Force Transmission in Granular Materials

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OUTLINE

- Review of Models
- Obtaining vector forces
- Force distributions and correlations
- Force Transmission (via an older experimental approach)
  Order, disorder, friction....
- Conclusion

Understanding Stress Balance—Ideally from Micromechanics

- Four unknown stress components (2D)
- Three balance equations
  - Horizontal forces
  - Vertical forces
  - Torques
- Need a constitutive equation

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \qquad \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \qquad \sigma_{xz} = \sigma_{zx}$$

Some approaches to describing stresses

- Elasto-plastic models (Elliptic, then hyperbolic)
- Lattice models
  - Q-model (parabolic in continuum limit)
  - 3-leg model (hyperbolic (elliptic) in cont. limit)
  - Anisotropic elastic spring model
- OSL model (hyperbolic)
- Telegraph model (hyperbolic)
- Double-Y model (type not known in general)

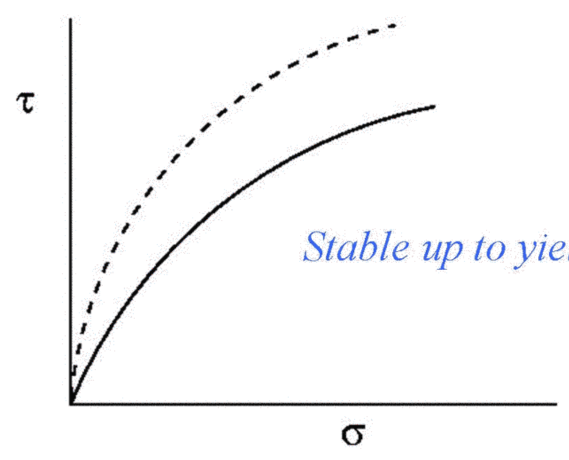
Features of elasto-plastic models

Conserve mass: $\partial\rho / \partial t + \partial_i(\rho v_i) = 0$

(Energy: lost by friction)

Conserve momentum: $\rho dv_i / dt = -\partial_j T_{ij}$

Concept of yield and rate-independence



Stable up to yield surface

τ => shear stress, σ => normal stress

Example of stress-strain relationship for deformation

$$T_{ij} = P\delta_{ij} kPV_{ij} / |V|$$

$$V_{ij} = -(\partial_j v_i + \partial_i v_j)/2$$

(Strain rate tensor with minus)

$$|V|^2 = \Sigma V_{ij}^2$$ $|V|$ = *norm of V*

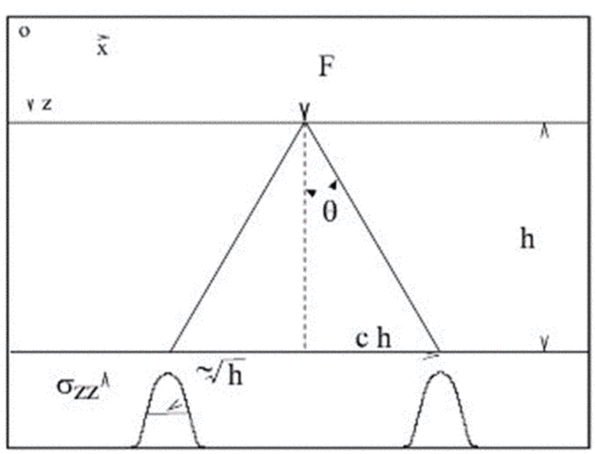
Contrast to a Newtonian fluid:

$$T_{ij} = P\delta_{ij} + 2\eta[V_{ij} - Tr(V)/3] + (2\zeta/3)Tr(V)$$

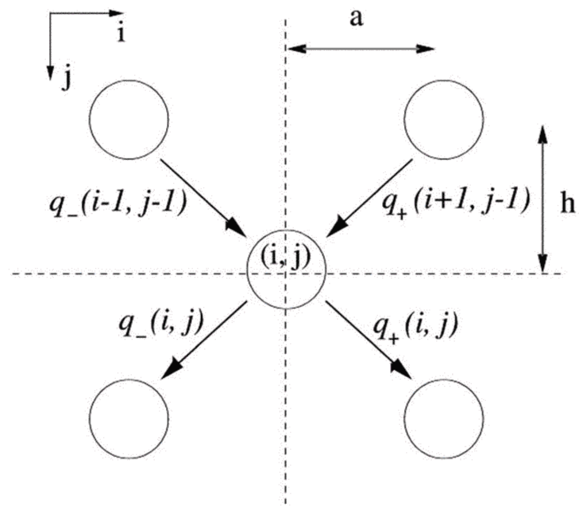
OSL model

$$\sigma_{xx} = \eta\sigma_{zz} + \mu\sigma_{xz}$$ *η, μ: phemonological parameters*

$$\sigma_{zz}(x,z) = \frac{F}{2}[\delta(x+cz) + \delta(x-cz)]$$



q-model (e.g. in 2D)



q's chosen from uniform distribution on [0,1]

Predicts force distributions ~ exp(-F/F_o)

Long wavelength description is a diffusion equation

$$\frac{\partial w(z,j)}{\partial z} = \beta[w(z,j+1) + w(z,j-1) - 2w(z,j)]$$

$$\frac{\partial w}{\partial z} = D\frac{\partial^2 w}{\partial x^2}$$

Expected stress variation with depth

$$\sigma_{zz}(x,z) = \frac{F}{2\sqrt{\pi D z}}\exp(-x^2/4Dz)$$

Convection-diffusion/3-leg model

Applies for weak disorder

$$O^{+}O^{-}\sigma = 0$$

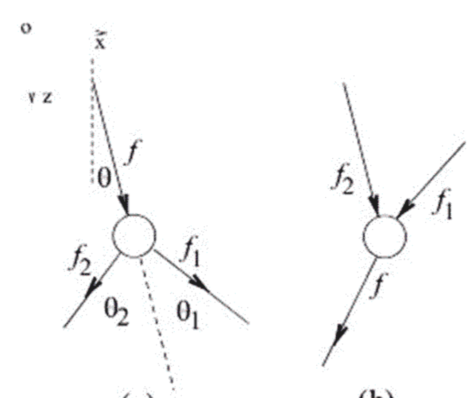
$$O^{\pm} = [\partial / \partial z \pm c\partial / \partial x - D\partial^2 / \partial x^2]$$

Expected response to a point force:

$$\sigma_{zz} = \frac{F}{2}\frac{1}{\sqrt{4\pi Dz}}\{\exp[-(x+cz)^2/4Dz] + \exp[-(x-cz)^2/4Dz]\}$$

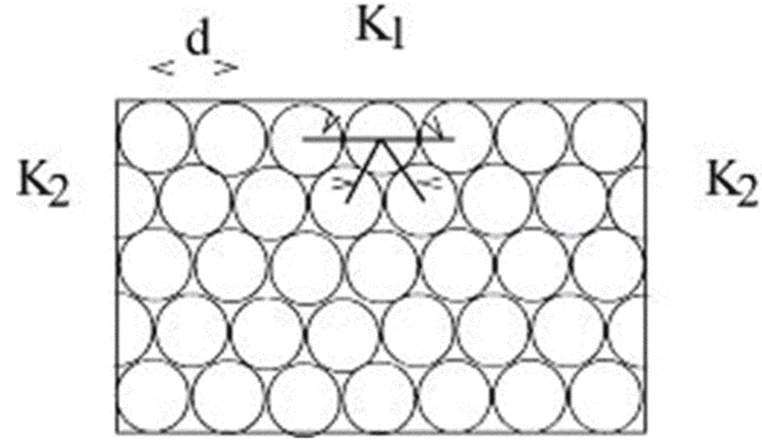
Double-Y model

Assumes Boltzmann equation for force chains



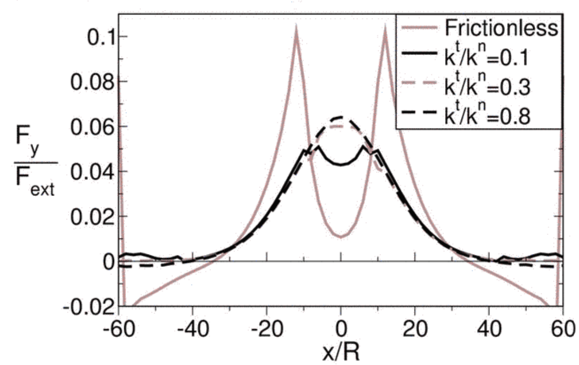
For shallow depths: One or two peaks
Intermediate depths: single peak-elastic-like
Largest depths: 2 peaks, propagative, with diffusive widening

Anisotropic elastic lattice model

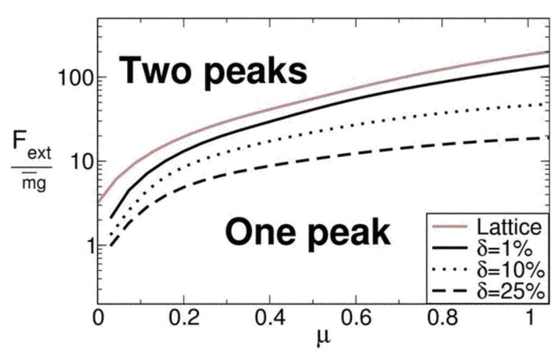


Expect progagation along lattice directions
Linear widening with depth—e.g. Goldenberg and Goldhirsch, Nature 435, 188 (2005)

Friction ‘enhances’ elasticity—e.g. tends to produce single-peaked response (Goldenberg/Goldhirsch)



Change from 2 to 1 Peak as friction increase



‘Phase diagram’ in Applied force, Friction space

Edwards Entropy-Inspired Models for P(f)

- Consider all possible states consistent with applied forces
- Compute Fraction where at least one contact force has value f→ P(f)

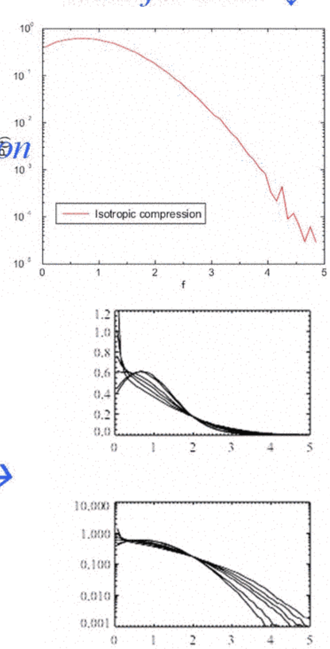
- E.g. Snoeier et al. PRL 92, 054302 (2004)
- Tighe et al. preprint (Duke University)

Some Typical Cases—isotropic compression and shear

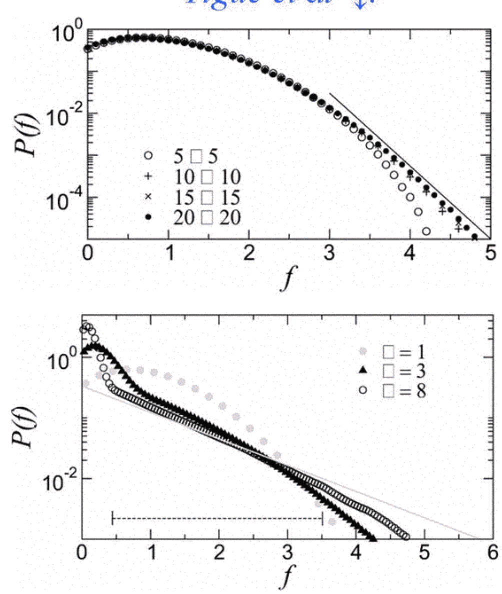
Snoeijer et al. ↓

Compression →

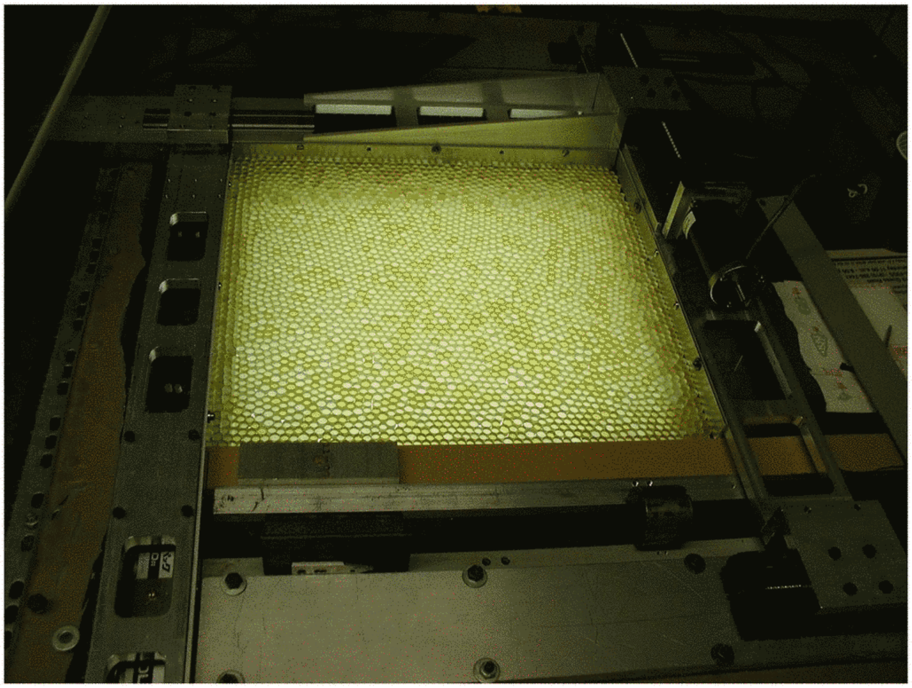
Shear →



Tigue et al ↓.



Experiments to determine vector contact forces (Trush Majmudar and RPB to appear, Nature)



Experiments Use Photoelasticity:

Biax schematic

Compression

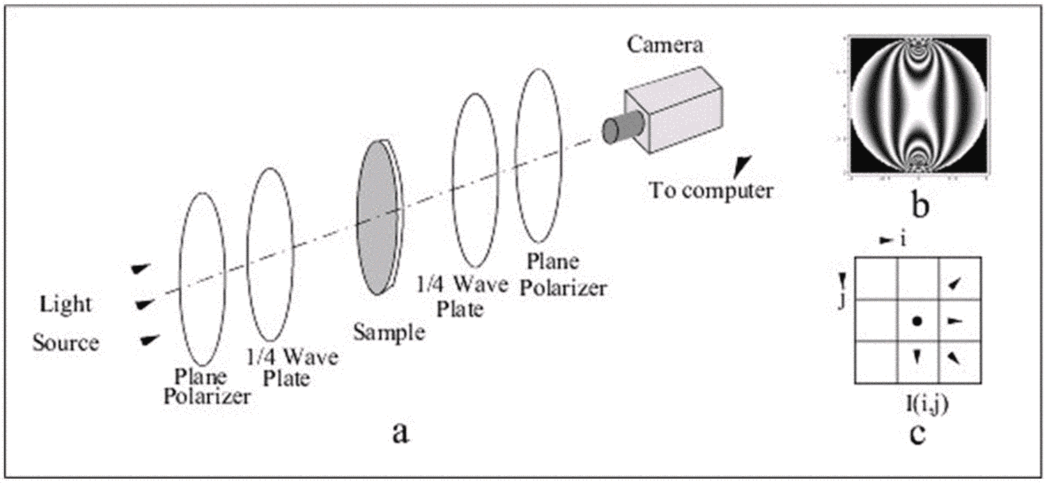
Figures temporarily not available

Image of Single disk

Shear

~2500 particles, bi-disperse, d_L*=0.9cm,* d_S*= 0.8cm,* $N_S/N_L = 4$

Measuring forces by photoelasticity



Basic principles of technique

- Process images to obtain particle centers and contacts
- Invoke exact solution of stresses within a disk subject to localized forces at circumference
- Make a nonlinear fit to photoelastic pattern using contact forces as fit parameters
- $I = I_o \sin^2[(\sigma_2 - \sigma_1)CT/\lambda]$
- In the previous step, invoke force and torque balance
- Newton's 3d law provides error checking

Examples of Experimental and 'Fitted' Images

Experiment *Fit*

Figures temporarily not available

Force distributions for shear and compression

Shear *Compression*

Figures temporarily not available

$\varepsilon_{xx} = -\varepsilon_{yy} = 0.04; \quad Z_{avg} = 3.1$

$\varepsilon_{xx} = -\varepsilon_{yy} = 0.016; \quad Z_{avg} = 3.7$

Distributions of frictional mobilization

Shear *Compression*

Figures temporarily not available

Distributions of contact angles for larger forces (Geometric anisotropy)

Shear *Compression*

Figures temporarily not available

Variation with angle of mean normal force (Force anisotropy)

Shear *Compression*

Figures temporarily not available

Spatial correlations of forces—angle dependent

Shear *Compression*

Figures temporarily not available

Chain direction

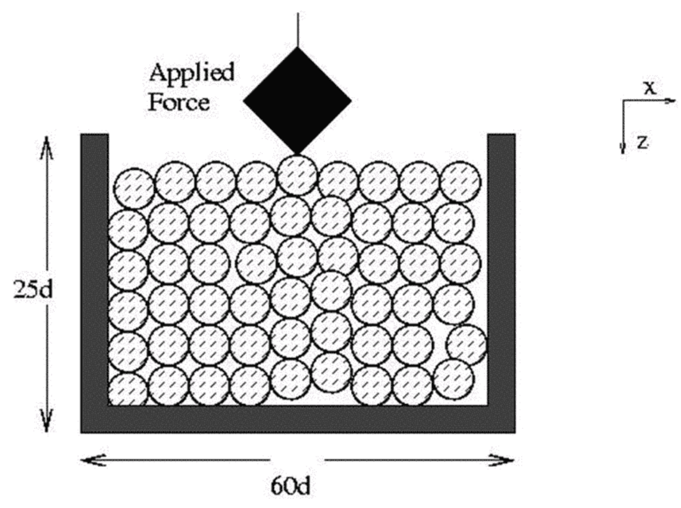
Both directions equivalent

Direction normal
To chains

Force response/transmission: what is the mechanical response to a small point force?

Physica D 182, 274 (2003)

Schematic of greens function apparatus



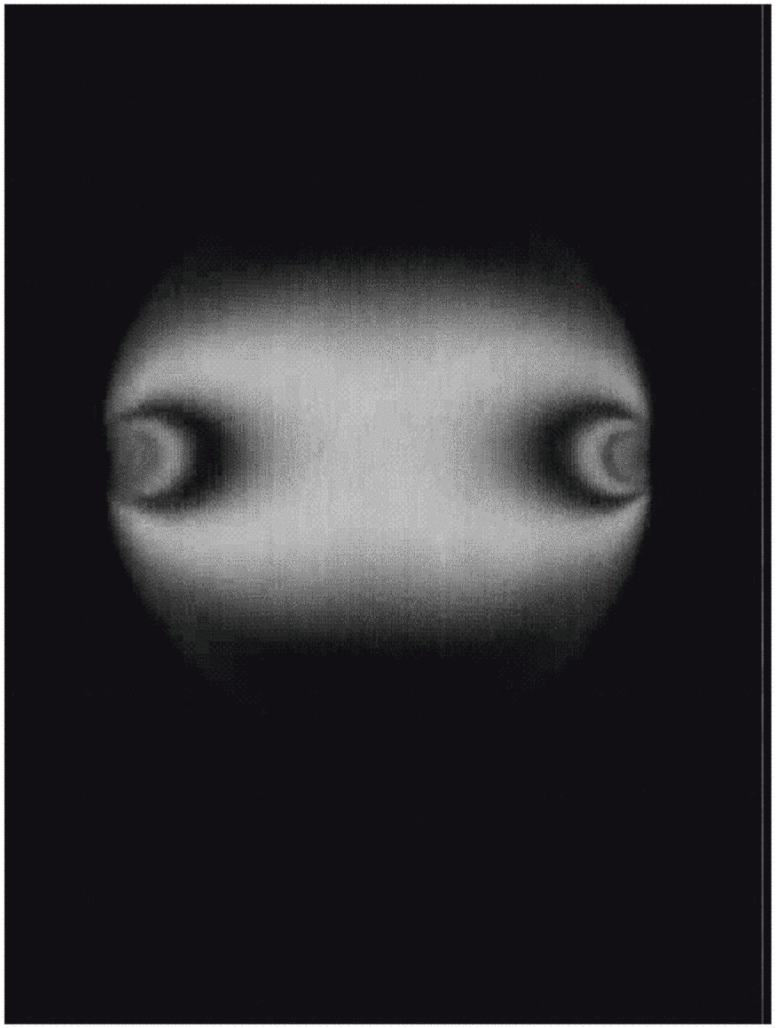
d = grain size

Use: 1) Monodisperse disks (spatially ordered)

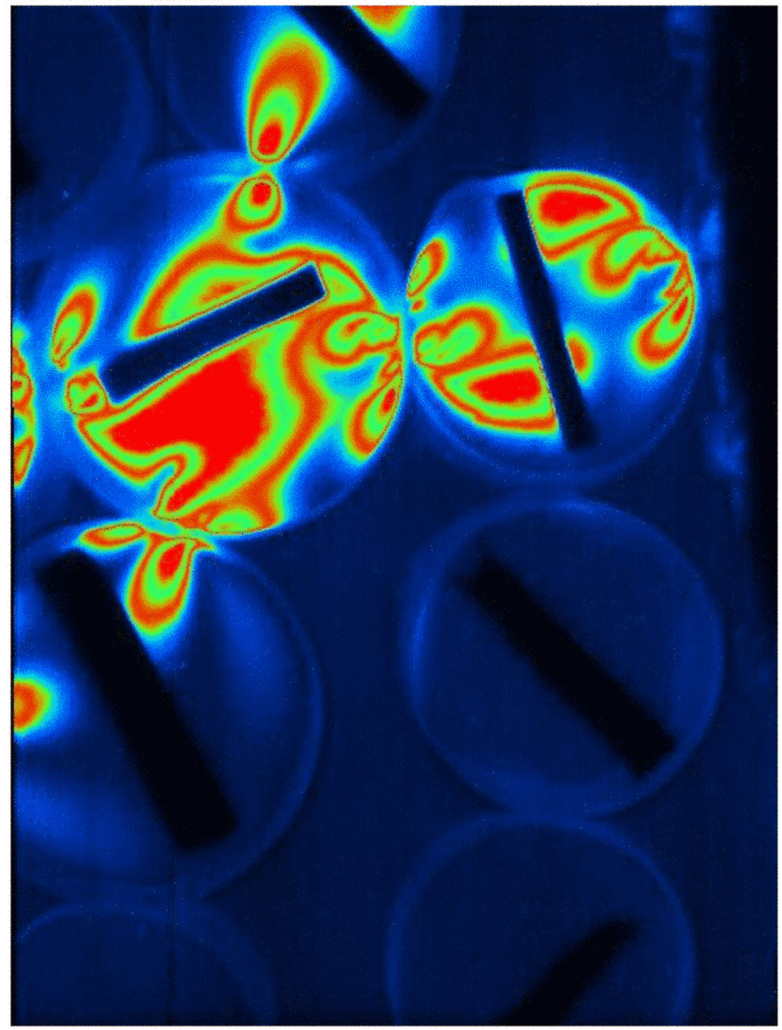
2) Bidisperse disks (weakly disordered)

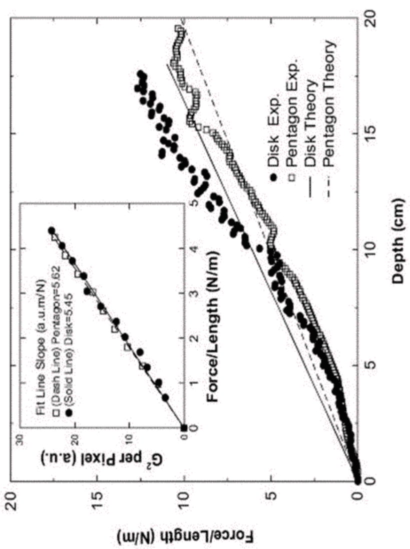
3) Pentagons (strongly disordered)

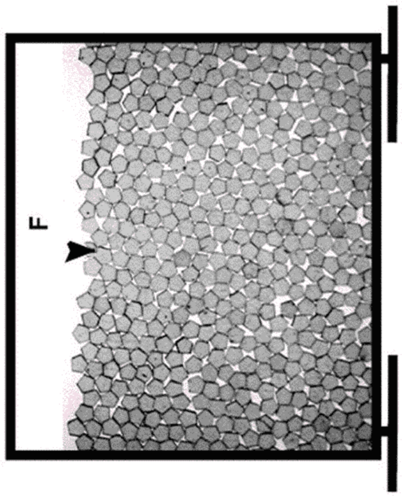
Diametrically opposed forces on a disk



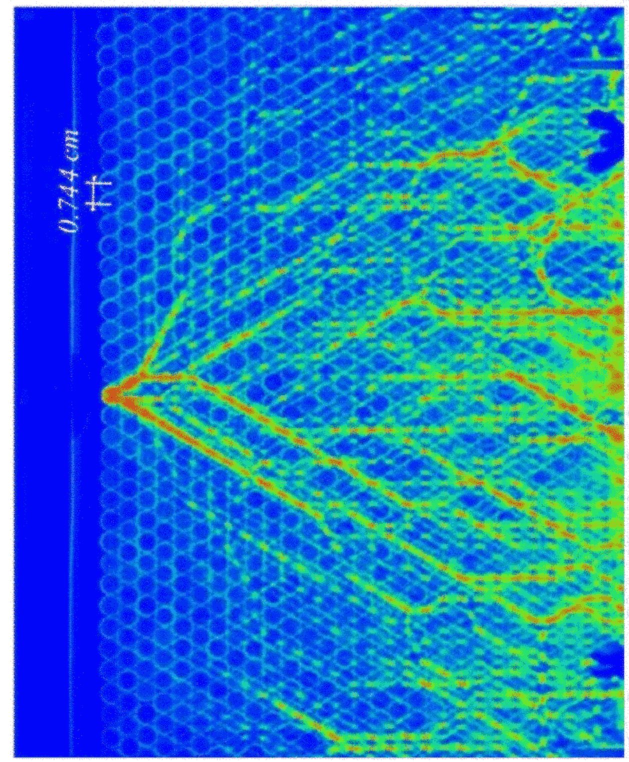
A gradient technique to obtain grain-scale forces





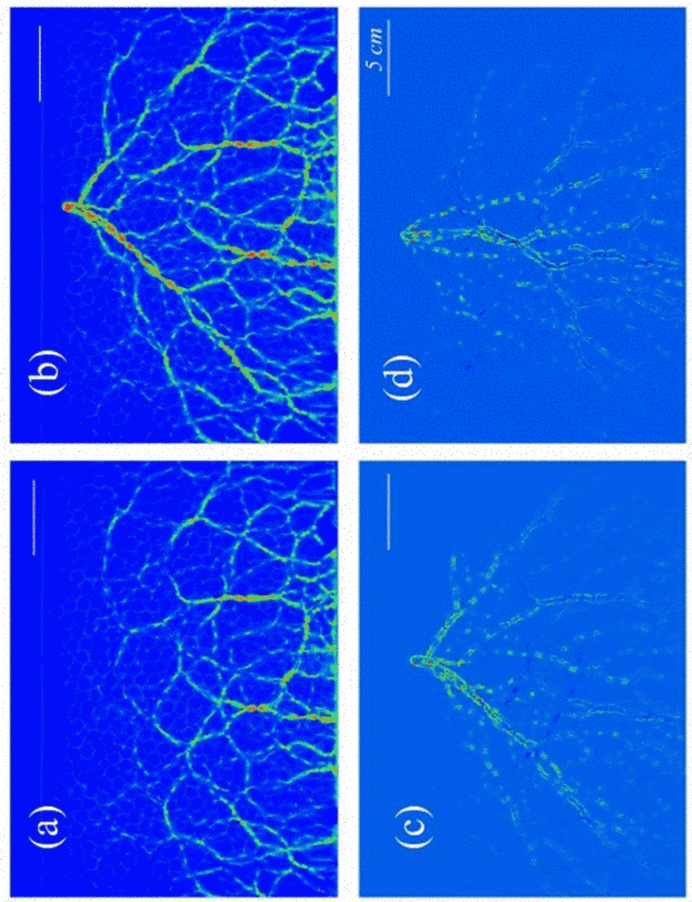


calibration

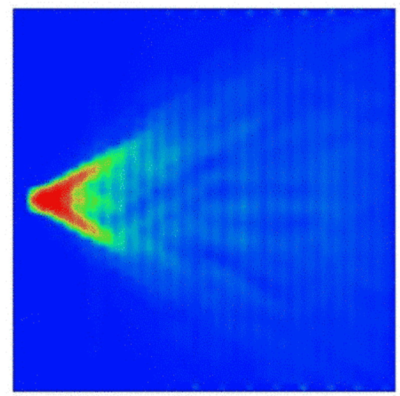


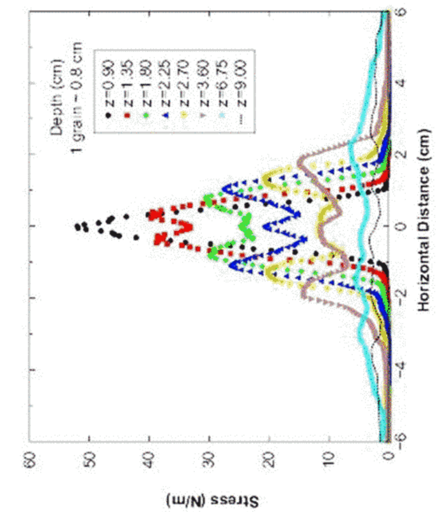
Disks-single response

Before-after

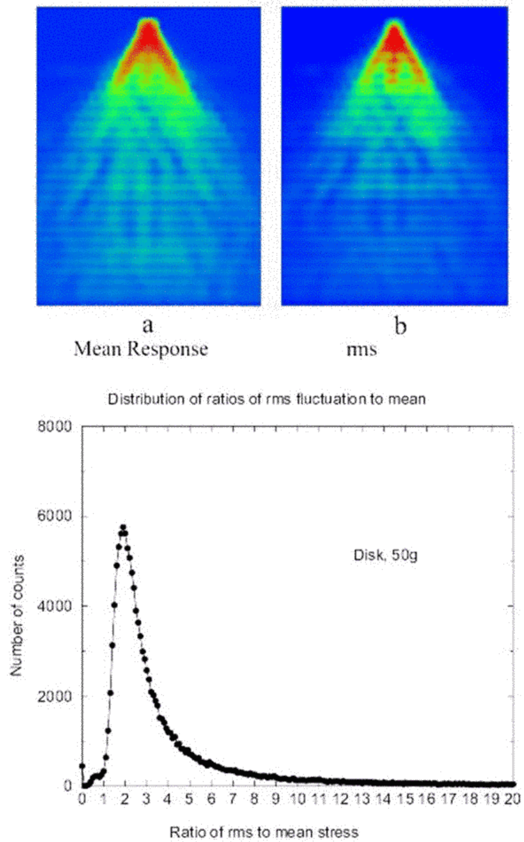


disk response mean





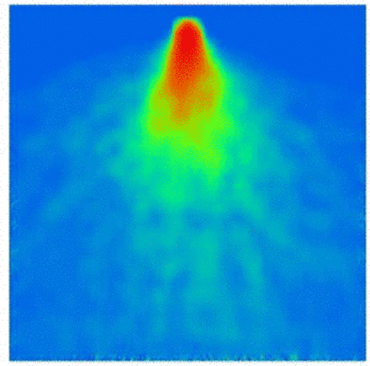
Large variance of distribution

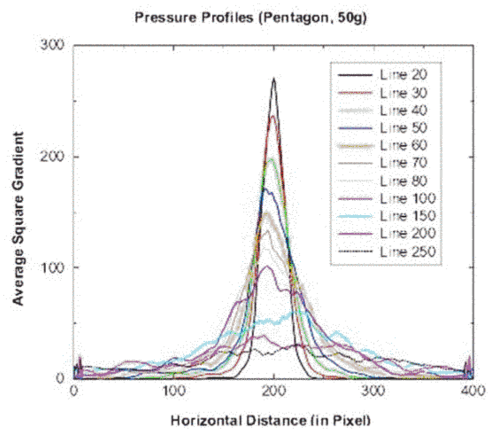


Organization of Results

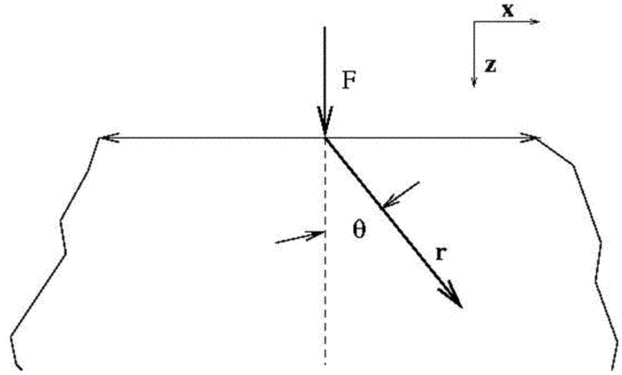
- Strong disorder: pentagons
- Varying order/disorder
  - Bidisperse disks
  - Reducing contact number: square packing
  - Reducing friction
- Comparison to convection-diffusion model
- Non-normal loading: vector/tensor effects
- Effects of texture

Pentagon response





Elastic response, point force on a semi-infinite sheet

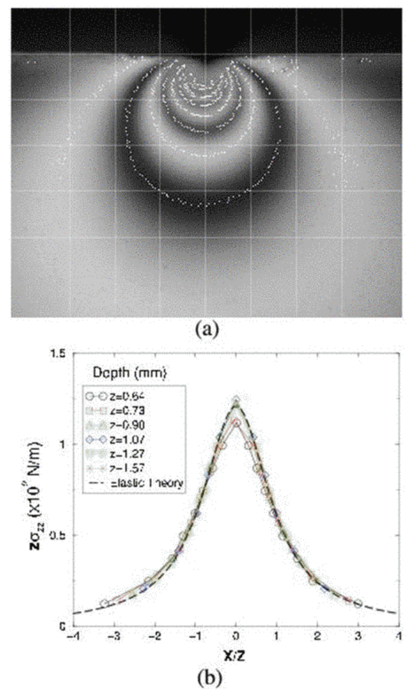


$$\sigma_{rr} = \frac{2F\cos\theta}{r\pi} \qquad \sigma_{r\theta} = \sigma_{\theta\theta} = 0$$

In Cartesian coordinates:

$$\sigma_{ii} = 1/[z(1+(x/z)^2]^p \qquad p = 1,2$$

Moment test



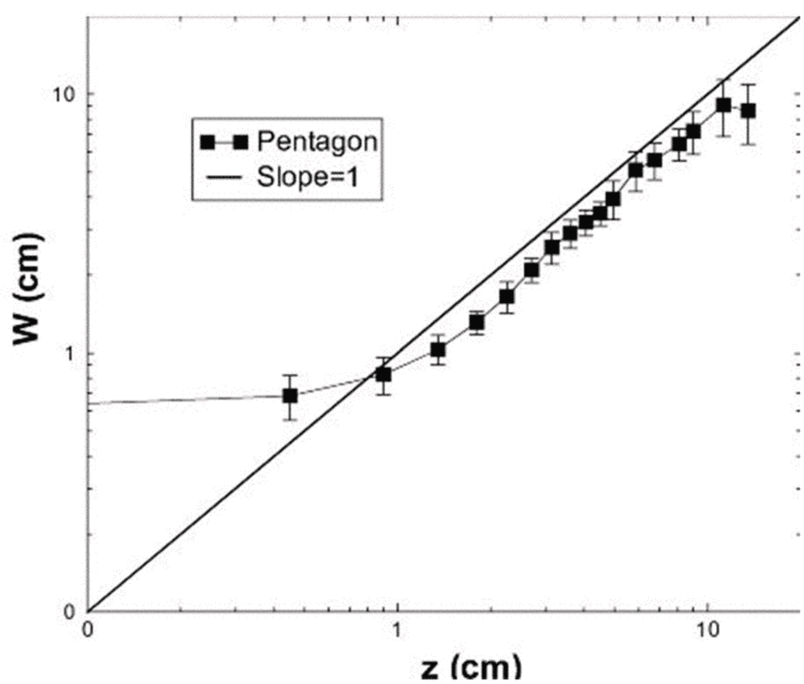
$$\sigma_{zz}(x,z) = \frac{2F}{z\pi}\frac{1}{[1+(x/z)^2]^2}$$

$$W^2 = \int x^2 \sigma_{zz}(x,z)\,dx$$

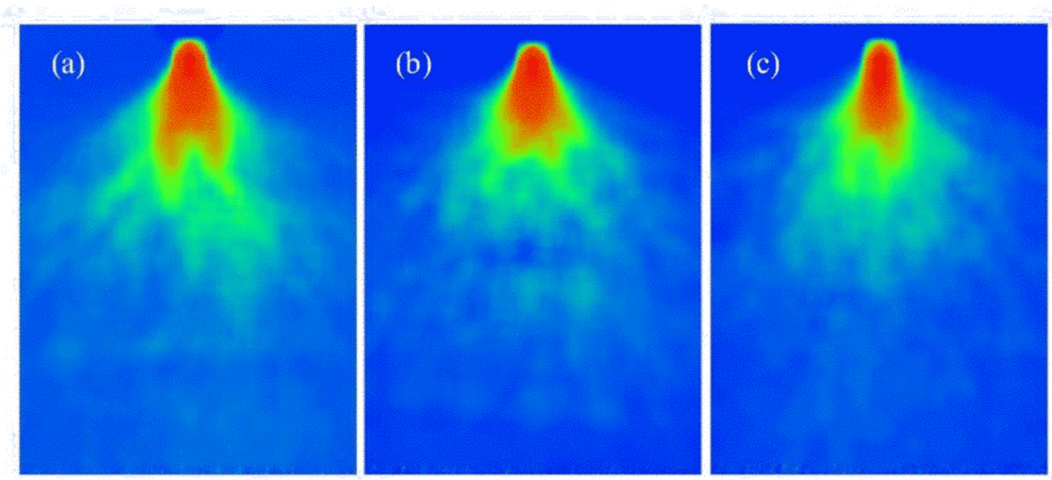
$$W(z) \propto z$$

(See Reydellet and Clement, PRL, 2001)

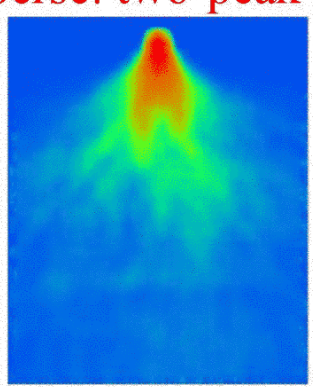
Pentagons, width vs. depth

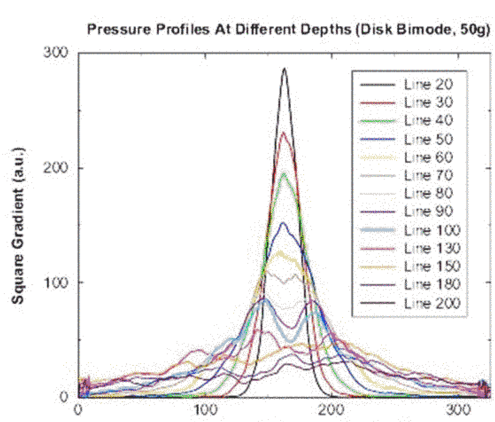


Bidisperse responses vs. A

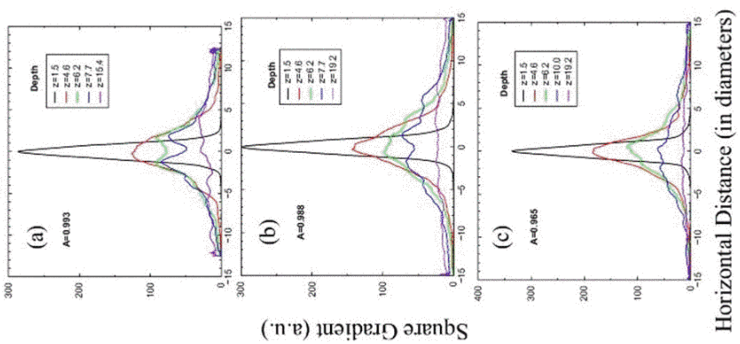


Weakly bi-disperse: two-peak structure remains

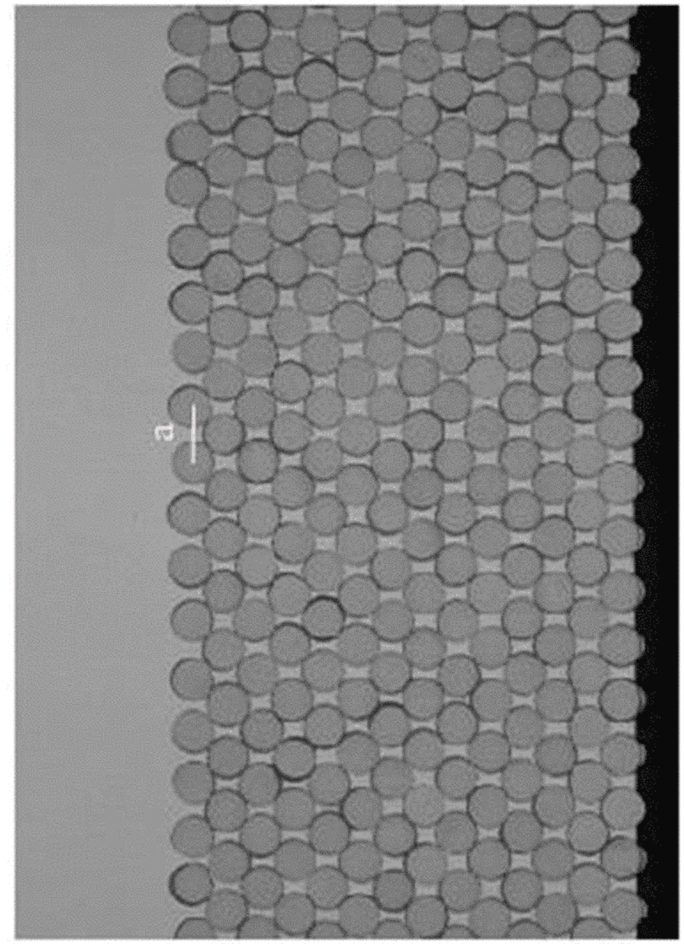




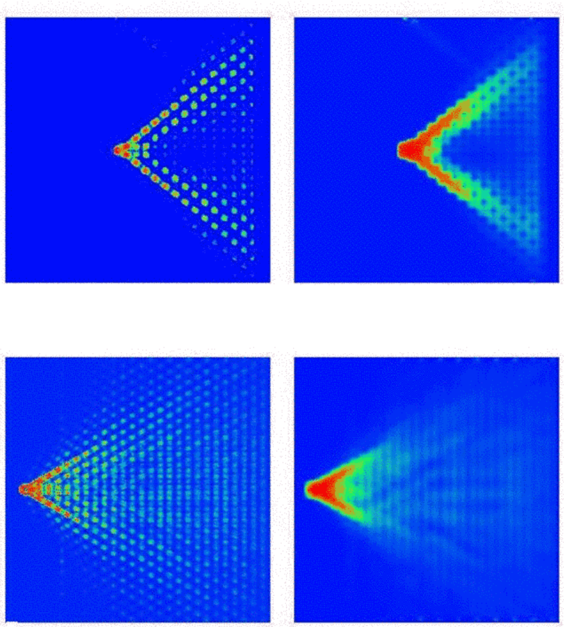
Bidisperse, data



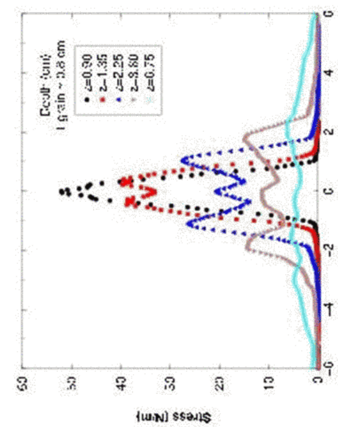
Rectangular packing reduces contact disorder



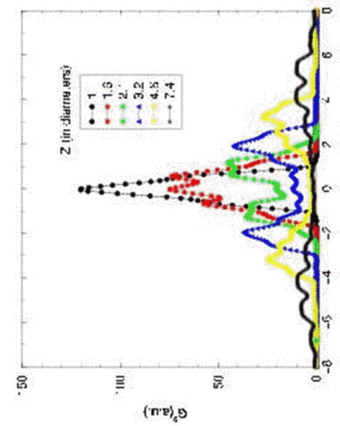
Hexagonal vs. square packing



Hexagonal vs. square, data

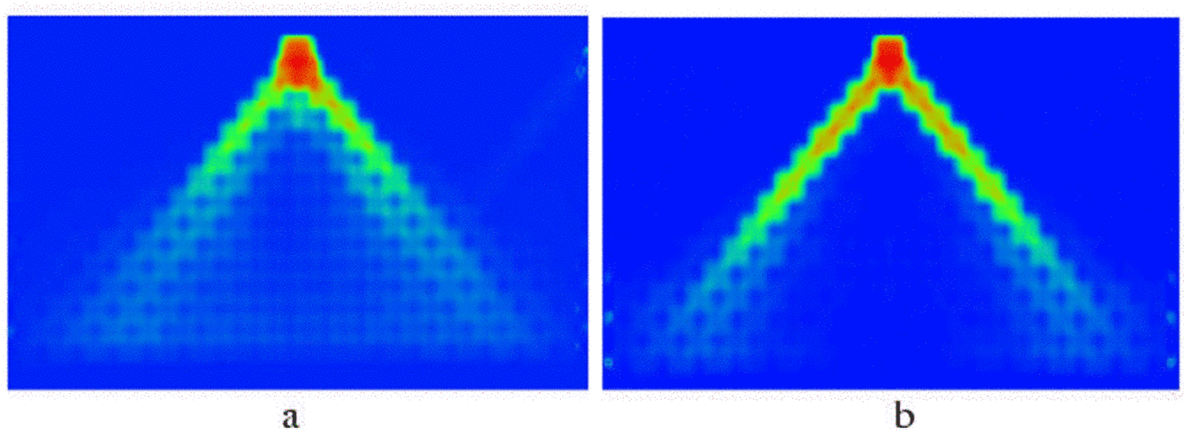


Hexagonal Packing

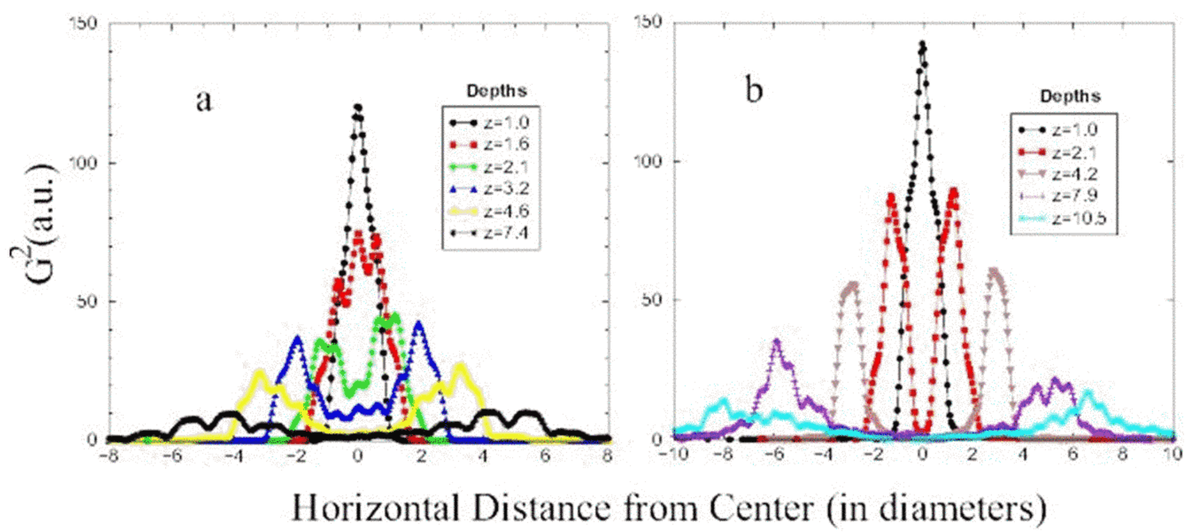


Square–lattice Packing

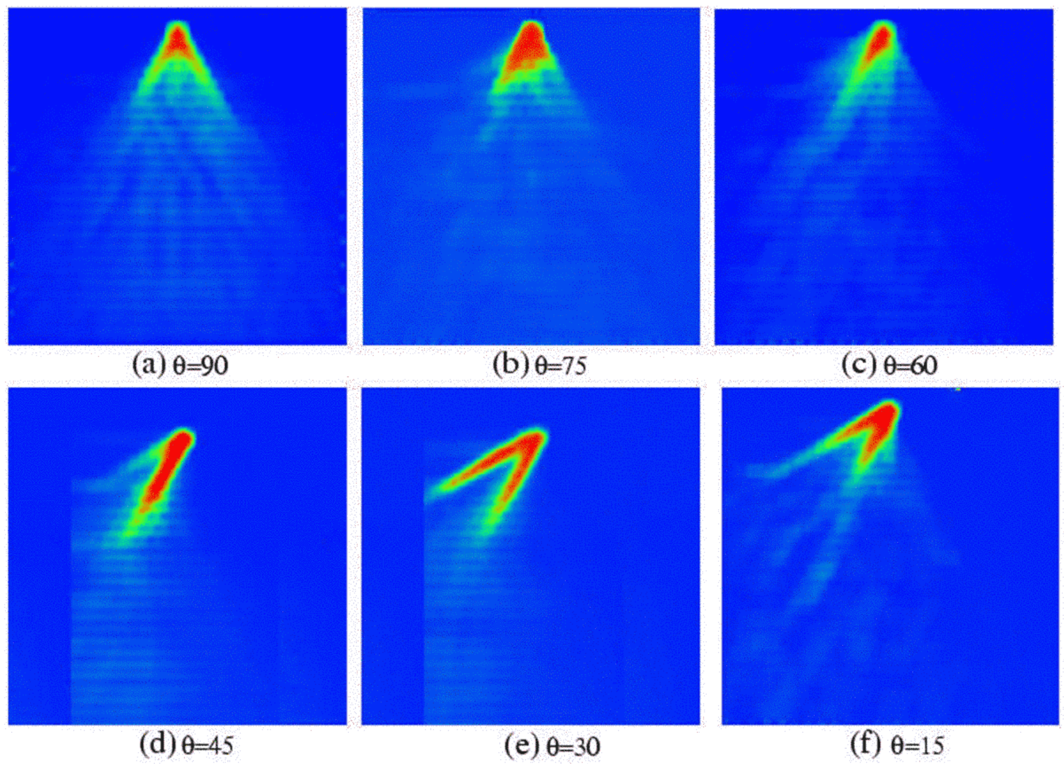
Square packs, varying friction



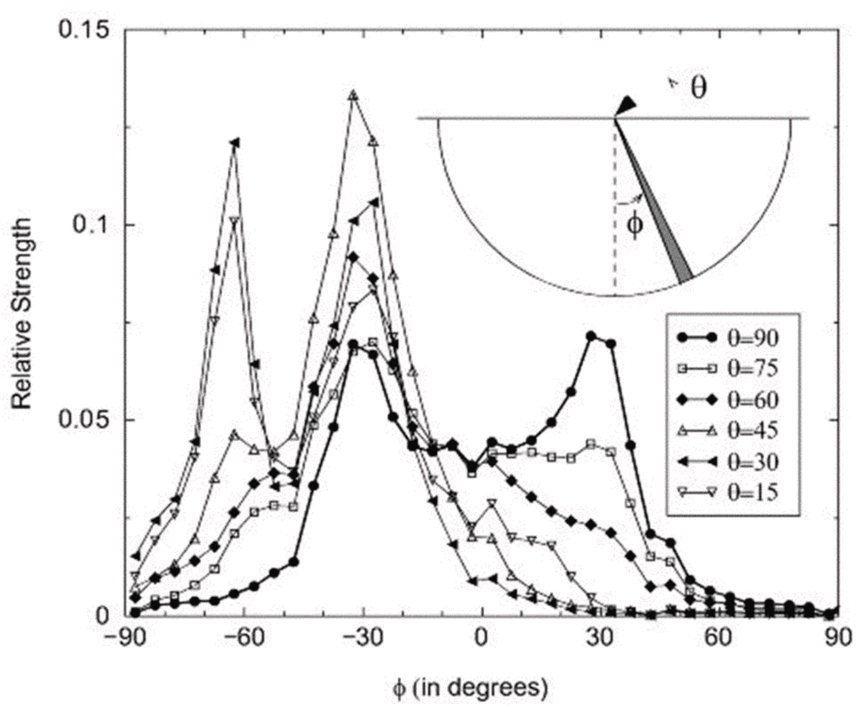
Data for rectangular packings



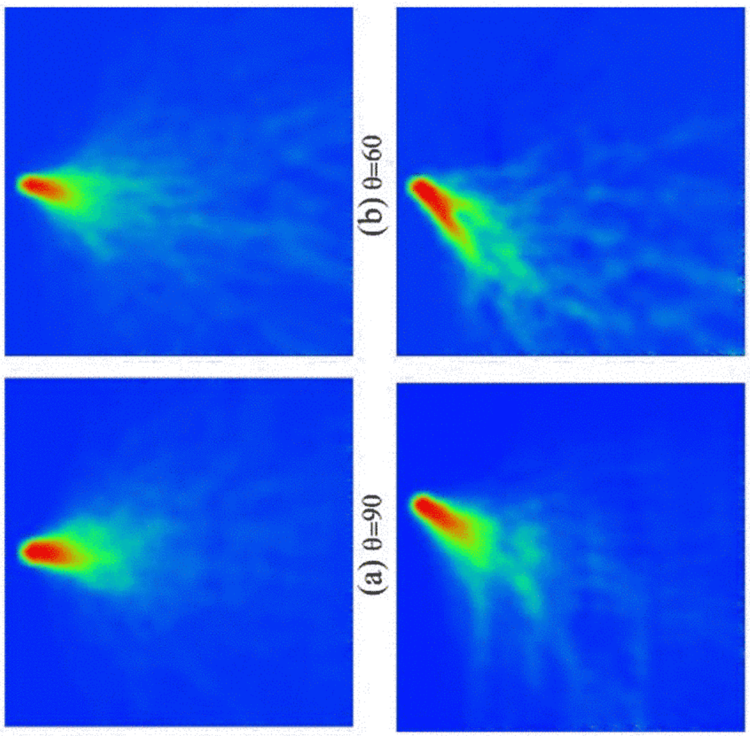
Non-normal response, disks, various angles



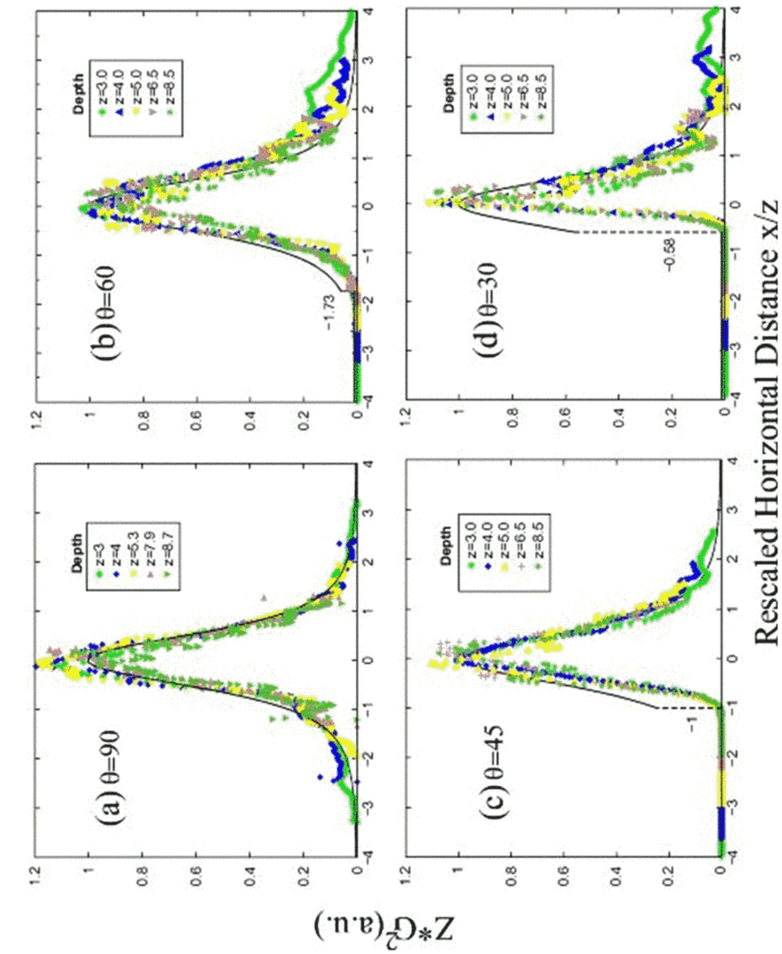
Non-normal response vs. angle of applied force

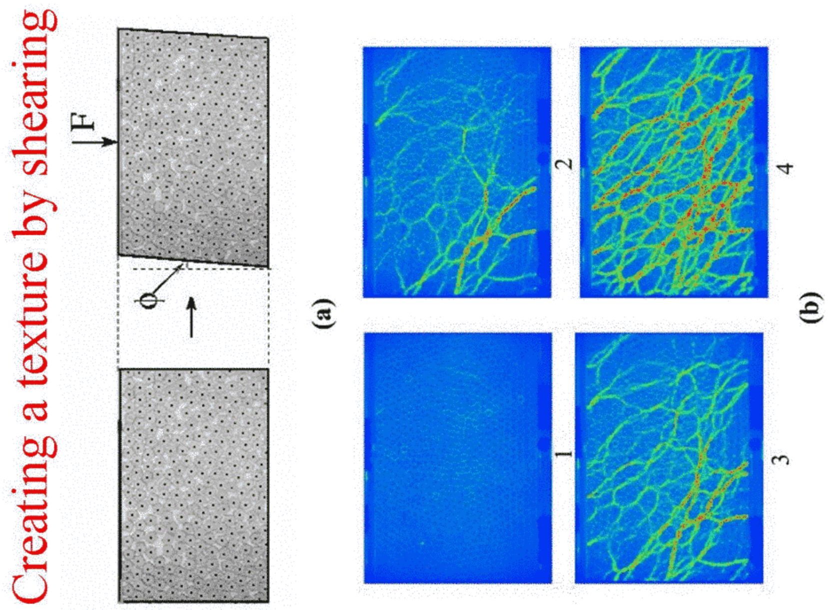


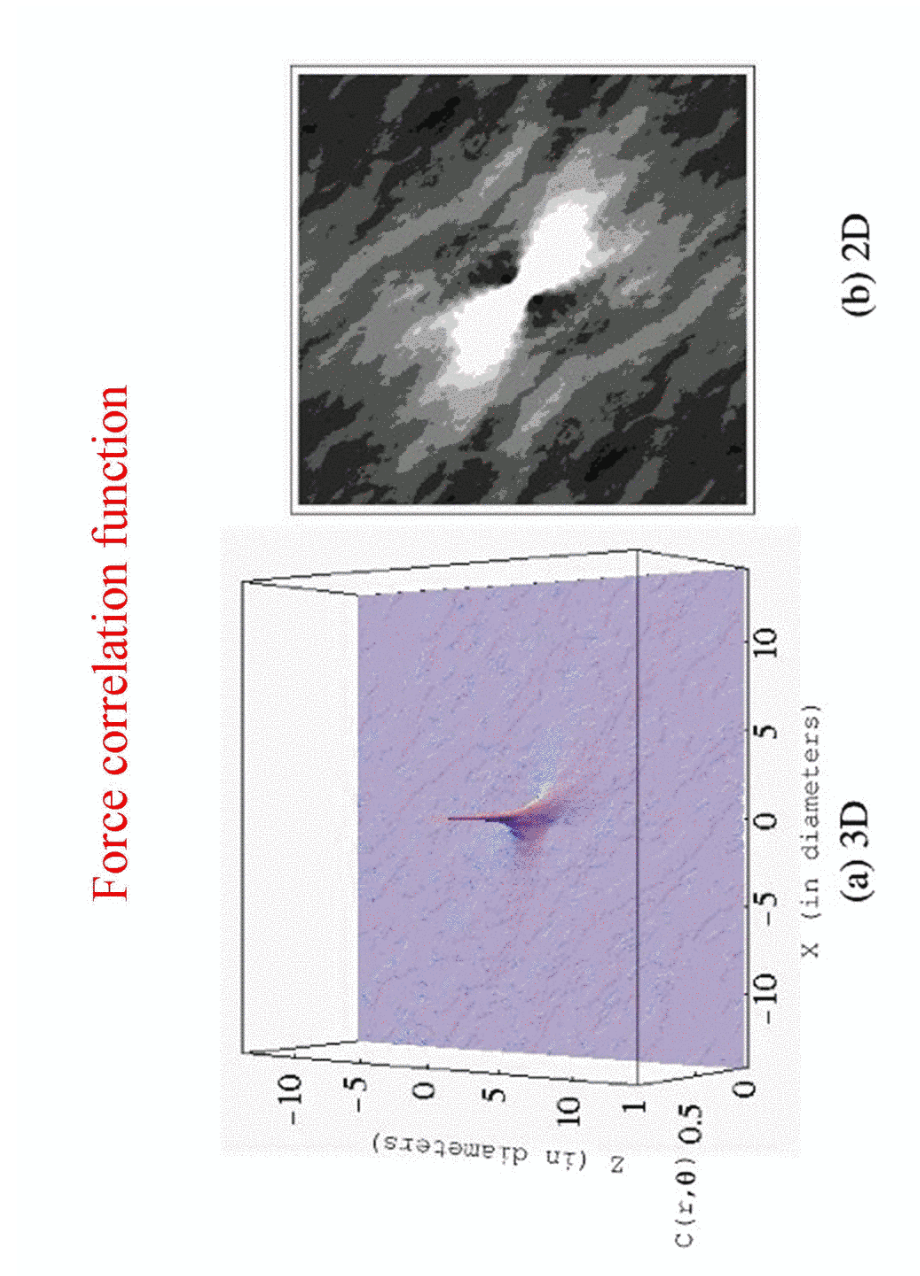
Non-normal responses, pentagons



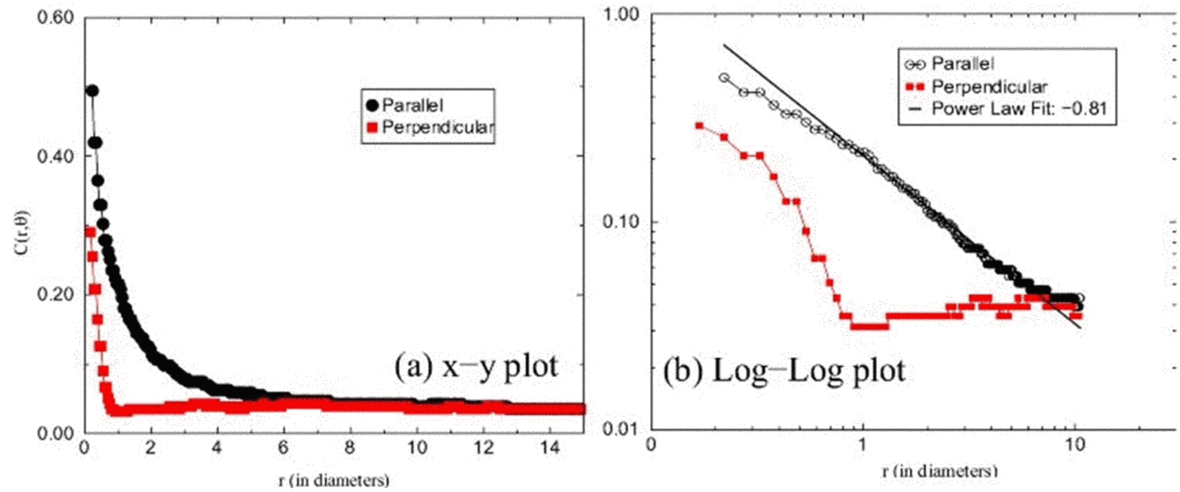
Non-normal response, pentagons, rescaled



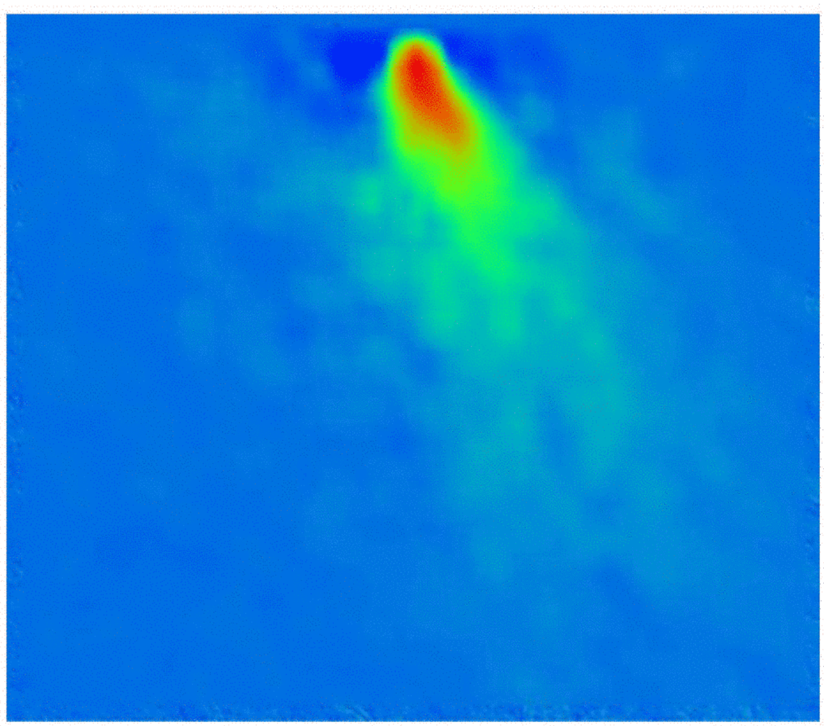

Creating a texture by shearing
F
φ
(a)
1
2
3
4
(b)


Force correlation function
C(r,θ)
Z (in diameters)
X (in diameters)
(a) 3D
(b) 2D

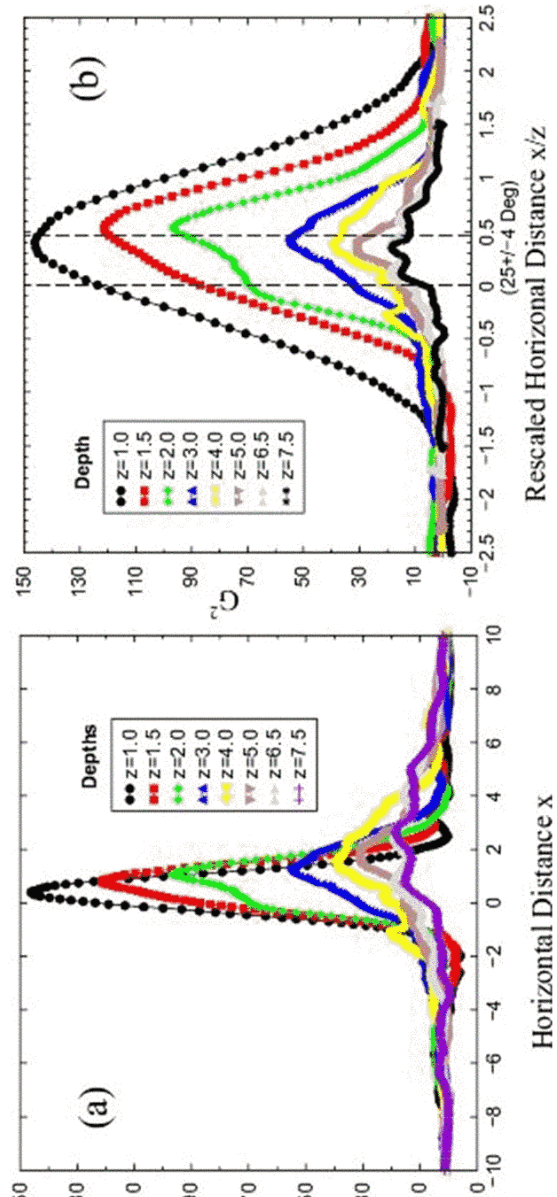
Correlation functions along specific directions



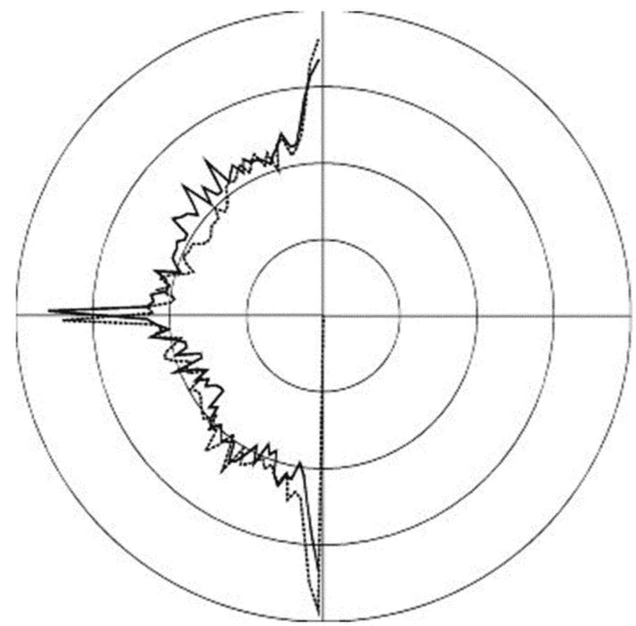
Response in textured system



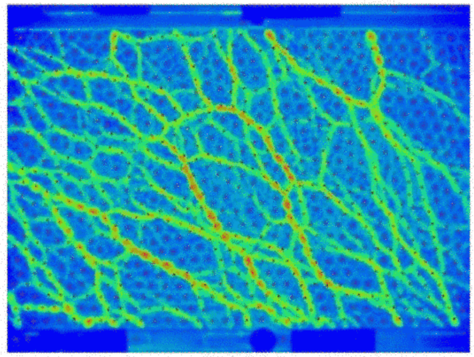
Response, textured system, data



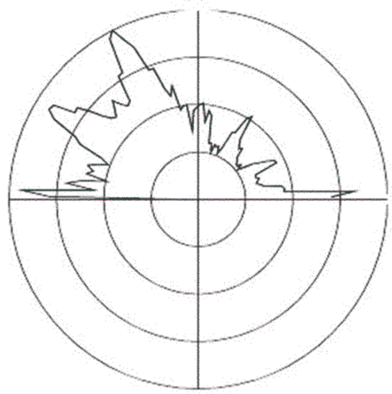
Fabric in textured system



"Fabric" from strong network



(a)

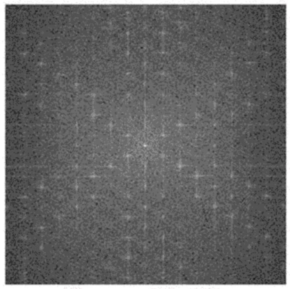


(b)

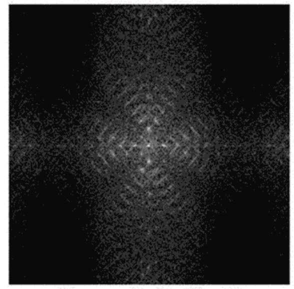
Conclusions

- Normal force distributions are sensitive to stress state
- Long-range correlations for forces in sheared systems
- Strong effects on transmission from order/disorder (spatial and force-contact)
- Ordered systems: propagation along lattice—competing viewpoints—Goldhirsch (elastic) vs. Bouchaud (scattering of propagating waves)
- Disorderd: roughly elastic response
- Textured systems
  - Power law correlation along preferred direction
  - Forces tend toward preferred direction

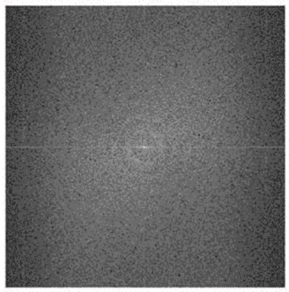
Spectra of particle density



Hexagonal Packing



Square–lattice Packing



Pentagonal Packing