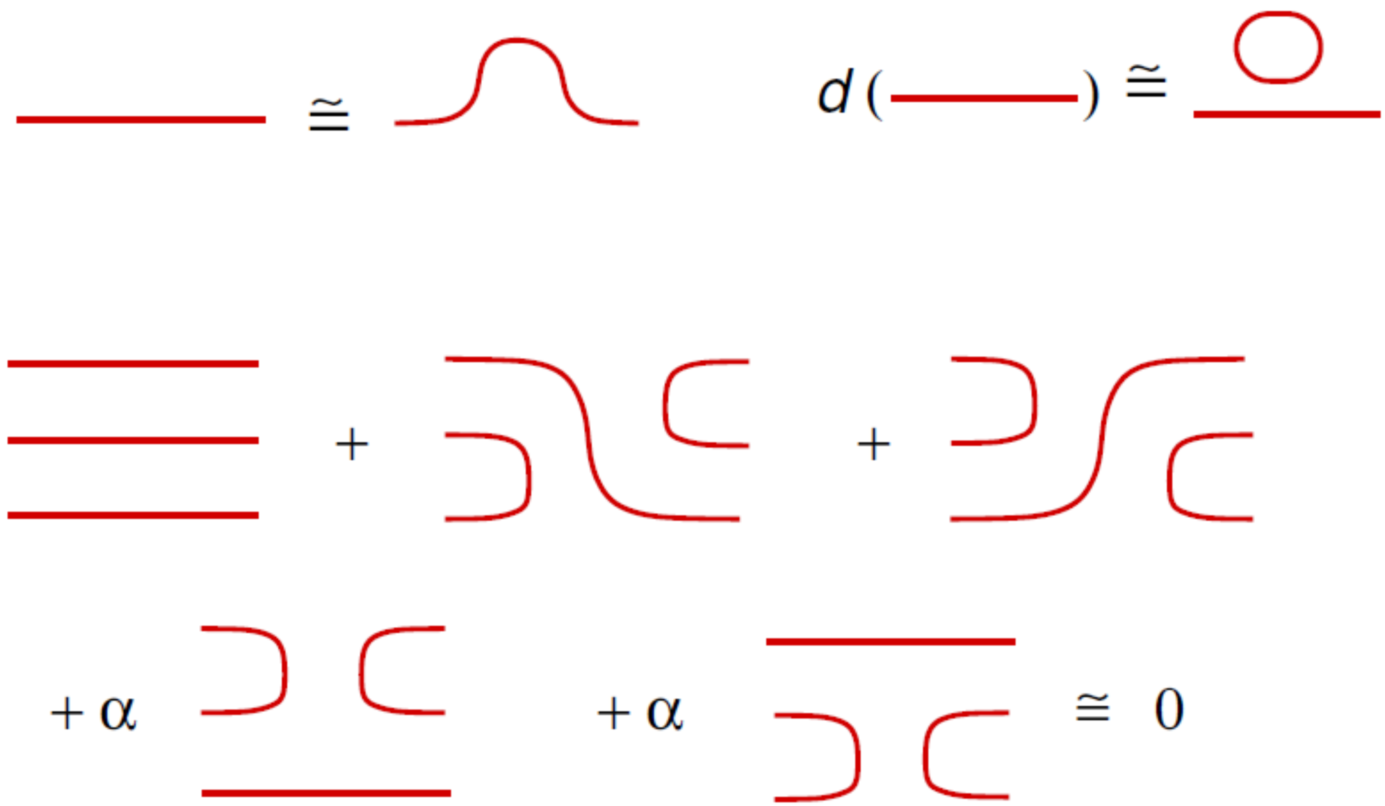


$$3d+\alpha d^2+\alpha=0$$

Can we go beyond Abelian theories?



$$3d + \alpha d^2 + \alpha = 0$$

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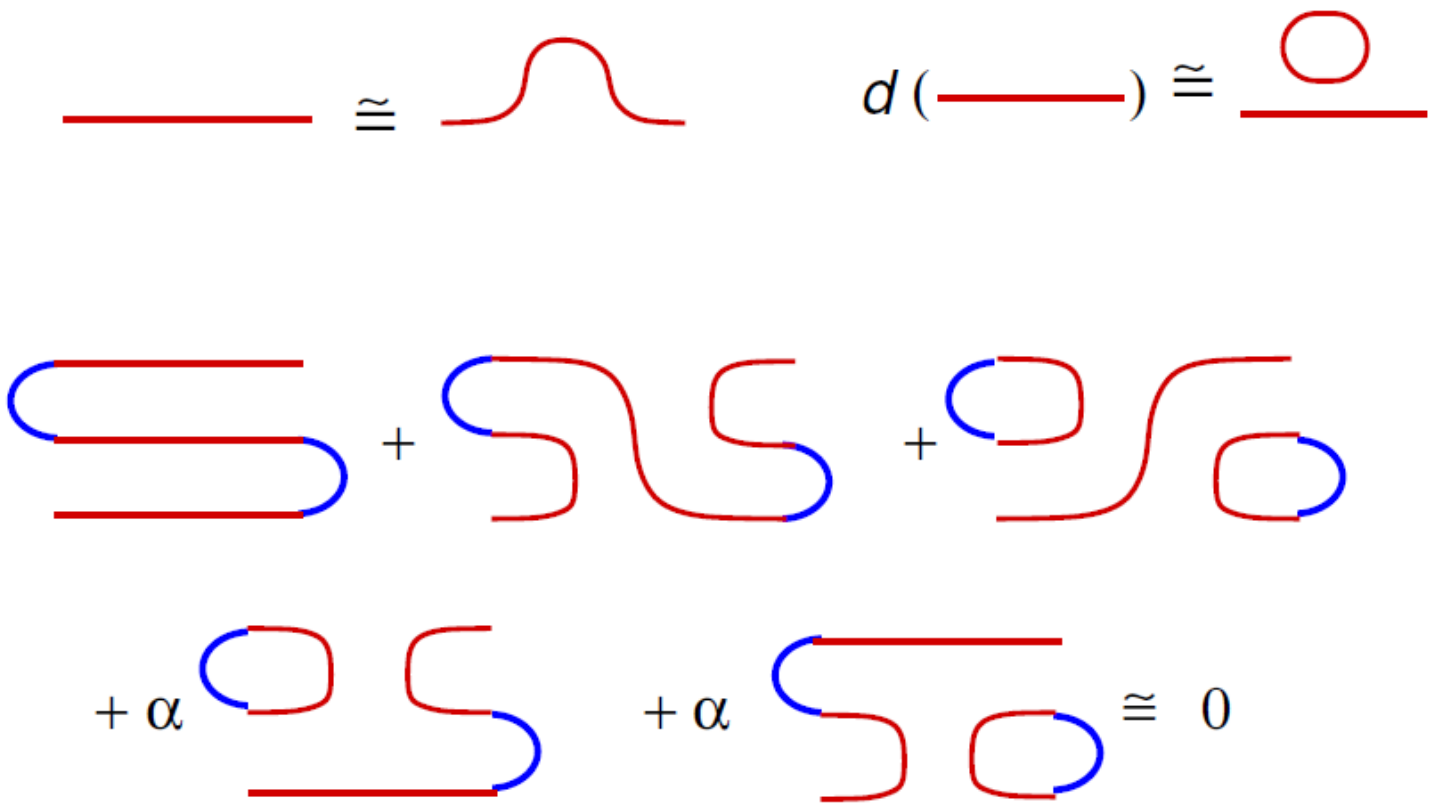

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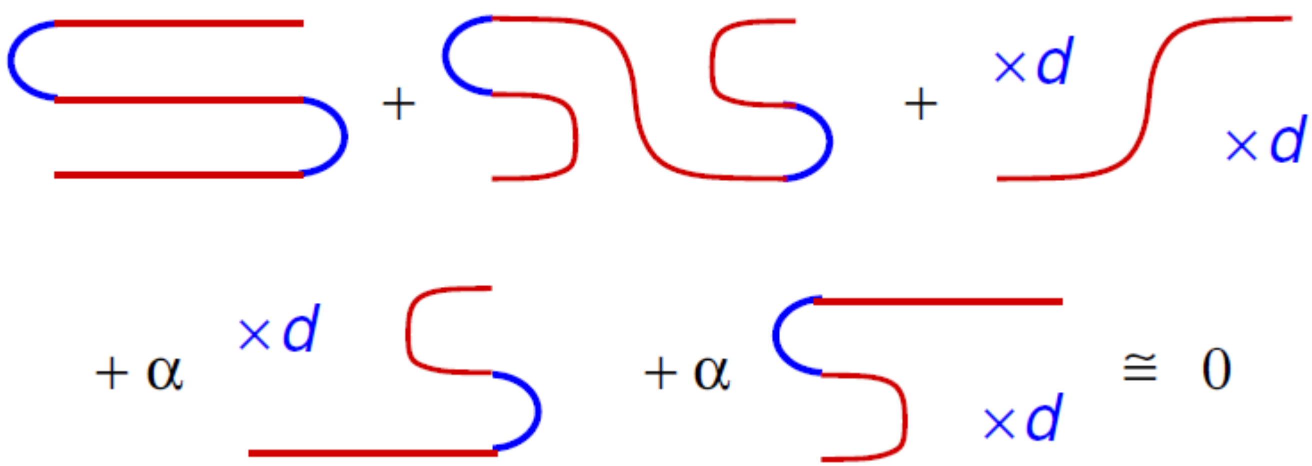
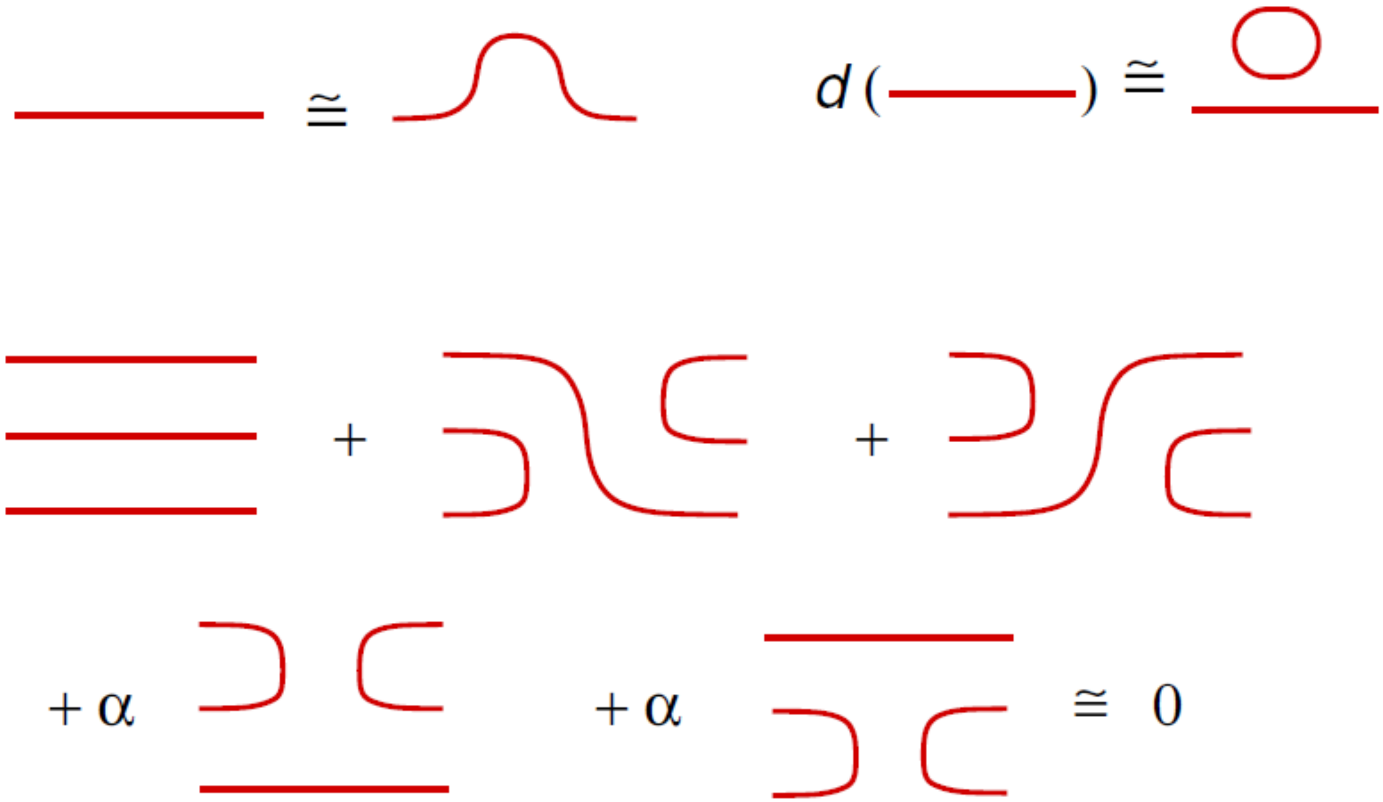


$$3d+\alpha d^2+\alpha=0$$

$$2+d^2+2\alpha d=0$$

Can we go beyond Abelian theories?



$$3d + \alpha d^2 + \alpha = 0$$

$$2 + d^2 + 2\alpha d = 0$$

$$\Rightarrow d = -\alpha = \sqrt{2}$$

(+ old roots)

Can we go beyond Abelian theories?

- These models are parametrised by $d = \pm 2\cos\left(\frac{\pi}{k+2}\right)$
  - the loop amplitude (quantum dimension)

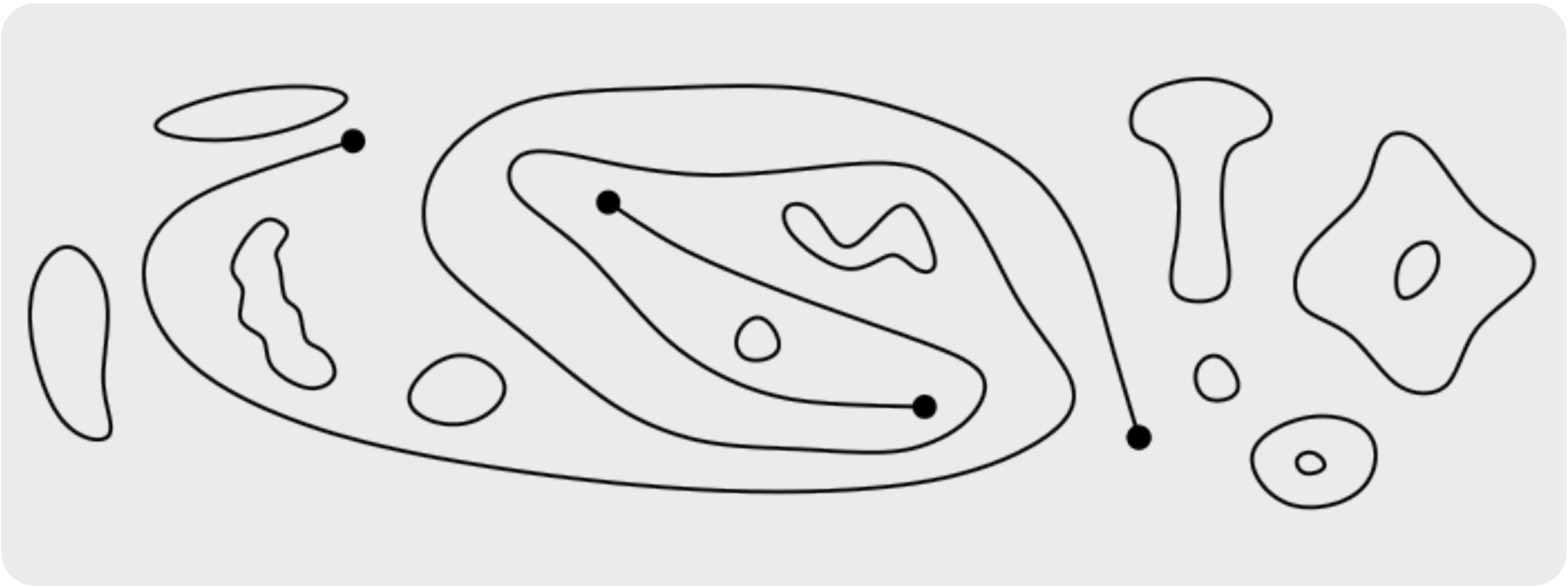
k	d
1	1
2	$\sqrt{2}$
3	$\left(1+\sqrt{5}\right)/2$
4	$\sqrt{3}$

- Skein relation on *k + 1* strands is required for finite dimensionality of Hilbert space.
- Related to doubled $SU(2)_k$ Chern-Simons Theory.

Freedman, Nayak, KS, Walker, Wang, Ann. Phys. 2003

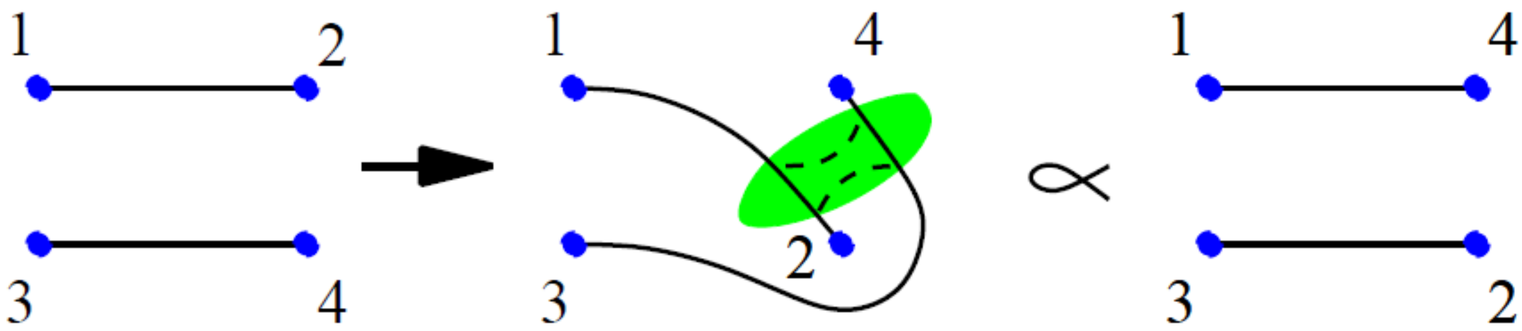
Anyons appear as excitations

- Excitations will be violations of the ground state conditions, such as broken loops

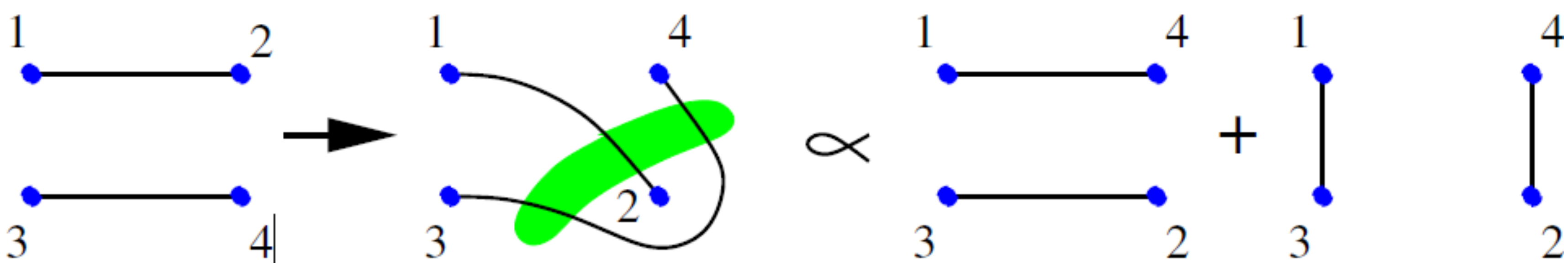


Abelian vs. non-Abelian Statistics

k = 1 (Abelian):



k = 2 (non-Abelian):



Can we go beyond Abelian theories?

How can we construct a microscopic model whose low-energy physics is described by a non-Abelian quantum loop gas?

Can we go beyond Abelian theories?

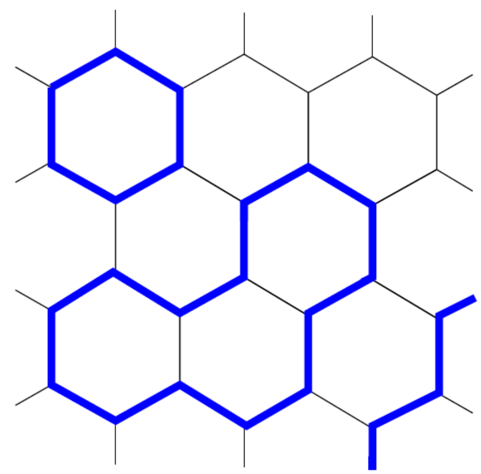
I. Can we learn anything from classical statistical mechanical models?

- Plazma analogy: O(n) loop model

$$Z = \prod_{\langle ij \rangle} (1 + \lambda\, \mathbf{S}_i \cdot \mathbf{S}_j) = \sum_{\text{loops}} (\lambda/n)^b n^l = \sum_{\text{loops}} x^b n^l$$

l – number of loops

b – their total perimeter

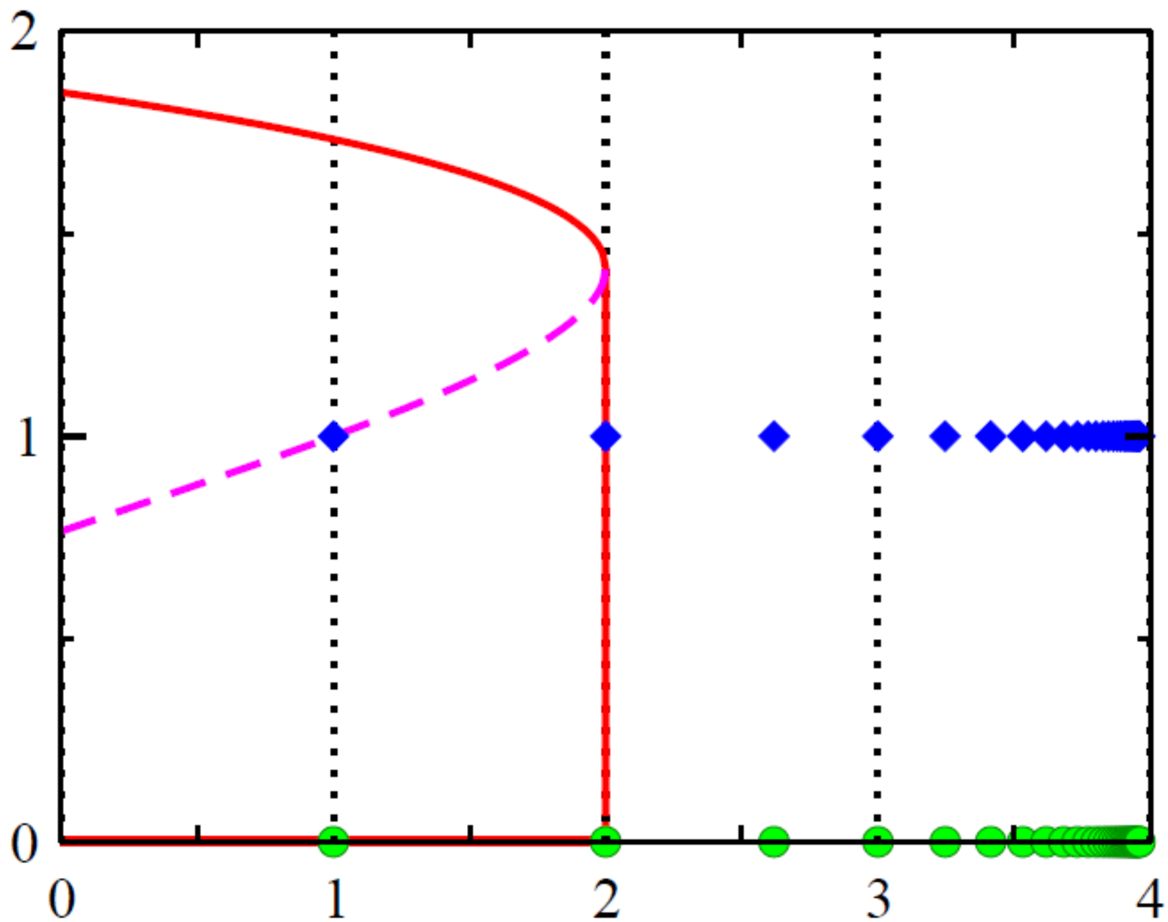


Can we go beyond Abelian theories?

Phase Diagram of O(n) loop model (Blöte, Nienhuis)

$$Z = \sum_{\text{loops}} x^b n^l$$

$$x^{-1} = \frac{n}{\lambda}$$



$$n = d^2$$

Requirements for a topological phase

- **Deconfinement**
  - Need a **critical loop gas** with a large, critical, fractal loop, otherwise the excitations are confined (only short strings exist)
  - Loop gases are described the O(n) loop model.
  - The O(n) model is critical for $n \le 2$ ” $\boldsymbol{d \le \sqrt{2}}$.
  - With Jones-Wentzl in mind, we are left only with d=√2 case.

- **Well-defined braiding statistics**
  - requires a **gap in the spectrum**, so that particles are localized.
  - **Surgery** is required to make loop gases gapped.
    Freedman, Nayak, KS (PRB 2008)

    However, *no three loops would ever (statistically) come close together* (O. Schramm) ” ***Jones-Wentzl is ineffective!***

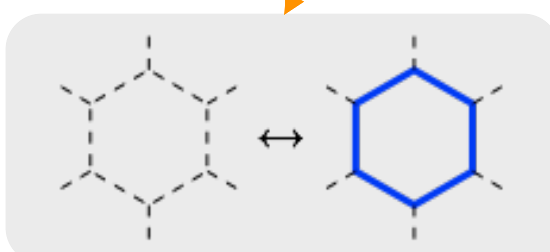
A d-isotopy loop gas model

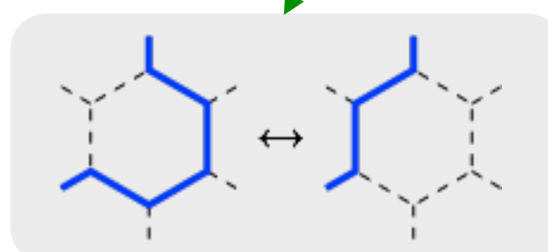
- Implement **vertex constraint** by a projector, forming a loop gas.
- Enforce d-isotopy (**fugacity** d and **isotopy**) by using another projector.

$$\mathcal{H}_0^{(d)} = J\sum_v \left(1 + \prod_{i\in e(v)} \sigma_i^z\right)$$

vertex constraint

$$+\frac{K}{2}\sum_p \left[\frac{2}{1+d^2}\left(d - F_p\right)\mathbb{P}_p^0\left(d - F_p\right) + \left(-F_p\right)\mathbb{P}_p^1\right]$$





- Use honeycomb lattice to avoid irrelevant complications from crossings.

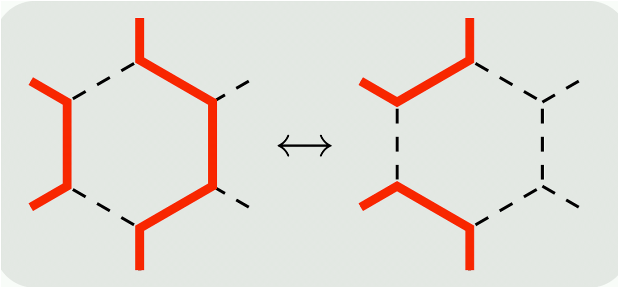
Troyer, Trebst, KS & Nayak, PRL 2008

Adding 2-strand surgery

- Loop gases without surgery are gapless
  - Local changes are inefficient in changing the loop length, giving rise to a slow mode
    Freedman, Nayak, KS (PRB 2008)

- Add a surgery term, keeping *d*-isotopy

$$\mathcal{H}_1 = \mathcal{H}_0^{(d)} + \frac{K}{2}\sum_p \left[\left(1 - d^{\Delta l} F_p\right)\left(\mathbb{P}_p^2 + \mathbb{P}_p^3\right)\right]$$



  - This is just the **toric code** for *d* =1.
  - Term is **non-local** for *d*≠1, since the change in loop number Δ*l* requires knowledge of the nonlocal connections of the “stubs”.

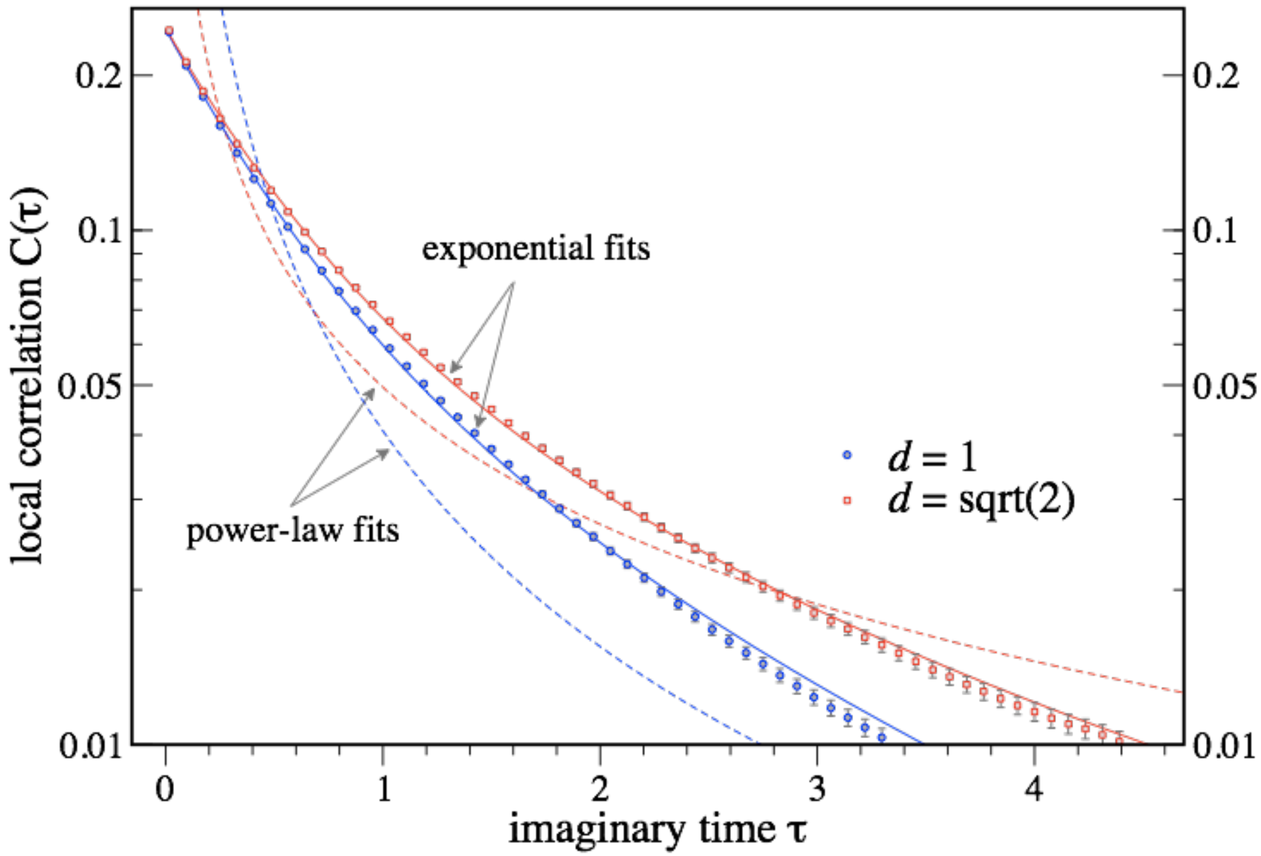
A local gapped Hamiltonian?

- Without surgery the Hamiltonian is gapless.
- What are the **instabilities**?
- Can a perturbation drive us into gapped, non-Abelian phase?
- Attempt the simplest perturbation:
  A tiny local surgery term (*d* =1), **violating *d*-isotopy**

$$\mathcal{H}_2 = \mathcal{H}_0^{(d=\sqrt{2})} + \frac{\epsilon}{2} \sum_p \left[(1 - F_p) \left(\mathbb{P}_p^2 + \mathbb{P}_p^3 \right) \right]$$

Opening a gap

- A tiny ε = 0.01 of the “wrong” surgery opens a gap!



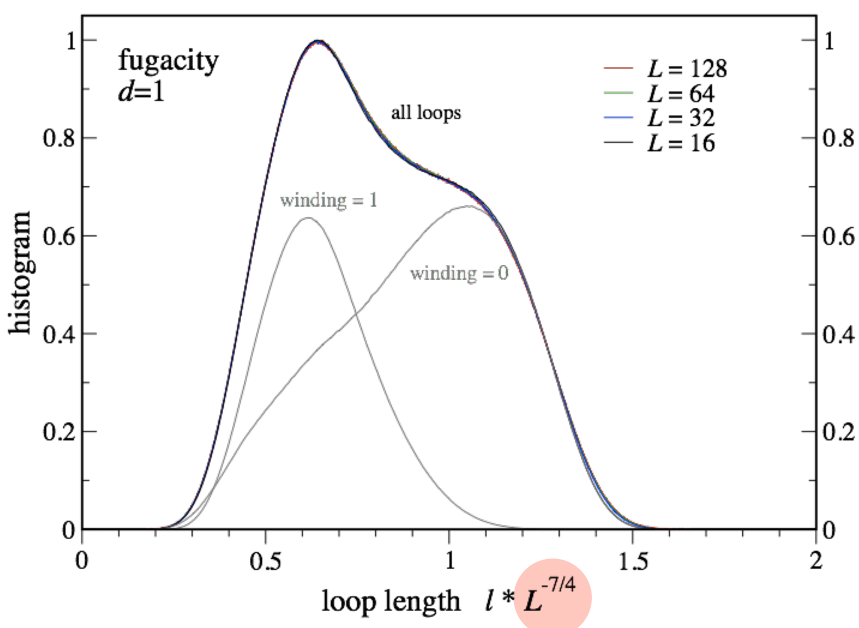
Troyer, Trebst, KS & Nayak, PRL 2008

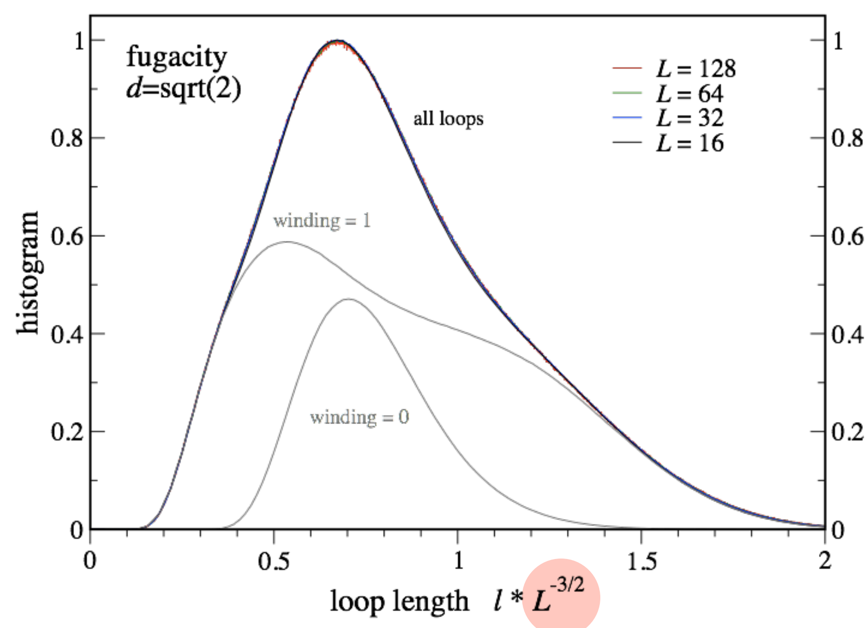
Detecting the non-Abelian phase

- Distinguish phases by
  the **Hausdorff dimension** of the longest loop

Abelian: $d_H = 7/4$

non-Abelian: $d_H = 3/2$



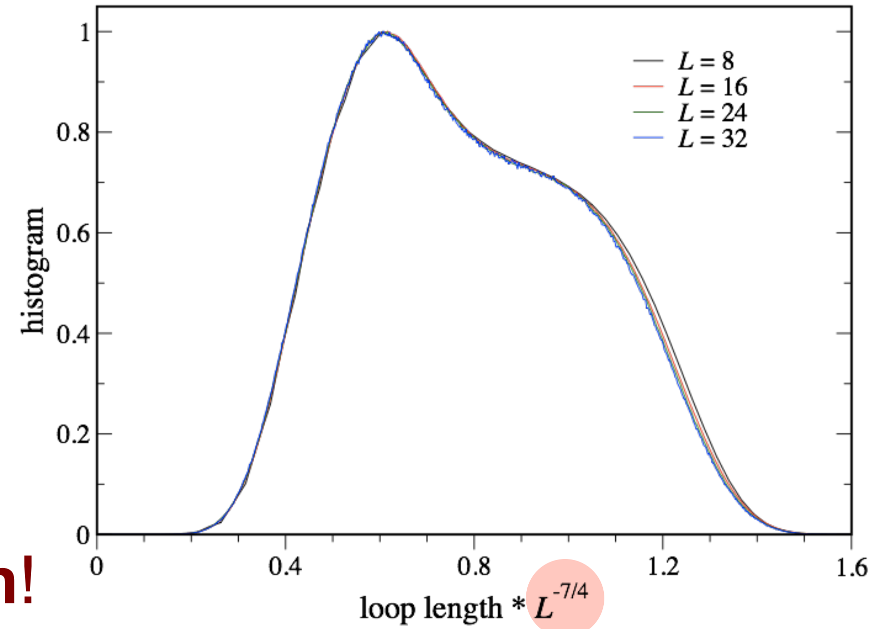


Troyer, Trebst, KS & Nayak, PRL 2008

The gapped phase is Abelian!

Take loop gas with $d=\sqrt{2}$ and a tiny surgery $\varepsilon = 0.01$

- find Abelian $d_H = 7/4$
- characteristic distribution of $d=1$ loop gas



The gapped phase is **Abelian**!

- **fugacity** *d* acts only on **microscopic loops**
- "wrong" **surgery** ε is tiny, but acts on **large loops**

" is a *relevant* perturbation and drives to the Abelian fixed point.

Troyer, Trebst, KS & Nayak, PRL 2008

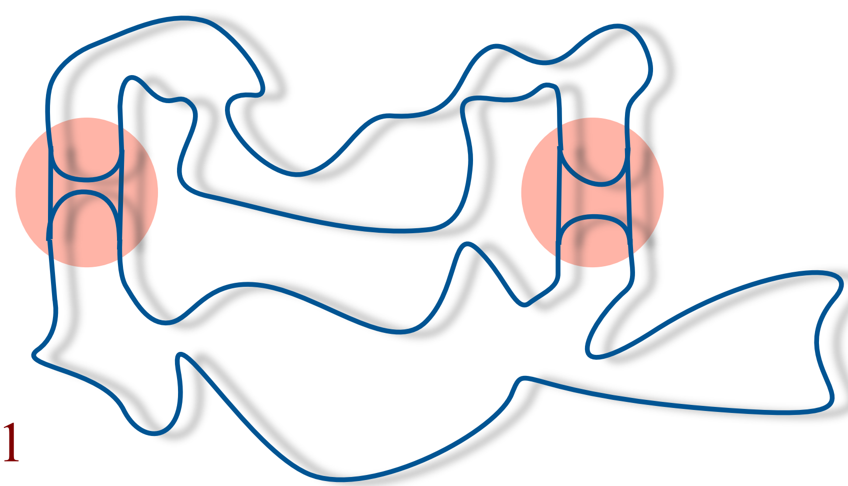
A no-go “theorem”

- Can other terms help?
- Can we have a local, gapped, non-Abelian loop gas?
  - Hastings (2004):
    All correlation functions of local operators decay exponentially in the ground states of local, gapped Hamiltonians.
  - But plaquette flips on the long critical loop are correlated:

left flip: $\Delta l_1 = +1$

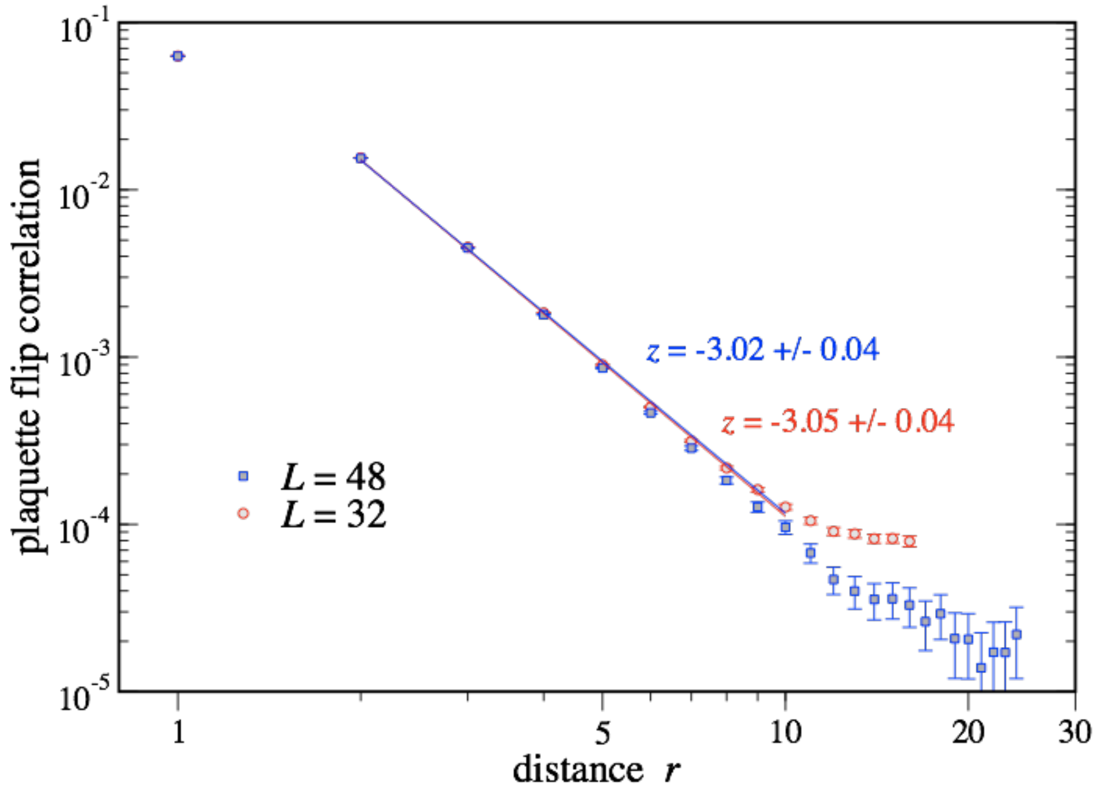
right flip: $\Delta l_2 = +1$

both flips: $\Delta l = 0 \neq \Delta l_1 + \Delta l_2$



Expect **critical correlations** for $d \neq 1$

Power-law correlations of flips



- **Power-law correlations** for $d \neq 1$
- Non-Abelian loop gases cannot be ground states of a gapped, local Hamiltonian.

Routes to Non-Abelian Phases

Non-Abelian topological phases in QLGs require either

- **Non-local Hamiltonians**

  – hard/impossible to implement

- **Modified inner products**

  P. Fendley, Ann Phys 2008

  – modifying the inner product we might open a gap

  – different loop configurations are no longer orthogonal

- **“Double-stranded” loops = trivalent graphs**

  (Turaev-Viro, Levin-Wen, Fendly-Fradkin, Velenich-Chamon-Wen)

  – allow double stranded loops to branch, forming trivalent graphs (AKA string-nets)

  – but so far only 12-site interactions known to work

  - can double strands form energeically from single strands?