DEGENERATE FERMI GASES AT ENS: THEORY AND EXPERIMENTS

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OUTLINE

- **Present experiments with fermions: why interesting ?**
- **Feshbach resonance: which model for the interaction potential ?**
- **A very simple model: a matter wave with one scattering center in a box**
- **An exact time dependent solution for the unitary quantum gas**

PRESENT EXPERIMENTS ON FERMIONS

Typical ENS experimental parameters:

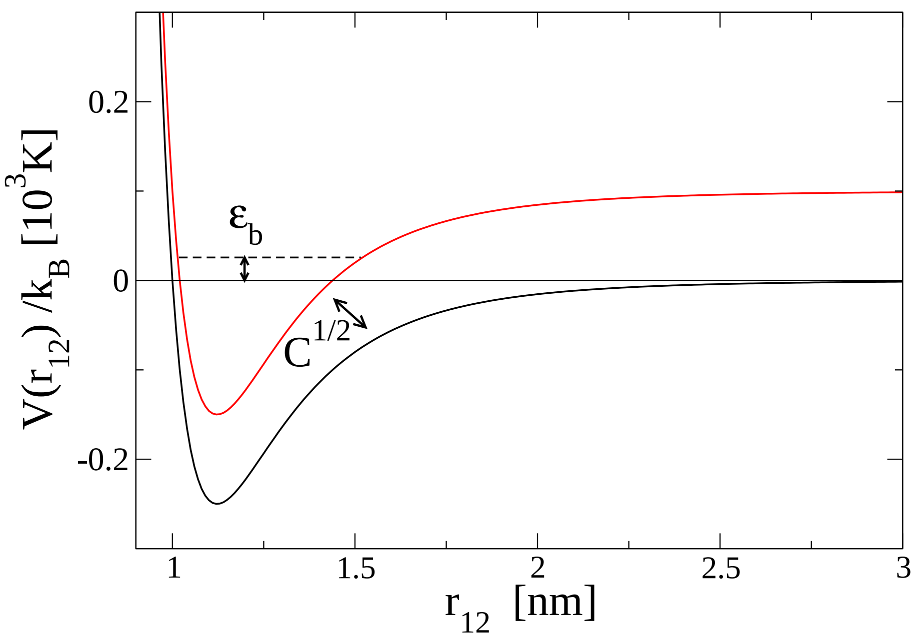
- fermionic ^{6}Li atoms: $F = 1/2$ atomic ground state
- optical trapping, evaporative cooling:
  $N \sim 10^5 \quad k_F \sim 1.6 \times 10^7 \mathrm{m}^{-1} \quad k_B T_F \sim 10 \mu\mathrm{K} \quad T < 0.2 T_F$

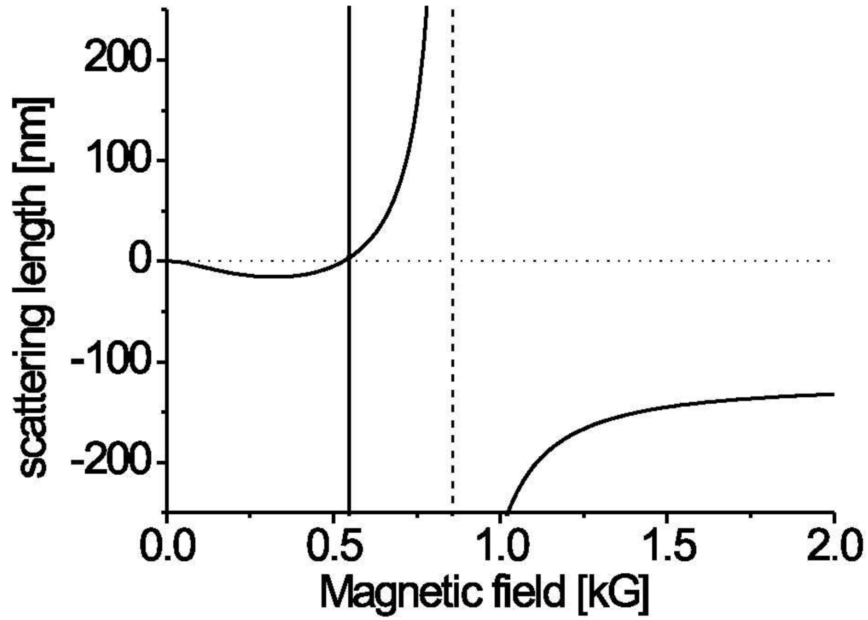
Why interesting ?

- strength and sign of interaction tunable with magnetic field
- scattering length $a > 0$: a 2-body bound state
  Bose condensate of dimers
- scattering length $a < 0$: Cooper pairs
  BCS state = condensate of pairs
- intermediate regime: $a = \pm\infty$
  strongly interacting regime $k_F |a| > 1$ stable

WHAT IS A FESHBACH RESONANCE ?

Schematic view:





Low energy scattering properties: $B_0 = 820$ Gauss

- scattering length

$$a(B) = a_{bg}\left(1 - \frac{\Delta B}{B - B_0}\right)$$

- Scattering amplitude for $k\times$ true potential range $\ll 1$:

$$f_k \simeq \frac{-1}{-\nu C^{-1} + ik + \hbar^2 k^2/2mC}$$

with effective detuning

$$\nu = \epsilon_b(B) + \Delta_b \propto B - B_0 \quad \text{for} \quad B \to B_0$$

and $C > 0$ coupling intensity

- Low k general scattering theory:

$$f_k = \frac{-1}{a^{-1} + ik - \frac{1}{2}k^2 r_e + \ldots}$$

gives

$$a = -\frac{C}{\nu} \qquad r_e = -\frac{\hbar^2}{mC} < 0$$

where r_e is effective range.

THE DILUTE GAS REGIME

Crucial assumption: gas dilute at the scale of r_e:

$$|r_e| \ll |a|, k_F^{-1}.$$

$\longrightarrow$ zero range model potential possible

interactions characterized by a only

Talk of Dima Petrov: regime of gas with effective long range interaction is possible

$$\text{true range} \ll k_F^{-1}, |a| \ll |r_e|$$

WHICH MODEL FOR THE INTERACTION POTENTIAL ?

Requirements for non-perturbative N-body problem:

- **hypothesis of thermal equilibrium applicable:**

$$\sigma_{1,\ldots,N} \propto e^{-\beta H}$$

so that e.g. Quantum Monte Carlo can be used.

rules out the 'true' interaction potential

- **leads to a well defined mathematical problem!**

good idea to check on e.g. 2-body problem!

Example of an ill-defined problem:

- **The Dirac delta interaction potential in 2D and 3D:**

$$V(\vec{r_1} - \vec{r_2}) = g\delta(\vec{r_1} - \vec{r_2})$$

- **2-body Schrödinger's equation in center of mass frame:**

$$E\psi = -\frac{\hbar^2}{m}\Delta_{\vec{r}}\psi + g\delta(\vec{r})\psi(\vec{r})$$

- **Either $\psi(\vec{r})$ vanishes or diverges in $\vec{r} = \vec{0}$:**

$$\text{3D} : \Delta\frac{-1}{4\pi r} = \delta(\vec{r})$$

$$\text{2D} : \Delta\frac{\log r}{2\pi} = \delta(\vec{r}).$$

WHAT IS ψ IN THIS TALK ?

- **for spinless bosons:**

$$\psi(\vec{r_1},\ldots,\vec{r_N}) = \langle \vec{r_1},\ldots,\vec{r_N}|\psi\rangle$$

and is totally symmetric

- **for spin 1/2 fermions:**

$$\psi(\vec{r_1},\ldots,\vec{r_N}) = \langle + : \vec{r_1},\ldots,+ : \vec{r_n}, - : \vec{r_{n+1}},\ldots,- : \vec{r_N}|\psi\rangle$$

and is totally antisymmetric with respect to $\vec{r_1},\ldots,\vec{r_n}$ **and with respect to** $\vec{r_{n+1}},\ldots,\vec{r_N}$.

MODEL 1: FERMI PSEUDO-POTENTIAL

Definition:

$$\langle \vec{r_1}, \vec{r_2} | V | \psi \rangle = \frac{4\pi\hbar^2 a}{m} \delta(\vec{r_1} - \vec{r_2}) \psi_{\text{reg}}(1 = 2)$$

where

$$\psi_{\text{reg}}(1 = 2) \equiv \left[\partial_{r_{12}}(r_{12}\psi(\vec{r_1}, \vec{r_2}))\right]_{r_{12}\to 0}$$

for fixed center of mass position $\vec{R_{12}}$ of 1 and 2

Basic assumption:

- ψ **diverges at most like Green's function**

$$\psi(\vec{r_1}, \ldots, \vec{r_N}) = O\left(\frac{1}{r_{ij}}\right) \quad \forall i \neq j$$

- **regularisation operator removes the r_{ij}^{-1} term**

MODEL 1: FERMI PSEUDO-POTENTIAL

Advantages:

- **Depends only on a**
- **Ideal for exact analytical calculations:**

$$(\Delta + k^2)\psi(\vec{r}) = 4\pi a \psi_{\mathrm{reg}} \delta(\vec{r})$$

$$\psi(\vec{r}) = \psi_0(\vec{r}) + 4\pi a \psi_{\mathrm{reg}} G(r)$$

where ψ_0 solves homogeneous equation, $G(r)$ is Green's function

$$G(r) = -\frac{\exp(ikr)}{4\pi r}.$$

Resulting unknown is a function of 3 less spatial coordinates:

$$\psi_{\mathrm{reg}} = \psi_{0,\mathrm{reg}} - ika\psi_{\mathrm{reg}}.$$

MODEL 1: FERMI PSEUDO-POTENTIAL

Disadvantages:

- Not intuitive: 2-body bound state only for $a > 0$
  $\int -\frac{\hbar^2}{2m}\psi^* \Delta \psi$ is not the kinetic energy
- A modified Hilbert space: equivalent to free waves with boundary conditions

$$\psi(\vec{r_1}, \ldots, \vec{r_N}) = A(a^{-1} - r_{ij}^{-1}) + o(1)$$

  when $r_{ij} \to 0$ for fixed $\vec{R_{ij}} = (\vec{r_i} + \vec{r_j})/2$.
- Variational calculation more tricky:
  - fermions: no Hartree-Fock, no BCS
  - bosons: no Hartree-Fock, no Hartree-Fock-Bogoliubov, no Jastrow ($\psi = [r_{12}r_{13}r_{23}]^{-1} \sim r_{12}^{-3}$ for $\vec{R}_{12} = \vec{r}_3$)

WHEN THREE PARTICLES MEET

Follow general method in free space:

- formal integration of Schrödinger equation relates $\psi(1,2,3)$ to $\psi_{\text{reg}}(i=j,k)$ and the Laplacian Green's function
- Fermi or Bose symmetry: only $\psi_{\text{reg}}(1=2,3)$ required
- in the center of mass frame:

$$\psi_{\text{reg}}(1=2,3) = \psi_{\text{reg}}(\vec{u} = \vec{R_{12}} - \vec{r_3})$$

- Fermions: when $\vec{u} = \vec{0}$ and $r_{12} \to 0$:

$$\psi = -\frac{a}{r_{12}}\psi_{\text{reg}}(\vec{0}\,) + o(1) \quad \longrightarrow \quad \psi_{\text{reg}}(\vec{0}\,) = 0.$$

- Bosons: same result using values $\vec{u} \neq \vec{0}$ and requirement of finite $\psi_{\text{reg}}(\vec{0}\,)$.

Never 3 particles at same point in model 1 !

MODEL 2: LATTICE MODEL

Definition: discrete δ on a lattice

- **Spatial coordinates discretized on a grid:**

$$\vec{r} = \sum_{\alpha=x,y,z} n_\alpha l \vec{e}_\alpha$$

- **Usual kinetic energy $\hbar^2 k^2/2m$ for wavevector $\vec{k}$, but**

$$\vec{k} \in \mathbf{D} \equiv [-\pi/l, \pi/l[^3.$$

- **Interaction potential:**

$$V(\vec{r_1} - \vec{r_2}) = \frac{g_0}{l^3}\delta_{\vec{r_1},\vec{r_2}}$$

- **l-dependence in exact state should disappear if**

$$k_F l \ll 1.$$

MODEL 2: LATTICE MODEL

How to choose the coupling constant g_0?

- **Exact scattering matrix on the grid:**

$$T_{\rm grid}(E+i\eta) = \frac{|\vec{r}=0\rangle\langle\vec{r}=0|}{g_0^{-1} - \int_{\rm D} d^3k(2\pi)^{-3}(E+i\eta-\hbar^2k^2/m)^{-1}}$$

- **Adjust g_0 to have scattering length a on the grid:**

$$g_0^{-1} = g^{-1} - \int_{\rm D} \frac{d^3k}{(2\pi)^3}\frac{m}{\hbar^2k^2}$$

$$g_0 = \frac{g}{1-2.442\,a/l}.$$

similar to 'usual' prescription (see e.g. Randeria).

- $l \ll |a|$ gives $g_0 = -5.14\,\hbar^2 l/m < 0$:

  not hard sphere but attractive for $|a| = \infty$

MODEL 2: LATTICE MODEL

Advantages:

- Regular Hilbert space: for fermions, BCS ansatz can be used
- Link with Hubbard Hamiltonian theory possible
- Negative coupling constant $g_0 < 0$:
  - The gas clearly experiences attraction
  - Quantum Monte Carlo possible for fermions: no sign problem

MODEL 2: LATTICE MODEL

Disadvantages for bosons: 3 particles can be on same site:

- breaks equivalence with model 1: has Efimov states not present in model 1

$$\psi_{\rm reg}(1=2,3) \stackrel{!}{\longrightarrow} \infty \text{ for } \vec{r_3} \to \vec{R_{12}}.$$

$$\psi\left[\vec{r_1},\vec{r_2},\vec{r_3}=\frac{1}{2}(\vec{r_1}+\vec{r_2})\right] \stackrel{!}{\sim} \frac{1}{r_{12}^2} \text{ for } r_{12}\to 0.$$

- breaks usability of thermodynamics: spectrum not bounded from below for $l \to 0$:

$$E_0 \leq -N\frac{2\pi\hbar^2}{2.442ml^2}(N-2.918)$$

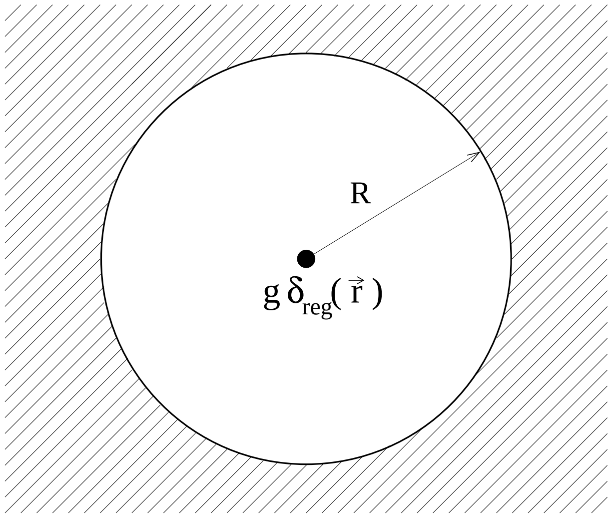
as obtained from variational ansatz $|N:\vec{r}=\vec{0}\rangle$.

A NAIVE BUT INSTRUCTIVE MODEL

For a given spin $\uparrow$ fermion:

- effect of nearest $\downarrow$ modeled by scattering center
- effect of other $N/2-1\downarrow$ modeled by box of size $\sim \rho^{-1/3}$
- effect of other $N/2-1\uparrow$: Fermi statistics, modeled by $\phi=0$ boundary conditions

Trick used in Jastrow MC calculations, see Pandharipande:
short range correlations crucial for $k_F|a|>1$



Relating the model to observables of the gas:

- **Energy of the gas vs energy in the box:**

$$E = \frac{1}{2} N \epsilon$$

- **Density of the gas vs radius in the box:**

$$a = 0 : \quad \frac{3}{5} N \epsilon_F = \frac{1}{2} N \frac{\hbar^2}{m} \left(\frac{\pi}{R} \right)^2 \qquad k_F R = \sqrt{5/3}\pi.$$

Solving the one-body problem:

$$-\frac{\hbar^2}{m} \Delta \phi = \epsilon \phi \qquad \partial_r \ln(r\phi)_{r=0} = -\frac{1}{a}$$

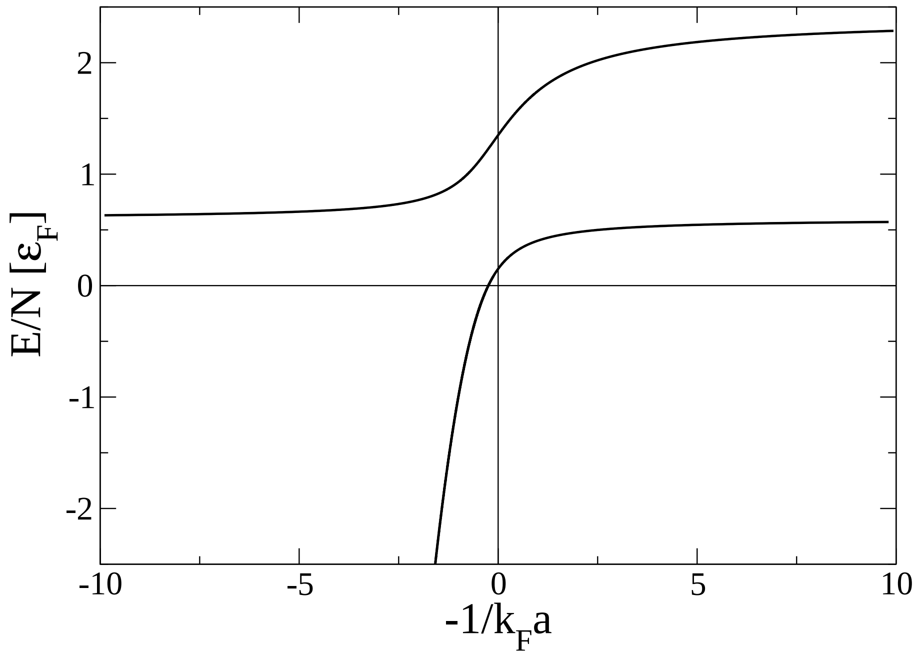
- **Positive energies** $\epsilon = \hbar^2 k^2 / m$:

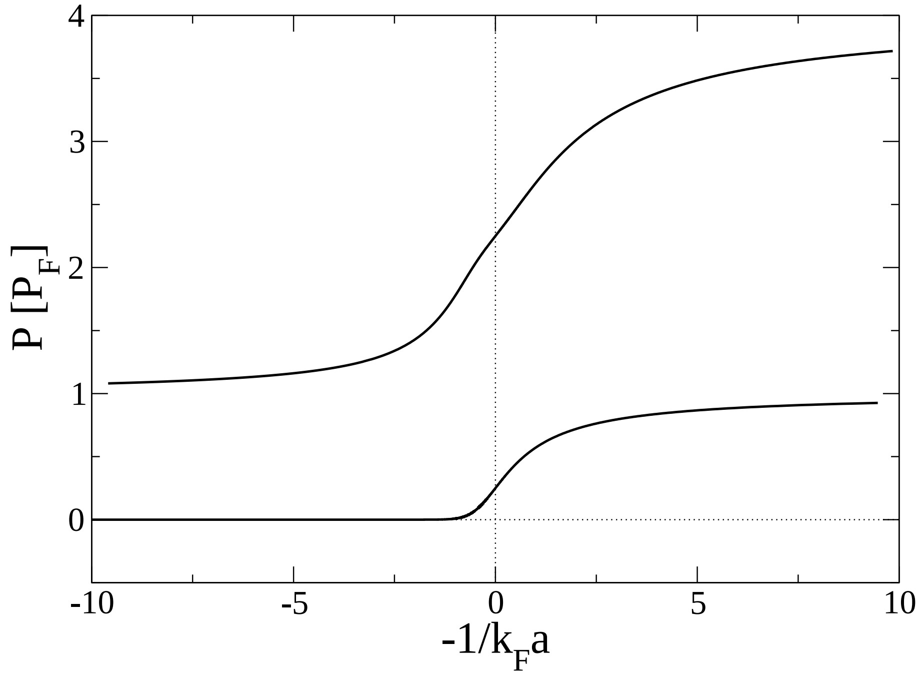
$$\phi(r) \propto \frac{\sin[k(r-R)]}{r} \qquad \tan kR = ka$$

• Negative energies $\epsilon = -\hbar^2\kappa^2/m$:

$$\phi(r) \propto \frac{\sinh[k(r-R)]}{r} \qquad \tanh \kappa R = \kappa a$$

PREDICTIONS OF NAIVE MODEL





PREDICTIONS OF NAIVE MODEL

Ground branch:

- **Connecting two interesting regimes:**
  - $k_F a \to 0^-$: **weakly attractive Fermi gas**
    BCS phase in a full N-body theory
  - $k_F a \to 0^+$:**dilute gas of dimers**
    BEC of dimers in a full N-body theory
- **Is stable. Even in unitary limit. Thanks to Fermi pressure.**
- **Energy less than the ideal Fermi gas. For $a = \pm\infty$: effective attraction.**
- **Crucial idea: adiabatic following**
  degenerate Fermi gas $\longrightarrow$ BEC of dimers
  BEC of dimers $\longrightarrow$ condensate of "Cooper" pairs

Upper branch:

- $k_F a \to 0^+$: weakly repulsive Fermi gas.
- For bosons: the standard state of BEC's !
- Is metastable. Relaxes to ground branch by three-body collisions:

  $$\mathbf{3\ atoms \longrightarrow 1\ dimer + 1\ atom}$$

  A second way to produce a BEC of dimers.

EXPERIMENTAL RESULTS AT ENS

Measuring the expansion energy of the gas:

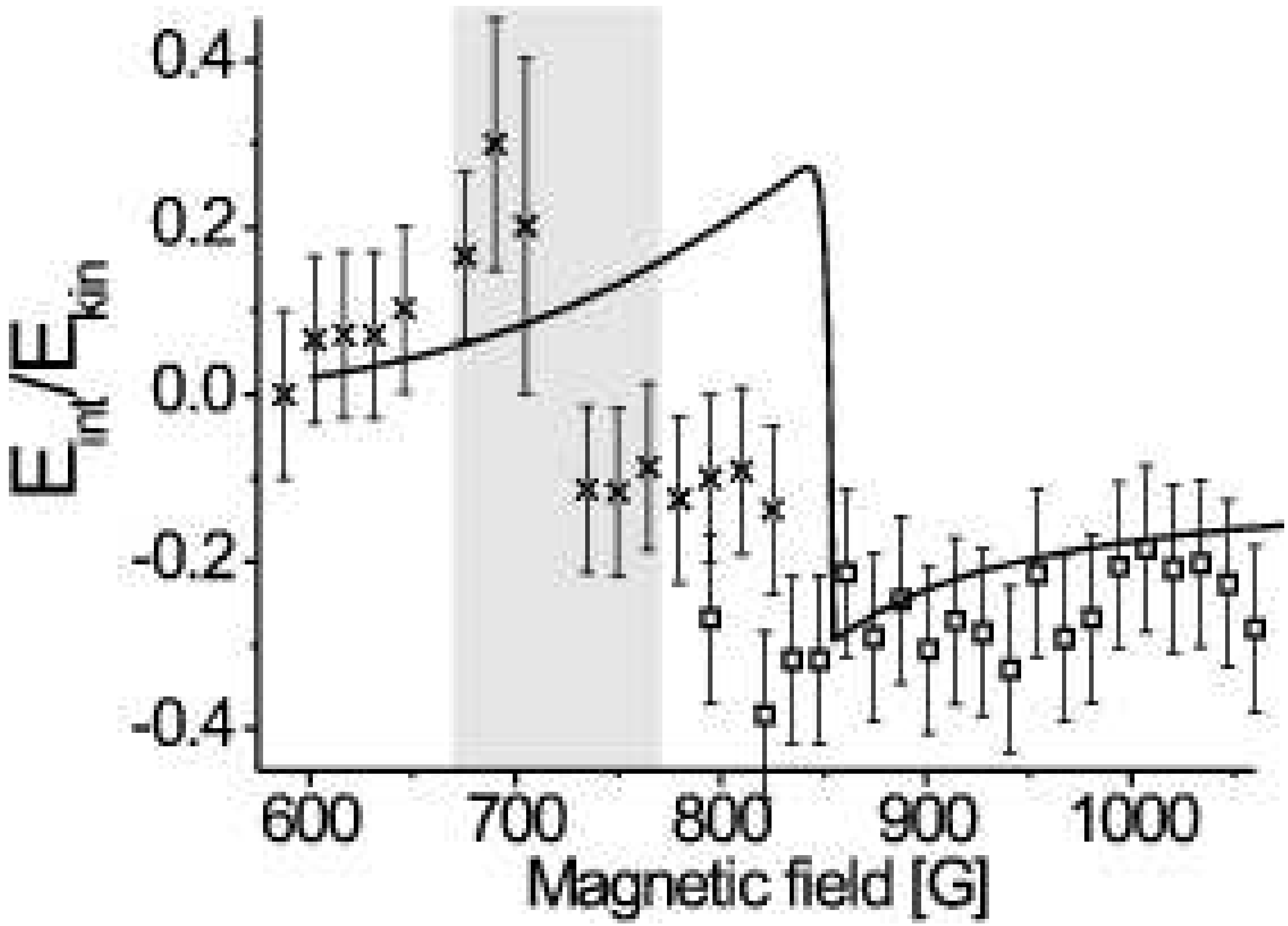
- **expansion with fixed a:**

$$E_{\text{expansion}} = E_{\text{kin}} + E_{\text{inter}}$$

- **expansion with $a = 0$ possible: gives momentum distribution**

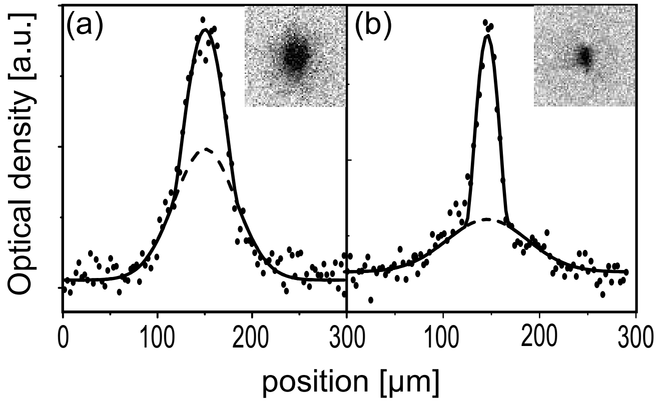
- **Breaking the molecules just before imaging:**

$$a > 0 \;\longrightarrow\; a < 0 \;\text{ then }\; a = 0.$$


E_{int}/E_{kin}
0.4
0.2
0.0
-0.2
-0.4
600
700
800
900
1000
Magnetic field [G]

A BEC of dimers

^{6}Li ^{7}Li



$$\frac{a_{\mathrm{mol}}}{a} = 0.56^{+0.3}_{-0.2} \text{ vs theory } 0.6.$$

Other groups:

- D. Jin, JILA (Boulder, USA), with ^{40}K
- R. Grimm, Innsbruck, with ^{6}Li
- W. Ketterle, MIT (Boston, USA), with ^{6}Li

A condensate of pairs for $k_F|a| > 1$ reported by D. Jin.

MORE DETAILS ON UNITARY LIMIT $|a| = +\infty$

Known properties:

- **Universality: depends only on T/T_F**

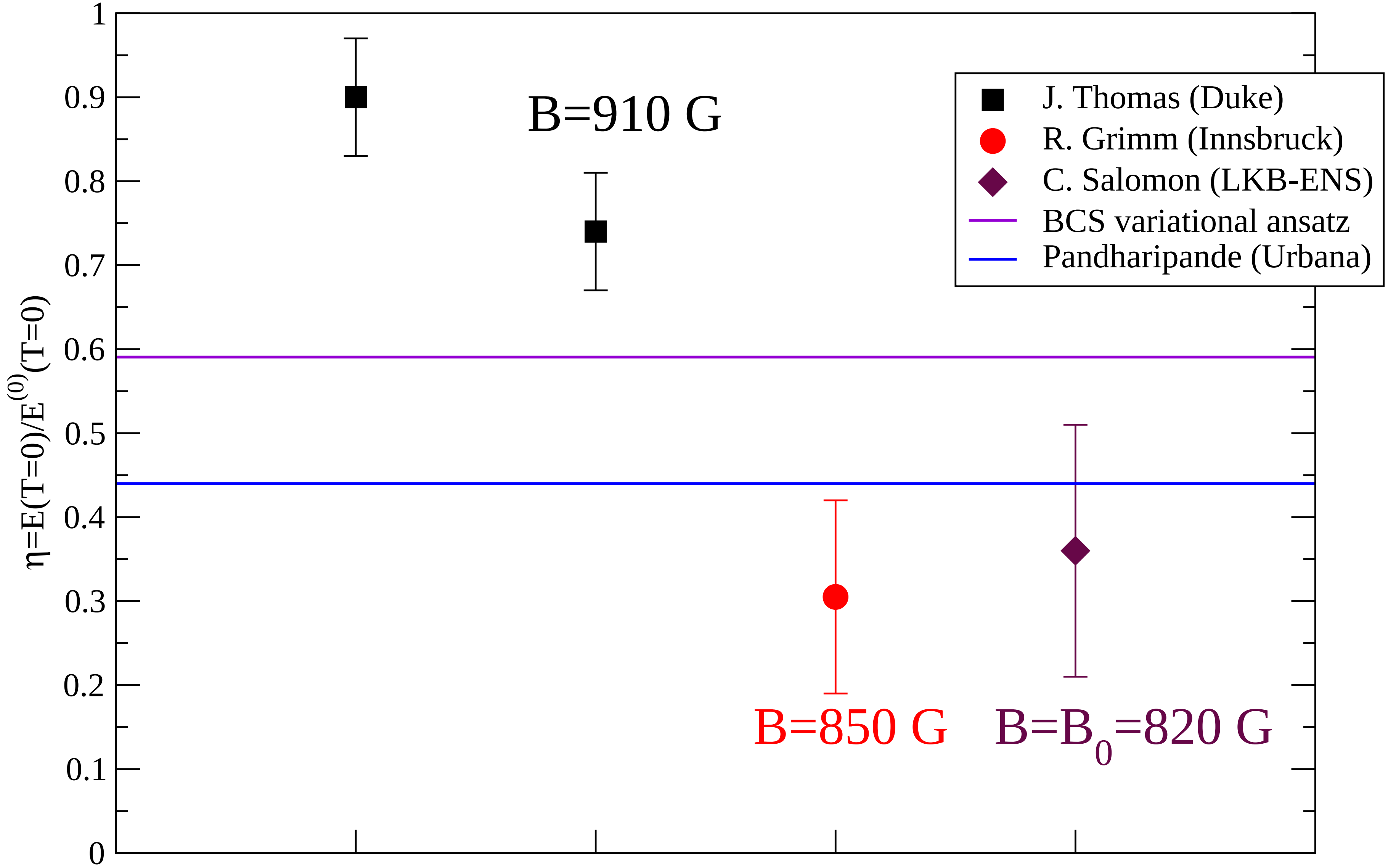
$$E(T=0) = \eta E^{(0)}(T=0)$$

- **Effective attraction: $\eta < 1$ from Hartree-Fock**
- **Upper bound from BCS (Randeria):**

$$\eta < 0.5906\ldots$$

- **Fixed Node Green's function Monte Carlo with trial Jastrow-BCS wavefunction (Pandharipande):**

$$\eta < 0.44 \pm 0.01$$



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
η=E(T=0)/E(0)(T=0)
B=910 G
B=850 G
B=B0=820 G
J. Thomas (Duke)
R. Grimm (Innsbruck)
C. Salomon (LKB-ENS)
BCS variational ansatz
Pandharipande (Urbana)

MORE DETAILS ON ADIABATIC FOLLOWING

Evolution of temperature:

- **Gas thermally isolated: for slow change of a, isentropic evolution:**

$$S(N, T_i, a_i) = S(N, T_f, a_f).$$

- **Bose condensate limit:**

$$k_F a \ll 1 \qquad k_B T \ll k_B T_c^{(0)} \ll \hbar^2/ma^2.$$

For $k_B T > \mu_{\mathrm{mol}}$

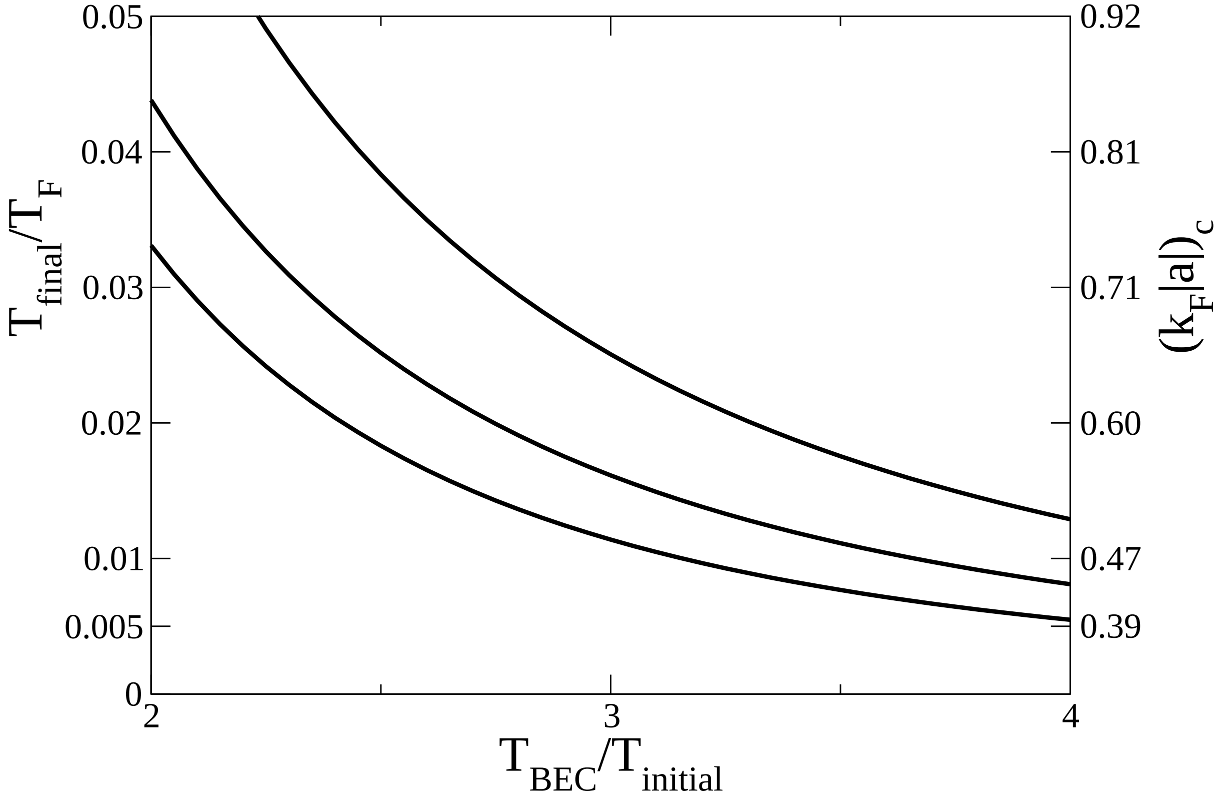
$$S \simeq 1.8 k_B N \left(\frac{T}{T_c^{(0)}}\right)^3$$

- **Ideal Fermi gas limit:**

$$k_F(-a) \ll 1 \qquad T > T_c^{\mathrm{BCS}}$$

$$S \simeq k_B N \pi^2 \frac{T}{T_F}$$

- **Moving from $a > 0$ to $a < 0$ can provide cooling.**



MORE ABOUT BALLISTIC EXPANSION

A very common experimental procedure:

- **prepare trapped gas in steady state**
- **switch off trapping potential abruptly**
- **gas freely expands for ~ 20 ms**
- **laser beam absorption imaging gives integrated density:**

$$\text{signal}(x, y) \propto \int dz\, \rho(x, y, z; t)$$

- **used as a 'magnification lens': e.g. to reveal a vortex lattice in a BEC (J. Dalibard)**

MORE ABOUT BALLISTIC EXPANSION (2)

Is it a faithfull ‘magnifying lens’ ?

- Yes if expanded density can be related to in situ observables
- A non-trivial problem because of interactions
- A sufficient condition: existence of scaling relation

$$\rho(x,y,z;t) = \frac{1}{\prod_\alpha \lambda_\alpha(t)} \rho_0 \left[\frac{x}{\lambda_1(t)}, \frac{y}{\lambda_2(t)}, \frac{z}{\lambda_3(t)} \right].$$

BRIEF HISTORY OF SCALING SOLUTIONS

- For an ideal gas in a harmonic potential
- For the Boltzmann equation in a harmonic isotropic potential: Boltzmann
- For the Gross-Pitaevskii equation in a harmonic trap:
  - in Thomas-Fermi regime: G. Shlyapnikov, E. Surkov, Yu. Kagan (1996), R. Dum, Y. Castin (1996)
  - in Thomas-Fermi regime for rotating traps: M. Olshanii, P. Storey (2000), Y. Castin, S. Sinha (2001)
  - in 2D in isotropic trap: G. Shlyapnikov, E. Surkov, Yu. Kagan (1996)
- For superfluid hydrodynamics in a harmonic trap with equation of state $\mu \propto \rho^\gamma$: Stringari, Menotti (2002)

- for N-body Schrödinger equation of 1D gas of impenetrable bosons in harmonic trap: is formally equivalent to ideal Fermi gas: Girardeau
- For N-body Schrödinger equation in 2D, isotropic harmonic trap, $1/r_{12}^2$ or $\delta(\vec{r_{12}})$ interaction potential: Pitaevskii, Rosch (1997).
- BUT required regularisation breaks scaling invariance: Olshanii, Pricoupenko (2002) so Pitaevskii result applies only to states with no particles at same point

$$\psi(\ldots, \vec{r_i} = \vec{r_j}, \ldots) = 0 \quad \forall i \neq j$$

like Laughlin state.

SCALING SOLUTION FOR THE 3D UNITARY QUANTUM GAS

The problem in an isotropic trap:

- **Free Schrödinger equation over domain $r_{ij} \neq 0$:**

$$i\hbar\partial_t\psi = \sum_{i=1}^{N}\left[-\frac{\hbar^2}{2m}\Delta_{\vec{r_i}} + \frac{1}{2}m\omega^2(t)r_i^2\right]\psi$$

- **plus contact conditions:**

$$\psi(\vec{r}_1,\ldots,\vec{r_N}) = \frac{A(\vec{R}_{ij},\{\vec{r_k}, k\neq i,j\})}{r_{ij}} + o(1).$$

- **Initially, stationary state in static trap $\omega = \omega_0$ with energy E.**

Ansatz: gauge plus scaling transform:

$$\psi(\vec{r}_1, \ldots, \vec{r_N}) = \frac{e^{-i\theta(t)}}{\lambda^{3N/2}(t)} \exp\left[\frac{im\dot{\lambda}}{2\hbar\lambda} \sum r_j^2\right] \psi_0(\vec{r_1}/\lambda, \ldots, \vec{r_N}/\lambda).$$

- **scaling preserves contact conditions**
- **gauge transform preserves contact conditions:**

$$r_i^2 + r_j^2 = 2R_{ij}^2 + \frac{1}{2} r_{ij}^2.$$

- **solves free Schrödinger equation if**

$$\ddot{\lambda} = \frac{\omega_0^2}{\lambda^3} - \omega^2(t)\lambda$$

$$\theta(t) = E \int_0^t \frac{d\tau}{\hbar\lambda^2(\tau)}.$$

CONSEQUENCES OF SCALING SOLUTION

- Linear response: undamped mode of frequency $2\omega_0$
- Existence of lowering operator:

$$L_- = -\frac{3N}{2} + \frac{E}{\hbar\omega_0} - \sum_{j=1}^{N} \vec{r_j} \cdot \partial_{\vec{r_j}} - \frac{m\omega_0}{\hbar} \sum_{j=1}^{N} r_j^2.$$

$L_-|\psi_0\rangle$ vanishes or has energy $E - 2\hbar\omega_0$.

- Virial theorem (F. Chevy):

$$E = 2E_{\text{harm}} > 0$$

$\longrightarrow$ spectrum semi-bounded, stability

NB. For isotropic trap hydrodynamic prediction gives same scaling as exact solution. For anisotropic traps experiments in disagreement with hydrodynamics.

CONCLUSION AND PERSPECTIVES

- **Crossover from BEC of composite bosons to BCS transition and strongly interacting regime is being studied experimentally with gases of fermionic atoms**

  A challenge for theorists !

- **To come: rotating superfluid Fermi gases, vortices**

- **To come: fermionic atoms in optical lattices and the Hubbard model**