The Eikonal Exponentiation and Soft Theorems

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Based on
2008.12743, 2101.05772, 2104.03256,
2105.04594, 2203.11915, 2204.02378
in various collaborations with
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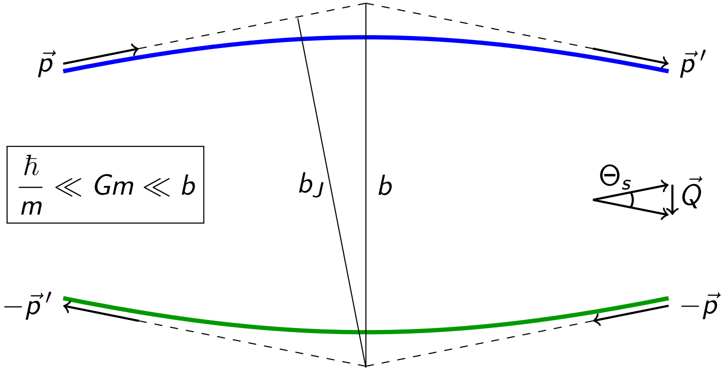
Outline

Outline

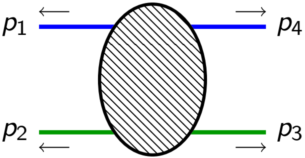
PM Black-Hole Scattering



$$\boxed{\frac{\hbar}{m} \ll Gm \ll b}$$

$$p\, b_J = J\,, \qquad Q = 2p\,\sin\frac{\Theta_s}{2}\,, \qquad s = E^2 = m_1^2 + 2m_1 m_2\,\sigma + m_2^2\,.$$

The Elastic Eikonal



$$s = -(p_1 + p_2)^2 = E^2 \,,$$
$$t = -(p_1 + p_4)^2 = -q^2 \,,$$
$$u = 2(m_1^2 + m_2^2) - s - t \,.$$

From q to b:

$$\tilde{\mathcal{A}}(s,b) = \frac{1}{4Ep} \int \frac{d^{D-2}q}{(2\pi)^{D-2}}\, e^{ib\cdot q} \mathcal{A}(s,q)\,, \qquad 1 + i\tilde{\mathcal{A}}(s,b) = e^{2i\delta(s,b)}\,,$$

in the classical limit, and the stationary-phase approximation gives

$$Q_\mu = \frac{\partial \,\mathrm{Re}\, 2\delta}{\partial b^\mu}\,.$$

Example: 1PM scattering in $D = 4 - 2\epsilon \to 4$

$$2\delta_0 = \frac{4Gm_1 m_2 \left(\sigma^2 - \frac{1}{2}\right)}{2\sqrt{\sigma^2 - 1}} \frac{\Gamma(-\epsilon)}{(\pi b^2)^{-\epsilon}} \implies Q_{1\mathrm{PM}} = \frac{4Gm_1 m_2 \left(\sigma^2 - \frac{1}{2}\right)}{b\sqrt{\sigma^2 - 1}}\,.$$

The 3PM Eikonal [2008.12743, 2101.05772, 2104.03256]

[Related work at 3PM: 1901.04424, 2005.04236, 2010.01641, 2104.03957, 2104.04510, 2105.05218, 2108.04216]

- Eikonal phase

$$\mathrm{Re}\, 2\delta_2 = \frac{4G^3 m_1^2 m_2^2}{b^2}\left\{ \frac{s\left(12\sigma^4 - 10\sigma^2 + 1\right)}{2m_1 m_2 \left(\sigma^2 - 1\right)^{\frac{3}{2}}} + \frac{2\left(\sigma - \frac{1}{2}\right)^2}{\sigma^2 - 1}\mathcal{I}(\sigma) - \frac{\sigma\left(14\sigma^2 + 25\right)}{3\sqrt{\sigma^2 - 1}} + \frac{-4\sigma^4 + 12\sigma^2 + 3}{\sigma^2 - 1}\,\mathrm{arccosh}\sigma \right\}$$

(color code: probe limit, potential, radiation reaction) with

$$\mathcal{I}(\sigma) = \frac{8 - 5\sigma^2}{3(\sigma^2 - 1)} + \frac{\sigma\left(2\sigma^2 - 3\right)}{\left(\sigma^2 - 1\right)^{\frac{3}{2}}}\,\mathrm{arccosh}\,\sigma\,.$$

- Infrared divergent exponential suppression:

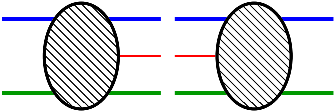
$$\mathrm{Im}\, 2\delta_2 = \frac{4G^3 m_1^2 m_2^2}{\pi\, b^2}\left[-\frac{1}{\epsilon} + \log(\sigma^2 - 1)\right]\frac{2\left(\sigma - \frac{1}{2}\right)^2}{\sigma^2 - 1}\mathcal{I}(\sigma) + \cdots\,,$$

where $\cdots$ are either $\mathcal{O}(\epsilon)$ or have no branch point at $\sigma = 1$.

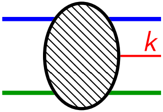
3PM Radiation Reaction from Soft Theorems [2101.05772]

- Analyticity: $i\log(1-\sigma^2 - i0) = i\log(\sigma^2-1)+\pi$
- Unitarity: $\text{Im}\, 2\delta_2 = [\text{Im}\,\tilde{\mathcal{A}}_2]_{3p.c.}$ and

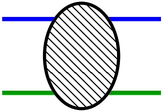
$$[\text{Im}\, 2\mathcal{A}]_{3p.c.} = \int d(\text{LIPS})$$



- Soft theorem: [Weinberg'64,'65]



$$\mu\nu \sim \left[\sum_n \frac{\kappa\, p_n^\mu p_n^\nu}{p_n\cdot k}\right]$$



For $\text{Im}\, 2\delta_2$ this gives $\frac{8G^3 m_1^2 m_2^2}{\pi\, b^2}\frac{2\left(\sigma-\frac{1}{2}\right)^2}{\sigma^2-1}\mathcal{I}(\sigma)$ times

$$\int_0^{\omega_{\max} b}\frac{d\omega}{\omega^{1+2\epsilon}} \sim -\frac{1}{2\epsilon}+\log(\omega_{\max} b) \sim \frac{1}{2}\left[-\frac{1}{\epsilon}+\log(\sigma^2-1)\right].$$

Smoothness of $\mathrm{Re}\, 2\delta_2$ at High Energy

- The infrared divergence in $\mathrm{Im}\, 2\delta_2$:

$$\mathrm{Im}\, 2\delta_2 = -\frac{G}{2\pi\epsilon}\, Q_{1\mathrm{PM}}^2\, \mathcal{I}(\sigma) + \mathcal{O}(\epsilon^0)\,.$$

- It determines the radiation-reaction term in $\mathrm{Re}\, 2\delta_2$,

$$\mathrm{Re}\, 2\delta_2^{RR} = \lim_{\epsilon\to 0} \left[-\pi\epsilon\; \mathrm{Im}\, 2\delta_2\right].$$

- At high energy, as $\sigma \to \infty$ and $s \sim 2m_1 m_2 \sigma$, i.e. in the massless limit, the complete eikonal phase is smooth

$$\mathrm{Re}\, 2\delta_2 \sim Gs\, \frac{\Theta_s^2}{4}\,, \qquad \Theta_s \sim \frac{4G\sqrt{s}}{b}$$

in agreement with [Amati,Ciafaloni,Veneziano'90].

Outline

Zero-Frequency Limit of the Energy Emission Spectrum

- ZFL of the energy emission spectrum:

$$W \equiv \lim_{\omega\to 0} \frac{dE_{\text{rad}}}{d\omega} = \lim_{\epsilon\to 0} \left[-4\epsilon \, \text{Im}\, 2\delta\right].$$

- To 3PM order,

$$W = \frac{2G}{\pi} \, Q_{\text{1PM}}^2 \, \mathcal{I}(\sigma) \, .$$

- At high energy, as $\sigma \to \infty$ and $s \sim 2m_1 m_2 \sigma$,

$$W \sim \frac{4G}{\pi} \, Q_{\text{1PM}}^2 \log \sigma \, .$$

Not smooth!

Infrared Divergences and the ZFL of the Spectrum

- The soft theorem determines the infrared divergences for any background hard process. [Weinberg'64,'65] [see also 2105.04594]

  $\mathcal{A} = e^{\mathcal{W}} \mathcal{A}^0$ with

$$\mathcal{W} = \frac{G}{2\pi\epsilon} \sum_{n,m} m_n m_m \left(\sigma_{nm}^2 - \tfrac{1}{2}\right) \frac{\eta_n \eta_m \operatorname{arccosh} \sigma_{nm} - i\pi\delta(n,m)}{\sqrt{\sigma_{nm}^2 - 1}} \,,$$

  where $\eta_n = +1$ if n is outgoing, $\eta_n = -1$ if n is incoming,

$$\sigma_{nm} = -\eta_n \eta_m \frac{p_n \cdot p_m}{m_n m_m} \,, \qquad \delta(n,m) = \delta_{\eta_n,\eta_m} - \delta_{nm} \,.$$

- This expression is independent of the details of the hard particles and only depends on their momenta.
- ZFL of the spectrum:

$$W \equiv \lim_{\omega \to 0} \frac{dE_{\text{rad}}}{d\omega} = \lim_{\epsilon \to 0} \left[4\epsilon \operatorname{Re} \mathcal{W}\right].$$

Eikonal Operator in the ZFL [2204.02378]

[Soft dressing: Bloch,Nordsieck'37, Thirring,Touschek'51, Weinberg'65, 1607.03120, 1712.04551, 2012.04208;
Operator exponentiation: 2107.12891, 2112.07556; Classical soft theorems: 1806.01872, 1808.03288, 1912.06413, 2105.08739.]

Operator dressing of the elastic eikonal in b space

$$S_{s.r.} = e^{\int_{\vec{k}} \left[f^{\mu\nu}(k) a^{\dagger}_{\mu\nu}(k) - f^{*\mu\nu}(k) a_{\mu\nu}(k) \right]} e^{i\,\mathrm{Re}\,2\delta} \,.$$

- Here $f^{\mu\nu}(k)$ is dictated by the Weinberg soft theorem ($\kappa = \sqrt{8\pi G}$)

$$f^{\mu\nu}(k) = \Pi^{\mu\nu}_{\rho\sigma}(k)\, F^{\rho\sigma}(k)\,, \qquad F^{\mu\nu}(k) = \sum_n \frac{\kappa\, p_n^\mu p_n^\nu}{p_n \cdot k}\,,$$

while $\int_{\vec{k}} = \int \frac{d^D k}{(2\pi)^D} 2\pi\delta(k^2)\theta(k^0)\theta(\Lambda - k^0)$, with Λ the cutoff, and $\Pi^{\mu\nu}_{\rho\sigma}$ is the TT projector.

- This step introduces the key identification

$$Q_\mu = e^{-i\,\mathrm{Re}\,2\delta} \left(-i\, \frac{\partial}{\partial b^\mu} \right) e^{i\,\mathrm{Re}\,2\delta} = \frac{\partial\, \mathrm{Re}\,2\delta}{\partial b^\mu}\,.$$

Using the Soft Eikonal Operator

- Infrared divergences: $\langle 0|S_{s.r.}|0\rangle = e^{2i\delta}$,

$$\mathrm{Im}\, 2\delta = \frac{1}{2}\int_{\vec{k}} F^{\mu\nu}\left(\eta_{\mu\rho}\eta_{\nu\sigma} - \tfrac{1}{D-2}\,\eta_{\mu\nu}\eta_{\rho\sigma}\right) F^{\rho\sigma} = -\,\mathrm{Re}\,\mathcal{W}\,.$$

- Probability of n soft emissions and average number of soft quanta

$$\mathcal{P}_n = \frac{1}{n!}\int_{\vec{k}_1}\cdots\int_{\vec{k}_n}\langle 0|S^\dagger_{s.r.}|n\rangle\langle n|S_{s.r.}|0\rangle = \frac{1}{n!}\,\left[2\,\mathrm{Im}\,2\delta\right]^n e^{-2\,\mathrm{Im}\,2\delta}\,,$$

$$\langle 0|S^\dagger_{s.r.}N\,S_{s.r.}|0\rangle = \mathcal{N}\,,\qquad \mathcal{N} = 2\,\mathrm{Im}\,2\delta\,.$$

- Energy and momentum in the ZFL:

$$\langle 0|S^\dagger_{s.r.}P^\mu S_{s.r.}|0\rangle = \mathcal{P}^\mu_{\mathrm{rad}}\,,\qquad W \equiv \lim_{\omega\to 0}\frac{dE_{\mathrm{rad}}}{d\omega} = \lim_{\epsilon\to 0}\left[-4\epsilon\ \mathrm{Im}\,2\delta\right].$$

ZFL of the Spectrum for a $2 \to 2$ Process

Let us define $\sigma_Q = \sigma - \frac{Q^2}{2m_1 m_2}$, so that

$$s = m_1^2 + 2m_1 m_2\,\sigma + m_2^2\,, \quad t = -Q^2\,, \quad u = m_1^2 - 2m_1 m_2\,\sigma_Q + m_2^2\,.$$

ZFL of the spectrum, with an exact dependence on σ, Q and $m_{1,2}$

$$W = \frac{4G}{\pi}\Bigg\{2m_1 m_2\left(\sigma^2 - \tfrac{1}{2}\right)\frac{\operatorname{arccosh}\sigma}{\sqrt{\sigma^2-1}} - 2m_1 m_2\left(\sigma_Q^2 - \tfrac{1}{2}\right)\frac{\operatorname{arccosh}\sigma_Q}{\sqrt{\sigma_Q^2-1}}$$

$$+ \sum_{j=1,2}\left[\frac{m_j^2}{2} - m_j^2\left(\left(1+\tfrac{Q^2}{2m_j^2}\right)^2 - \tfrac{1}{2}\right)\frac{\operatorname{arccosh}\left(1+\frac{Q^2}{2m_j^2}\right)}{\sqrt{\left(1+\frac{Q^2}{2m_j^2}\right)^2-1}}\right]\Bigg\}.$$

$$= \lim_{\epsilon\to 0}\left[-4\epsilon\ \operatorname{Im} 2\delta\right]$$

PM Expansion at Low and High Energy

- Standard PM regime: when $Q^2 \ll m_{1,2}^2$, we recover [2101.05772]

$$W \sim \frac{2G}{\pi} Q^2 \mathcal{I}(\sigma)\,, \qquad \operatorname{Im} 2\delta \sim -\frac{G}{2\pi\epsilon} Q^2 \mathcal{I}(\sigma)\,.$$

This gives W, $\operatorname{Im} 2\delta$ up to $\mathcal{O}(G^5)$, and hence $\operatorname{Re} 2\delta_2^{RR}$, for any background hard, including the spinning case [2203.13272].

- UR regime: when $m_{1,2}^2 \ll Q^2 < s$, we recover [1901.10986]

$$W \sim \frac{4G}{\pi}\left[s \log \frac{s}{s-Q^2} + Q^2 \log \frac{s-Q^2}{Q^2}\right]$$

and when $Q \sim \frac{\sqrt{s}}{2}\Theta_s$ with $\Theta_s \sim \frac{4G\sqrt{s}}{b} \ll 1$ [2105.08739]

$$W \sim \frac{4G}{\pi}\left(\frac{\Theta_s}{2}\right)^2 \left[\log\left(\frac{2}{\Theta_s}\right)^2 + 1\right].$$

- Sharp convergence radius of the PM expansion: $\boxed{Q < 2m_{1,2}}$ (agrees with the bound in [Kovacs,Thorne'77,'78, D'Eath'78]).

Outline

$-i0$ Prescription

- Stress-energy tensor for an idealized scattering process:

  [Weinberg'72,1702.00095,1907.05187]

$$\mathcal{T}^{\mu\nu}(x) = \sum_n \theta\left(\eta_n x^0\right) m_n u_n^\mu u_n^\nu \int_{-\infty}^{+\infty} \delta^{(D)}\left(x - u_n \tau\right) d\tau\,,$$

  with $p_n^\mu = \eta_n m_n u_n^\mu,\ \ u_n^\mu u_{n,\mu} = -1,\ \ u_n^0 > 0.$

- Fourier transform:

$$\tilde{\mathcal{T}}^{\mu\nu}(k) = \int d^D x\, \mathcal{T}^{\mu\nu}(x) e^{-ik\cdot x} = \frac{1}{i\kappa} F^{\mu\nu}(k)\,,$$

  with

$$F^{\mu\nu}(k) = \sum_n \frac{\kappa\, p_n^\mu p_n^\nu}{p_n \cdot k - i0}\,.$$

- This is also the Feynman $-i0$ prescription carried by the propagator of the nth line.

Warm-Up: Memory Effect [2203.11915,2204.02378]

[Kovacs,Thorne'77,'78, 1411.5745, 2101.12688, 2102.08339, 2107.10193]

- Waveform: $\kappa \langle 0 | S^\dagger_{s.r.} H_{\mu\nu}(x) S_{s.r.} | 0 \rangle = W_{\mu\nu}(x)$ with

$$H_{\mu\nu}(x) = \int_{\vec{k}} \left[a_{\mu\nu}(k) e^{ikx} + a^\dagger_{\mu\nu}(k) e^{-ikx} \right].$$

- Asymptotic limit: set $x^\mu = (u + r, r\hat{x})$ and send $r \to \infty$ for fixed u, $\hat{x}$.
- In the approach [2203.11915], the $-i0$ fixes the u-independent ST ambiguity [cf. 2010.01641,2201.11607], i.e. in the Fourier transform

$$\int_{-\infty}^{+\infty} \frac{d\omega}{i2\pi} \frac{e^{-i\omega u}}{-\eta_n \omega - i0} = \int_{-\infty}^{+\infty} \frac{d\omega}{i2\pi} \frac{e^{i\omega\eta_n u}}{\omega - i0} = \theta\left(\eta_n u\right).$$

- Letting $p_n = \eta_n(E_n, \vec{k}_n)$,

$$W^{\mu\nu} \sim \frac{2G}{r} \sum_n \theta\left(\eta_n u\right) \frac{(p_n^\mu p_n^\nu)_{TT}}{E_n - \vec{k}_n \cdot \hat{x}} = \frac{2G}{r} \sum_{\text{in}} \frac{(p_n^\mu p_n^\nu)_{TT}}{E_n - \vec{k}_n \cdot \hat{x}} + \mathcal{O}(G^2).$$

The PM limit thus agrees with [2010.01641].

$\mathcal{J}^{\rm sc}_{\alpha\beta}$ for a Massless Scalar [2203.11915]

- Charge associated to Lorentz transformations:

$$iJ^{\rm sc}_{\alpha\beta} = \frac{1}{2}\int_{\vec{k}} \left[a^\dagger(k) k_{[\alpha} \frac{\partial a(k)}{\partial k^{\beta]}} - k_{[\alpha} \frac{\partial a^\dagger(k)}{\partial k^{\beta]}} a(k) \right].$$

- Classical contribution due to gravitons with $\omega < \Lambda$: $\mathcal{J}^{\rm sc}_{\alpha\beta} = \langle 0| S^\dagger_{s.r.} J^{\rm sc}_{\alpha\beta} S_{s.r.} |0\rangle$ gives

$$i\mathcal{J}^{\rm sc}_{\alpha\beta} = \frac{1}{2}\int_{\vec{k}} \left(f^* k_{[\alpha} \frac{\partial f}{\partial k^{\beta]}} - k_{[\alpha} \frac{\partial f^*}{\partial k^{\beta]}} f \right), \qquad f = \sum_n \frac{g_n}{p_n \cdot k - i0}$$

- Again the $-i0$ prescription is important, and the result localizes to $\omega = 0$,

$$\int_{-\Lambda}^{\Lambda} \frac{\omega d\omega}{(-\eta_n \omega + i0)(-\eta_m \omega - i0)} = -\frac{i\pi}{2}(\eta_n - \eta_m)$$

$$\int_{-\Lambda}^{\Lambda} \frac{\omega d\omega}{(-\eta_m \omega - i0)^2} = -i\pi\eta_m\,.$$

$\mathcal{J}_{\alpha\beta}$ for the Graviton [2203.11915]

- Total angular momentum/mass dipole operator:

$$iJ_{\alpha\beta} = \int_{\vec{k}} a^{\dagger}_{\mu\nu}(k)\left(P^{\mu\nu,\rho\sigma} k_{[\alpha} \frac{\overleftrightarrow{\partial}}{\partial k^{\beta]}} + 2\eta^{\mu\rho}\delta^{\nu}_{[\alpha}\delta^{\sigma}_{\beta]}\right) a_{\rho\sigma}(k)$$

with $P^{\mu\nu,\rho\sigma} = \frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}\right)$.

- Classical average: $\mathcal{J}_{\alpha\beta} = \langle 0|S^{\dagger}_{s.r.} J_{\alpha\beta} S_{s.r.}|0\rangle$

$$i\mathcal{J}_{\alpha\beta} = \int_{\vec{k}} F^{*}_{\mu\nu}\left[\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \frac{1}{D-2}\eta^{\mu\nu}\eta^{\rho\sigma}\right) k_{[\alpha} \frac{\overleftrightarrow{\partial}}{\partial k^{\beta]}} + 2\eta^{\mu\rho}\delta^{\nu}_{[\alpha}\delta^{\sigma}_{\beta]}\right] F_{\rho\sigma}$$

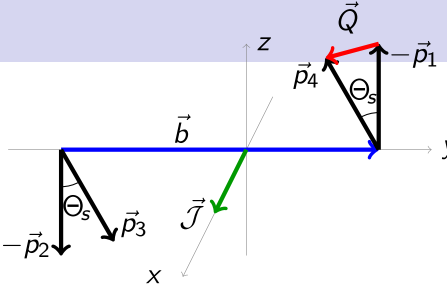
in agreement with [2203.04283].

The final expression in $D = 4$ is

$$\mathcal{J}^{\alpha\beta} = \frac{G}{2}\sum_{n,m}\left[\left(\sigma^2_{nm} - \frac{1}{2}\right)\frac{\frac{\sigma_{nm}\,\mathrm{arccosh}\,\sigma_{nm}}{\sqrt{\sigma^2_{nm}-1}} - 1}{\sigma^2_{nm} - 1} - \frac{2\sigma_{nm}\,\mathrm{arccosh}\,\sigma_{nm}}{\sqrt{\sigma^2_{nm}-1}}\right](\eta_n - \eta_m)\, p_n^{[\alpha} p_m^{\beta]}\,.$$

Angular Momentum Loss

The formula captures the loss of mechanical angular momentum $\vec{\mathcal{J}}$ completely up to $\mathcal{O}(G^2)$ [2010.01641, 2101.12688,2102.08339,2110.08681,2203.04283]



Analyticity vs Linear Response:

$$\mathcal{J}^{yz} \sim \frac{4p}{Q} \lim_{\epsilon\to 0}[-\pi\epsilon \operatorname{Im} 2\delta] + \mathcal{O}\left(G^4\right)$$

ensures that/explains why [2104.03256]

$$\Theta_{3\mathrm{PM}}^{\mathrm{RR}} = -\frac{1}{p}\frac{\partial \operatorname{Re} 2\delta_2^{\mathrm{RR}}}{\partial b} = \frac{2}{pb}\lim_{\epsilon\to 0}\left[-\pi\epsilon \operatorname{Im} 2\delta_2\right]$$

agrees with [1210.2834,2010.01641]

$$\Theta_{3\mathrm{PM}}^{\mathrm{RR}} \simeq -\frac{1}{2p}\frac{\partial\Theta_{1\mathrm{PM}}}{\partial b}\mathcal{J}^{yz} \simeq \frac{Q}{2p^2 b}\mathcal{J}^{yz}\,.$$

Angular Momentum Loss

- The formula captures the loss of mechanical angular momentum to $\mathcal{O}(G^2)$ also for spinning particles, for generic spin alignments [2203.13272].
- It also captures the $\mathcal{O}(G^n)$ loss due to zero-frequency gravitons attached to the elastic process. Cross-checked to $\mathcal{O}(G^3)$ against [2203.04283].
- In analogy with W, the PM limit and the high-energy limit do not commute. In the high-energy limit $m_i^2 \ll Q^2 = s\sin^2\frac{\Theta_s}{2}$,

$$\mathcal{J}^{yz} \sim 2Gs\sin\Theta_s \log\frac{\cos\frac{\Theta_s}{2}}{\sin\frac{\Theta_s}{2}}$$

and for small Θ_s

$$\mathcal{J}^{yz} \sim Gs\Theta_s \log\frac{4}{\Theta_s^2}\,.$$

Mass-Dipole Loss

- We find an $\mathcal{O}(G^2)$ loss for the ty component

$$\frac{\mathcal{J}_{ty}}{b\,(E_1 - E_2)} \sim \frac{\mathcal{J}_{yz}}{2bp}$$

in agreement with [2203.04283]. Solving

$$\Delta\,(b_1 E_1 - b_2 E_2) = -\mathcal{J}_{ty}\,, \qquad \Delta\,(b_1 + b_2)\,p = -\mathcal{J}_{yz}$$

yields

$$\Delta b_1 p = \Delta b_2 p = -\mathcal{J}_{yz}/2\,.$$

- There is an $\mathcal{O}(G^3)$ loss for the tz component

$$\frac{\mathcal{J}_{tz}}{b\,(E_1 - E_2)} \sim \frac{\Theta_s}{8}\frac{\mathcal{J}_{yz}}{bp}\,.$$

- The formula does not capture $\mathcal{O}(G)$ and $\mathcal{O}(G^2)$ terms in the mass dipole loss due to the long-range nature of the gravitational force [2110.08681].

Summary and Outlook

- $\mathrm{Re}\, 2\delta_2$ is smooth at high energy
- $S_{s.r.}$ gives a unitary description for soft radiation on top of any process:
  - yields the ZFL of the energy emission spectrum W
  - predicts/subtracts divergent part of $\mathrm{Im}\, 2\delta_2$
  - is crucial to restore smoothness at high energy
- Including the appropriate $-i0$ prescription, it also yields the $\mathcal{O}(G^n)$ $\mathcal{J}_{\alpha\beta}$ due to the "zero-frequency gravitons", or rather to static field effects, on top of any process.

For the future:

- Beyond the ZFL [cf. 2112.07556] (in progress...)
- $\mathcal{O}(G^n)$ angular momentum from the subleading soft theorem
- Learning more about RR effects at $\mathcal{O}(G^4)$ [cf. 2203.04283].