Long-range interactions and the Yang-Baxter equation

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based on work with C. Paletta, B. Pozsgay, A. Pribyl, A. L. Retore, P. Ryan
Integrability

- Tower of commuting independent charges (QM, classical)

\{ Q_i \}

- Factorized scattering (FT)

Yang-Baxter equation (YBE)
Combining Framework:

**Algebraic Bethe Ansatz**

(Faddeev, Reshetikhin,...)

Solution of YBE \((R_{ij} = R_{ij}(u_i, u_j))\)

\[ R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \]

Generates charges for spin chain

\(\Rightarrow\) sometimes solve
YBE plays central role in integrability

Appears in:
- QFT
- QM
- Maths
- String theory
- Quantum circuits

Plan:
- find all solutions of YBE (of certain type)
- can we find R for a given system?
Solving the Yang-Baxter equation
Historically

- Via symmetry (Quantum group, Temperley-Lieb, ...) 
  
  😊 easy (linear eqns)
  
  😞 only highly (manifestly) symmetric models

- Direct solution
  
  😊 complete
  
  😞 very hard & technical

- Boost approach
  
  [Vieira, Izumi, ...]

[Our work]
Boost approach

Bottom up approach

II starting point

$\exists \mathcal{Q}_i \rightarrow R$

😊 complete, doable

😢 regular R-matrices, still some technical challenges
Idea

Let $R(u,v)$ be solution of YBE

$\rightarrow$ integrable spin chain $R\rightarrow L$ Lax operator

Def monodromy $T(u,\theta) = R_{(L,4\theta)} \cdots R_{(4,\theta)} a \frac{1}{2} \cdots \frac{1}{2}$

transfer $t(u,\theta) = tr_a T \rightarrow$ operator on $V^a$

$\Rightarrow \left[ t(u,\theta), t(v,\theta) \right] = 0$

$\Rightarrow t(u) = \exp (u Q_2 + u^3 Q_3) \rightarrow \frac{3}{2} Q_i \{ Q_j \} \text{ commuting}$
Regular $R$-matrix $R(u,u) = P$

$\Rightarrow t(\theta,\theta) = U = e^{iP}$

$Q_i = d \log t = \sum_n H_{n,m}^1$ \quad Here $\frac{P}{\partial u} R \bigg|_{u=0}$

Nearest Neighbour interactions

Alternative way to generate $Q_i \Rightarrow$ Boost

[Tedeman]

[Gold, Linke, Zhang]
Boost operator

\[ B = \sum_n n X_{n,n+1} + d_\theta \]

\( Q_{in} \sim [B, Q_i] \)

\( Q_3 (x, jx) , Q_4 (x, jx, jx) , \ldots \)

\( H \) is a fundamental building block
Idea 1

Take $\mathcal{H}(\theta)$ general

↓ Boost

Compute $Q_3(\theta)$

↓ Impose $[Q_2, Q_3] = 0$

Potential integrable $\mathcal{H}$'s

(Extra constraints from $[Q_2, Q_4]$ $[Q_2, Q_5]$ $\cdots$ ?)

$R$?
Idea 2

Expand YBE at $u_1 = u_2$

$$[R_{13} R_{23}, \mathbb{H}_{12}] = \dot{R}_{13} R_{23} - R_{13} \dot{R}_{23}$$

Overdetermined set of diff eqns.

B.C. \( R(u, u) \sim P \)

\( \dot{R}(u, u) \sim P \cdot \mathcal{H}(u) \)

\( \mathcal{H} \Rightarrow R \Rightarrow \text{check YBE} \quad \text{done} \)
Observation

- Any potential integrable $\mathcal{H}$ is integrable
  $(Q_2, Q_3) = 0$ is enough, Sutherland is sufficient

- Complete classification

- Some duplicate solutions
  $R$'s related by basis transformation
  - reparameterization
  - transposition
  ;
What did we classify?

- $4 \times 4$, difference form $R_{12}(u,-u)$

  $\Rightarrow$ non-trivial & new non-hermitian, non diagonalizable models

- $4 \times 4$, 8v type

  $\Rightarrow$ 2 new models; elliptic deformations of $AdS_2$ & $AdS_3$ models
- 9x9 Satisfying ice rule \( (U1)^3 \)
- 16x16 \( SU(2) \times SU(2) \) Symmetry
  - Hubbard like models
- 16x16 Integrable Lindblad operators [Prosen]
  - Solvable systems with interactions with environment
- N\times N New flag like structure
  - Generalization of graded algebras [Mśl, Retore, Nepomechie]
- Open for suggestions...
Long-range interactions
Reverse Problem

Give $H_{nn} \Rightarrow$ - integrable?
- $R$-matrix?

Not easy from previous classification.
- $H(u) \Rightarrow H$
- identifications
Question: is there \( L_u \) and \( R(u,v) \) such that

\[
R_{ab}(u,v) L_{an}(u) L_{bn}(v) = L_{bn}(v) L_{an}(u) R_{ab}(u,v)
\]

and \( R \) satisfies \( YBE \)

\[
L(0) = P
\]

\[
L'(0) = P \cdot \mathcal{H}
\]
Now define \( t \) via \( L \)

\[
\begin{align*}
  t &= \text{tr } h_a L \ldots h_a \\
  \Rightarrow \\
  t(\alpha) &= U = e^{\mathbf{P}} \\
  t'(\alpha) &= \mathbf{P} \cdot H
\end{align*}
\]

Can again define boost operator
Write:

\( \Omega(u) = P \left( 1 + u \chi + \sum_{n} \chi^{(n)} u^{n} \right) \)

\( \uparrow \quad \uparrow \quad Q_{2} \quad u \)

\( Q_{3} = \sum_{n} [ \chi_{n-1}, \chi_{n} ] + \chi_{n}^{2} - Q^{(2)} \)

All new information in range 2 term

Impose \( [Q_{2}, Q_{3}] = 0 \) \( \Rightarrow \) \( Q^{(2)} \)
All information size $\Rightarrow$

very easy to compute $\mathbb{A}^m$ (we did $\mathbb{Qzo}$)

$\Rightarrow$ if they exist $\Rightarrow$ integrable

Usually (with some tricks) enough to

find $\mathbb{A}(u)$ explicitly

$\Rightarrow \mathbb{R}$ via $\mathbb{RLLR} = \mathbb{LLR}$ (linear!)
L not unique (e.g. \( u \rightarrow u \cdot au^2 \ldots \))

Check that any 6r model

\[
H = \begin{pmatrix}
    h_1 & h_2 & h_3 \\
    h_4 & h_5 & h_6 \\
    h_7 & h_8 & h_9
\end{pmatrix}
\]

is integrable \( \Rightarrow \) found \( R \)
Long-range deformations

Perturbative formalism:
\[ H(\lambda) = H_{NN} + \lambda \, H_{NNNN} + \lambda^2 H_{NNN} + \ldots \]

Integrability defined to some order in \( \lambda \)

\[ [Q_i(\lambda), Q_j(\lambda)] \sim O(\lambda^{1+\epsilon}) \]

Structure of AdS/CFT spin chains
Deformation equation

\[ \frac{d}{dl} Q_{ij} = \left[ X(l), Q_{ij} \right] \]

\[ \Rightarrow \text{check for } g N \text{ and } f V \]

All integrable pert. long range spin chains are generated by
Defined on level of charges

→

Are there L, R that generates them?

• Important to help understand and solve these types of spin chains
Approach

Group sites together

$\mathcal{X}_{a,a+1,a+2}$

$\mathcal{X}_{A,A+1}$

$\text{A NN interaction}$
Lax operator factorizes

$$\hat{h}_{a_1, a_2, b_1, b_2}^A \hat{h}_{a_1, a_2, b_2}^B = \hat{h}_{a_1, a_1, b_2}^A \hat{h}_{a_2, a_2, b_1}^B$$

generates $\mathcal{H}$

$\Rightarrow$ Only need to double auxiliary space

$\Rightarrow$ repeat procedure with $\hat{h}_{A,b}^A$
Loops in $N=4\ SYM$

SU(2) sector

\[ \mathcal{H}_{123} = (1 - P_{12}) + g^2 P_{13} + g^4 (\ldots) \]

\[ \Rightarrow \]

\[ L_{123} = P_{13} P_{23} \left( 1 + u (1 - P_{12}) + 2 g^2 \frac{u P_{13}}{(u-2)(u^2-1)} \right) \]
$R$-matrix $16 \times 16$ not nice

$R_{12,34}(u,v) \big|_{g=0} = R_{14}(u,0) R_{13}(u,v) R_{24}(0,0) R_{2,3}(0,v)$

At order $g^2$ no nice factorized structure

$RLL$ and $YBE$ only valid at $g^2$

For $g^4$ need range 4 term etc.
- Also works at $g^4$
- Integrability not enough to fix $g^4$ contribution
- Size of $R.L$ grows
Do all long-range deformations admit $(L,R)$?

- Look at $6V$ classification

Every deformation comes from $L,R$

$L,R$ depend on $6 + 5$ parameters

NN

NNN
Summary:

- Boost method to solve YBE
- Classified several cases
  \[\rightarrow\text{new solutions}\]
- How to lift \(X\) to \(R\)
- Found \(L, R\) for long-range
Open Problems

- Understand \([Q_2, Q_3] = 0\)
- Deformation equation \(\leftrightarrow L, R\)
- Other cases to study
- Non-regular
- Other long-range?
- Implicit R-matrix?
- Properties of new models
- Wrapping

[Somber]
Thank You