Jim Hartle's Generalized Quantum Mechanics: Where does what happens happen?

Fay Dowker
Theoretical Physics Group,
Imperial College, London

Jim Hartle's Legacy
KITP
12 Feb 2024

Jim's Generalized Quantum Mechanics: a path integral approach

Jim set out his Generalized Quantum Mechanics (GQM) in two major foundational lecture series: **The Quantum Mechanics of Cosmology** (1990)[1] and **Spacetime quantum mechanics and the quantum mechanics of spacetime** (1992)[2]. And subsequently in a pioneering series of works, developing and championing the Feymannian view that the path integral is the foundation of quantum theory in which what happens, happens in spacetime.

GQM is related to, but distinct from, the operator form of the Decoherent Histories approach of Jim and Murray Gell-Mann and of the Consistent Histories approach of Griffiths and Omnès.

Outline of Talk:

- Reframe QM in terms of a Double Path Integral: Copenhagen Redone.
- Apply to closed quantum systems: Copenhagen Outdone.
- GQM: a structure for as yet unknown theories
- Hilbert space from GQM (if Jim's axioms are slightly strengthened)
- Summary

---

[1] 7th Jerusalem Winter School 1990 on Quantum Cosmology and Baby Universes

[2] Les Houches Summer School on Gravitation and Quantizations, Les Houches, France, 6 Jul - 1 Aug 1992

Dirac's 1933 Fork in the Road [3]

Jim's GQM is based on the path integral and its roots are almost at the beginning:

- "Quantum mechanics was built up on a foundation of analogy with the Hamiltonian theory of classical mechanics.
- "There is an alternative [...] provided by the Lagrangian. [...T]here are reasons for believing that the Lagrangian one is the more fundamental.
- "There is no action principle [...] of the Hamiltonian theory
- "The Lagrangian method can easily be expressed relativistically; the Hamiltonian method is essentially non-relativistic in form, since it marks out a particular time variable as the canonical conjugate of the Hamiltonian function."

- **Hamiltonian** $\to$ Canonical quantisation $\to$ Ψ.
- **Lagrangian** $\to$ Path integral, a *spacetime* approach $\to$ no need for Ψ.

Let us, with Jim, take the Path Integral Road

[3] Dirac "The Lagrangian in Quantum Mechanics" 1933

The Double Path Integral

Jim's GQM is based on a *double* path integral (of Schwinger-Keldysh type). Why?

The textbook calculation for the probability for the outcomes of a sequence of measurements at $\{t_1, t_2, \ldots t_n\}$ between $t = 0$ and T gives

$$\mathrm{Prob}(\Delta_1, \ldots \Delta_n) = \| \, |\Delta\rangle \, \|^2 \tag{1}$$

where Born Rule & Collapse Postulate give

$$|\Delta\rangle := U(T, n) \, P_{\Delta_n} \ldots\ldots P_{\Delta_2} \, U(2,1) \, P_{\Delta_1} \, U(1,0) \mid \Psi\rangle \, .$$

In 1-d QM, for **position** measurements, $P_{\Delta_i} = \int_{a_i}^{b_i} dx \mid x\rangle\langle x \mid$ and

$$\mathrm{Prob}(\Delta) = \int_{\gamma \in \Delta} [d\gamma] \int_{\gamma' \in \Delta} [d\gamma'] \; D(\gamma, \gamma')$$

where

$$D(\gamma, \gamma') = \mathrm{amp}(\gamma)^* \mathrm{amp}(\gamma') \; \delta(\text{final endpoints}) \quad \text{and} \quad \mathrm{amp}(\gamma) = e^{i \frac{S[\gamma]}{\hbar}} \; \Psi(\gamma(0))$$

Define the restricted propagator (we'll need it later):

$$U_\Delta(T, 0) := U_\Delta(T, n) P_{\Delta_n} \ldots\ldots P_{\Delta_2} \, U(2,1) \, P_{\Delta_1} \, U(1,0) \, .$$

Double path integrals: Copenhagen redone

- The double path integral (DPI)

$$\text{Prob}(\Delta) = \int_{\gamma \in \Delta} [d\gamma] \int_{\gamma' \in \Delta} [d\gamma'] D(\gamma, \gamma')$$

  can be generalized to N particles: now a history γ is a set of N spacetime trajectories.

- The double path integral gives everything the Born rule & collapse postulate give when Δ is a sequence of position measurement outcomes.
- The DPI gives *less*: no direct answer for other observables (see however next slide).
- The DPI also gives *more*:

  Δ can be any set of histories[4] such as the famous Feynman-type “passing through spacetime region R”.

- The *decoherence functional* championed by Jim:

$$D(A, B) = \int_{\gamma \in A} [d\gamma] \int_{\gamma' \in B} [d\gamma'] D(\gamma, \gamma')$$

[4] up to the knotty question, familiar from the Wiener process, of which sets are ‘measurable’

Diagram from The Quantum Mechanics of Cosmology

Illustration of the histories summed over in $D(A, B)$

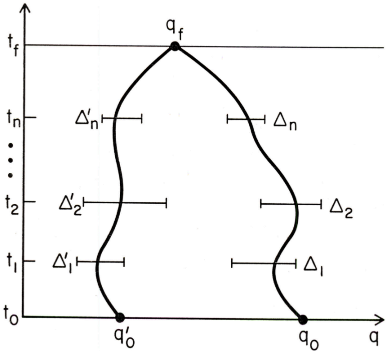

Fig. 2: *The sum-over-histories construction of the decoherence functional.*

Double path integral for a **closed** system: Copenhagen outdone

Now suppose an N particle history, γ, describes the microsystem, the instruments, and the environment together. *No external anything*.

- Each possible sequence of outcomes registered by the instruments corresponds to a set X of histories of the closed system.
- Following the terminology of stochastic processes we call X an *event*.
- Claim: For X an instrument event, then For All Practical Purposes (FAPP)

$$D(X, X) = \text{Copenhagen Prob(sequence of outcomes} \leftrightarrow X)$$

  Jim gives convincing arguments for the claim, based on the assumption the instruments work as they should and on the efficiency of environmental decoherence to make $D(X, Y)$ negligible for distinct instrument events.

Copenhagen has been outdone because the same predictions have been made but while treating the whole world quantum mechanically.

Path integral for the Universe released from instrumentalism

Consider now $D(X, Y)$ for all events X, Y (not just instrument events). The initial Ψ in the definition of D is the "wave function of the universe".

- How do we do physics with D(X,Y)? I know of two proposals.
  1. Jim's proposal in GQM is based on *decoherent partitions* of the set of all histories.
  2. Another proposal, Quantum Measure Theory, due to Rafael Sorkin[5], is based on *events of measure zero*.
- Where is our beloved Hilbert space in such a path-integral theory? How do we connect D to the familiar canonical framework?

[5] Rafael D. Sorkin, "Quantum Mechanics as Quantum Measure Theory" gr-qc/9401003

Axioms for Generalized Quantum Mechanics and Quantum Measure Theory

Any closed quantum theory has

- A space of histories Ω
- A Boolean algebra $\mathfrak{A}$ of events (subsets of Ω)
- A decoherence functional $D(A,B)$ satisfying
  - **Hermiticity**: $D(A,B) = D(B,A)^* \quad \forall A,B \in \mathfrak{A}$
  - **Positivity**: $D(A,A) \geq 0 \quad \forall A \in \mathfrak{A}$
  - **Normalization**: $D(\Omega,\Omega) = 1$
  - **Linearity**: $D(A \cup B, C) = D(A,C) + D(B,C) \quad \forall A,B,C \in \mathfrak{A}$ s.t. $A \cap B = \emptyset$

If the positivity axiom is strengthened to

- **Strong Positivity**: For any finite collection of events $\{B_1, \dots B_n\}$, $D(B_i, B_j)$ is a positive semi-definite $n \times n$ matrix ,

then the GNS construction gives a Hilbert space[6] in which every event $A \to |A\rangle$ s.t.

$$\langle A \,|\, B \rangle = D(A,B)\,.$$

[6] Martin, O'Connor, Sorkin, gr-qc/0403085

The Event Hilbert space

If the histories in Ω *accrete* in time then the Event Hilbert space, EH, typically grows with time – a form of non-unitarity.

Theorem (Dowker, Johnston & Sorkin arxiv:1002.0589)

If D comes from a unitary path integral in non-rel QM with a reasonable Hamiltonian then the Event Hilbert space EH *is* the canonical Hilbert space H.

Sketch of proof in non-rel QM:
Consider the linear map $\phi : EH \to H$ defined by

$$\phi : |A\rangle_{EH} \mapsto U_A(T, 0)|\Psi\rangle_H$$

This map is $1-1$ as it preserves the inner product (by definition).
Characteristic (top hat) functions are an over-complete set of wavefunctions in the canonical H. For each such function f, there is a sequence of vectors in EH s.t. the image of the sequence under ϕ converges to f.

So ϕ is onto and therefore ϕ is an isomorphism.

Why should the Decoherence Functional be Strongly Positive?

We say 2 systems with D_1 and D_2 **compose** if the tensor product $D_1 D_2$ is Positive.

If D_1 and D_2 are SP then $D_1 D_2$ is also SP and the systems compose. But if D_1 and D_2 are only Positive and not SP, then they don't necessarily compose.

Theorem (Dowker& Wilkes arXiv:2011.06120)

If a collection $\mathcal{T}$ of Positive systems is

1. Closed under composition
2. Full: Any Positive system that can be composed with *every* system in $\mathcal{T}$ is in $\mathcal{T}$

then $\mathcal{T}$ is the set of SP systems $\mathcal{S}$.

Sketch of proof

- The collection $\mathcal{S}$ of SP systems satisfies 1 and 2 above.[7]
- If a system can be composed with itself n times for arbitary n, then it is SP or its D has only positive values.
- $\mathcal{T} \subseteq \mathcal{S}$ or $\mathcal{T} \subseteq \mathcal{PV}$
- No subset of $\mathcal{PV}$ is full.

[7] Boes & Navascues arXiv:1609.09723

Summary

- Jim's seminal work on GQM is a formalization of a view of quantum theory that Feynman espoused but didn't formalize.
- In a path integral approach to quantum foundations, events happen in spacetime, not in Hilbert space.
- If the decoherence functional of a system is Strongly Positive, there is a Hilbert space in the theory.

Partial to do list:

- Technical: what are the histories and event algebra $\mathfrak{A}$ for known unitary theories? (Like construction of Wiener measure from the propagator, but much harder)
- Is decoherence needed (GQM) or can one get by with "events of measure zero don't happen" (QMT)?
- Quantum Gravity: Jim built GQM for this purpose. What is the decoherence functional? If spacetime is discrete, we may need only finite path sums not pesky path integrals and QG may turn out to be simpler than QM!

Thank you Jim, for everything