Long-range interactions and the Yang-Baxter equation

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based on work with
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THE ROYAL SOCIETY

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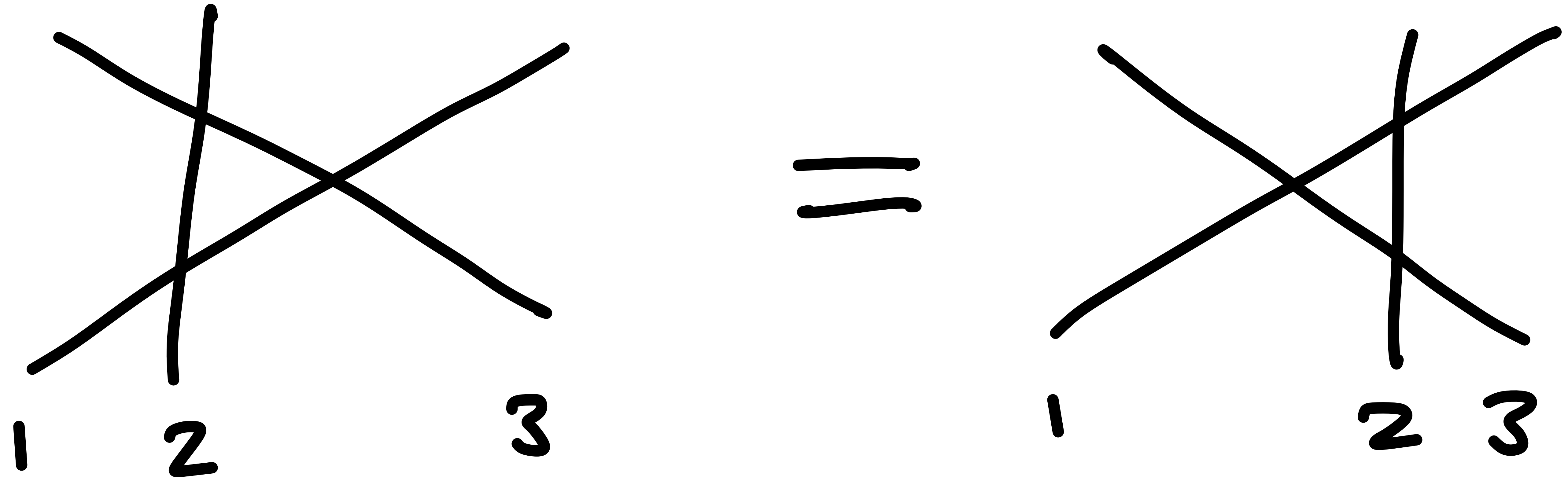
Integrability

- Tower of commuting independent charges (QM, classical)

$$\{Q_i\}$$

- Factorized scattering (FT)

Yang-Baxter equation (YBE)



Combining framework:

Algebraic Bethe Ansatz (Faddeev, Reshetikhin, ...)

Solution of YBE ($R_{ij} = R_{ij}(u_i, u_j)$)

$$\boxed{R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}}$$

Generates charges for spin chain

$\Rightarrow$ & sometimes solve

YBE plays central role in integrability

Appears in:

- QFT
- QM
- Maths
- String theory
- quantum circuits

⋮

Plan: - find all solutions of YBE (of certain type)
- can we find R for a given system?

Solving the Yang-Baxter equation

Historically

- via symmetry (Quantum group, Temperley-Lieb, ...)
  - easy (linear eqns)
  - only highly (manifestly) symmetric models
- direct solution [Vieira, Izumi, ...]
  - complete
  - very hard & technical
- boost approach [our work]

Boost approach

Bottom up approach

$\mathcal{H}$ starting point

$$\{Q_i\} \longrightarrow R$$

☺ complete, doable

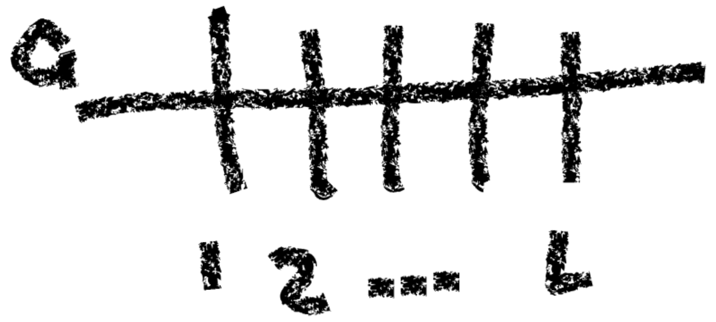
☹ regular R-matrices, still some technical challenges

Idea

Let $R(u,v)$ be solution of YBE

$\rightsquigarrow$ integrable spin chain $R \sim \mathcal{L}$ Lax operator

Def monodromy $T(u,\theta) = R_{aL}(u,\theta) \dots R_{a1}(u,\theta)$



transfer $t(u,\theta) = \mathrm{tr}_a T \leftarrow$ operator on $V^{\otimes L}$

$\overset{YBE}{\Longrightarrow}$ $[t(u,\theta), t(v,\theta)] = 0$

$\Longrightarrow t(u) = \exp(u Q_2 + u^3 Q_3 + \dots) \rightsquigarrow \boxed{\{Q_i(\theta)\} \text{ commuting}}$

Regular R-matrix $\boxed{R(u,u) = P}$

$$\Rightarrow \quad t(\theta,\theta) = \mathcal{U} = e^{iP}$$

$$Q_2 \equiv d \log t = \sum_n \mathcal{H}_{n,n+1} \qquad \mathcal{H} \sim P \frac{d}{du} R \Big|_{u\to 0}$$

$\vdots$

Nearest Neighbour interactions

Alternative way to generate Q: $\Longrightarrow$ Boost [Tetelman] [Gould, Links, Zhang]

Boost operator

$$\boxed{\mathcal{B} \equiv \sum_n n\, \mathcal{H}_{n,n+1} + \partial_\theta}$$ ← only defined for $L=\infty$

$Q_{i+1} \sim [\mathcal{B}, Q_i]$ ← consistent reduction to $L<\infty$

Observation: $Q_3(\mathcal{H}, \dot{\mathcal{H}})$, $Q_4(\mathcal{H}, \dot{\mathcal{H}}, \ddot{\mathcal{H}})$, ...

$\boxed{\mathcal{H} \text{ is a fundamental building block}}$

Idea 1

Take $\mathcal{H}(\theta)$ general

$\Downarrow$ Boost

Compute $Q_3(\theta)$

$\Downarrow$ Impose $[Q_2, Q_3] = 0$ ← coupled ODEs

Potential integrable $\mathcal{H}$'s (Extra constraints from $[Q_2, Q_4]$ $[Q_2, Q_5]$ $\vdots$?)

$\Downarrow$

R ?

Idea 2

Expand YBE at $u_1 = u_2$

$$\boxed{[R_{13}R_{23}, \mathcal{H}_{12}] = \dot{R}_{13}R_{23} - R_{13}\dot{R}_{23}} \quad *$$

Overdetermined set of diff eqns.

B.C. $R(u,u) \sim P$

$\dot{R}(u,u) \sim P \cdot \mathcal{H}(u)$

$\mathcal{H} \overset{*}{\Longrightarrow} R \Longrightarrow$ check YBE <u>done</u>

Observation

- Any potential integrable $\mathcal{H}$ is integrable
  ($[Q_2, Q_3] = 0$ is enough, Sutherland is sufficient)

- Complete classification

- Some duplicate solutions
  R's related by - basis transformation
  - reparameterization
  - transposition
  ⋮

What did we classify?

- 4×4, difference form $R_{12}(u_1-u_2)$

  $\Rightarrow$ non-trivial & new non-hermitian, non diagonalizable models

- 4×4 8v type

  $\Rightarrow$ 2 new models; elliptic deformations of AdS_2 & AdS_3 models

- 9×9 satisfying ice rule ($(U(1))^3$)
- 16×16 $su(2) \times su(2)$ symmetry
  - Hubbard like models
- 16×16 Integrable Lindblad operators [Prosen]
  - Solvable systems with interactions with environment
- $N \times N$ New flag like structure
  - generalization of graded algebras [MdL, Retore, Nep. mechie]
- open for suggestions...

Long-range interactions

Reverse Problem

Give $\mathcal{H}_{NN} \Longrightarrow$ - integrable?

- R-matrix?

Not easy from previous classification.

- $\mathcal{H}(\underline{\underline{u}}) \longrightarrow \mathcal{H}$
- identifications

<u>Question</u> : is there $\mathcal{L}(u)$ and $R(u,v)$ such that

$$R_{ab}(u,v)\,\mathcal{L}_{an}(u)\,\mathcal{L}_{bn}(v) = \mathcal{L}_{bn}(v)\,\mathcal{L}_{an}(u)\,R_{ab}(u,v)$$

and R satisfies <u>YBE</u>

$$\mathcal{L}(0) = P$$

$$\mathcal{L}'(0) = P \cdot \mathcal{H}$$

Now define t via $\mathcal{L}$

$$t = \operatorname{tr} \mathcal{L}_{a_L} \cdots \mathcal{L}_{a_1}$$

$\Longrightarrow$

$$t(0) = \mathcal{U} = e^{iP}$$

$$t'(0) = P \cdot \mathcal{H}$$

$$\vdots$$

Can again define boost operator

Write:

$$\mathcal{L}(u) = P\Big(\underset{\substack{\uparrow \\ \mathcal{U}}}{1} + u \underset{\substack{\uparrow \\ Q_2}}{\mathcal{H}} + \sum_n \mathcal{L}^{(n)} u^n \Big)$$

$$Q_3 = \sum_n \left[\mathcal{H}_{n-1,n} , \mathcal{H}_{n,n+1} \right] + \mathcal{H}_n^2 - \mathcal{L}^{(2)}$$

All new information in range 2 term

Impose $[Q_2, Q_3] = 0 \overset{?}{\Longrightarrow} \mathcal{L}^{(2)}$

All information 2-site $\Rightarrow$

very easy to compute $h^{(n)}$ (we did Q20)

$\rightsquigarrow$ if they exists $\Rightarrow$ integrable

Usually (with some tricks) enough to find $h(u)$ explicitly

$\Rightarrow$ R via $RLL = LLR$ (linear!)

↳ not unique (e.g. $u \to u + \alpha u^3 + \dots$)

Check that any 6v model

$$\mathcal{H} = \begin{pmatrix} h_1 & & & \\ & h_2 & h_3 & \\ & h_4 & h_5 & \\ & & & h_6 \end{pmatrix}$$

is integrable $\Rightarrow$ found R

Long-range deformations

Perturbative formalism:

$$\mathcal{H}(\lambda) = \mathcal{H}_{NN} + \lambda\, \mathcal{H}_{NNN} + \lambda^2\, \mathcal{H}_{N^4} + \dots$$

Integrability defined to some order in λ

$$[Q_i(\lambda), Q_j(\lambda)] \sim \mathcal{O}(\lambda^{r+1})$$

Structure of AdS/CFT spin chains

Deformation equation

[Borgheer, Beisert, Loebbert]

$$\boxed{\frac{d}{d\lambda} Q(\lambda) = [X(\lambda), Q(\lambda)]} \quad *$$

$\Rightarrow$ check for $\mathfrak{gl}_N$ and δV

[Mdl, Beisert, Fievet Loebbert]

$\boxed{\text{All integrable pert. long range spin chains are generated by } *}$

Defined on level of charges

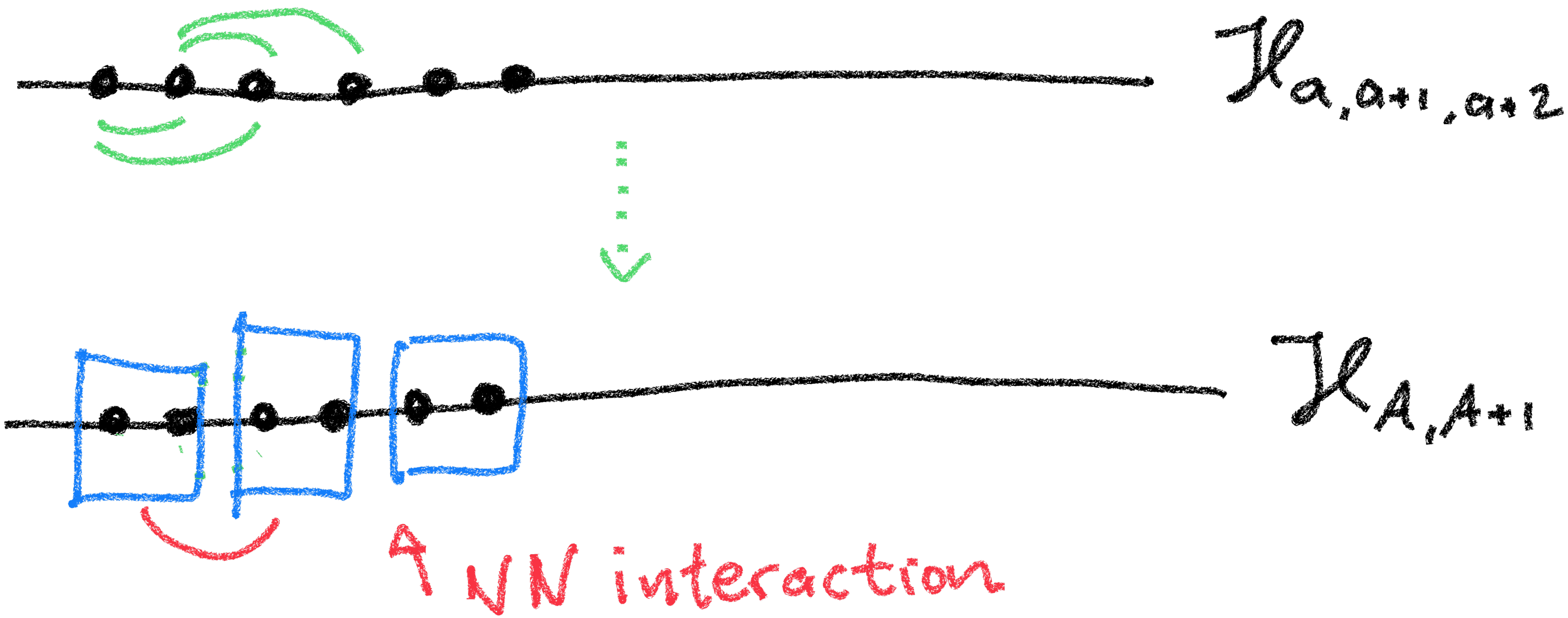
$\longrightarrow$

Are there L, R that generates them?

- Important to help understand and solve these types of spin chains

Approach

[Pocsgay, Gombor]



Group sites together

Lax operator factorizes

$$\mathcal{L}_{\underbrace{a_1 a_2}_{A},\,\underbrace{b_1 b_2}_{B}} = \underset{\substack{\uparrow \\ \text{generates } \mathcal{H}}}{\mathcal{L}_{a_1 a_2,\, b_2}}\; \mathcal{L}_{a_1 a_2,\, b_1}$$

$\Rightarrow$ Only need to double auxiliary space

$\Rightarrow$ repeat procedure with $\mathcal{L}_{A,b}$

Loops in $\mathcal{N}=4$ SYM

SU(2) sector

[Beisert, Kristjansen, Staudacher]

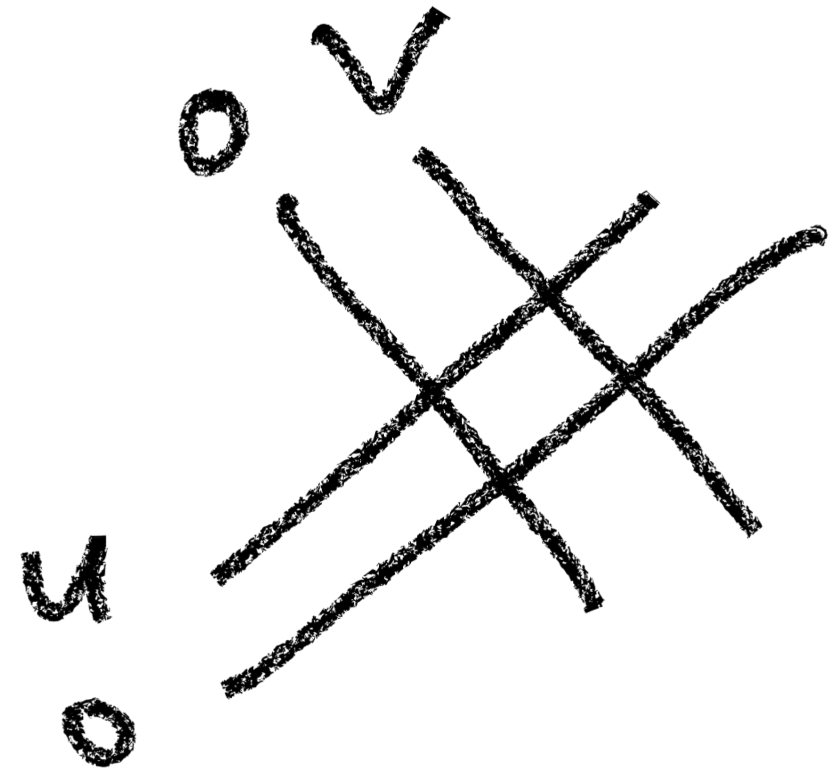
$$\mathcal{H}_{123} = (1-P_{12}) + g^2 P_{13} + g^4(\dots)$$

$$\Downarrow$$

$$h_{123} = P_{13} P_{23} \left(1 + u(1-P_{12}) + 2g^2 \frac{u\, P_{13}}{(u-2)(u^2-1)} \right)$$

R-matrix 16×16 not nice

$$R_{12,34}(u,v)\Big|_{g=0} = R_{14}(u,0)\,R_{13}(u,v)\,R_{24}(0,0)\,R_{2,3}(0,v)$$



At order g^2 no nice factorized structure

$\boxed{\text{RCC and YBE only valid at } g^2}$

For g^4 need range 4 term etc

- Also works at g^4
- integrability not enough to fix g^4 contribution

- Size of R.G grows

Do all long-range deformations admit $\mathcal{L}, R$?

- Look at 6v classification

Every deformation comes from $\mathcal{L}, R$

$\mathcal{L}, R$ depend on 6 + 5 parameters

NN NNN

Summary

- Boost method to solve YBE
- Classified several cases
  - → new solutions
- How to lift $\mathcal{H}$ to R
- Found $\mathcal{L}$, R for long range

Open Problems

- Understand $[Q_2, Q_3] = 0$
- Deformation equation $\longleftrightarrow$ $\mathcal{L}, R$
- Other cases to study
- non-regular
- Other long-range?
- Implicit R-matrix?
- Properties of new models
- wrapping

[Gombor]

Thank

You