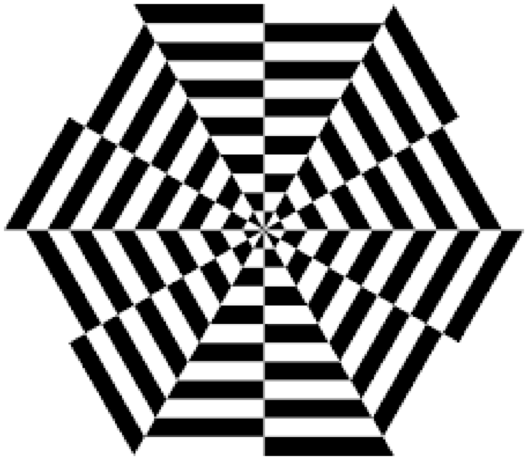
Tilting the Cusp Anomalous Dimension in Planar N=4 SYM



Lance Dixon (SLAC)

B. Basso, LD, G. Papathanasiou, 2001.05460

and in progress with BB, GP, Yu-Ting Liu, Gerald Dunne

KITP Conference “Talking Integrability”

September 1, 2022

image:10binary

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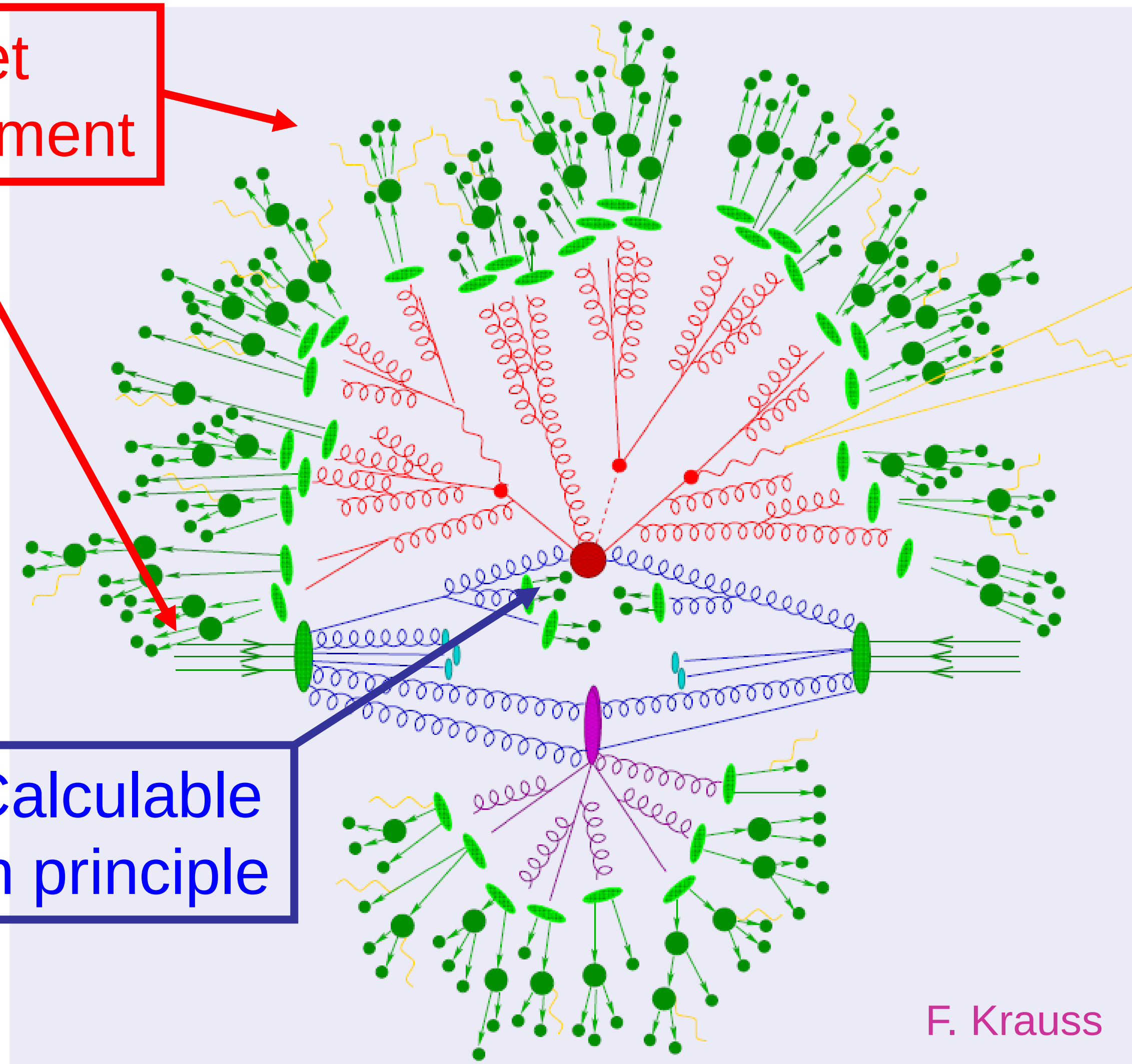
Typical Collision at Large Hadron Collider

Model or get from experiment



Calculable in principle

Scattering Amplitudes

- Physics at Large Hadron Collider dominated by scattering of gluons and quarks in a non-Abelian gauge theory (QCD)
- Planar N=4 SYM is an excellent testing ground for methods also used in QCD, as both are massless gauge theories
- However, planar N=4 SYM is also conformal and quantum integrable

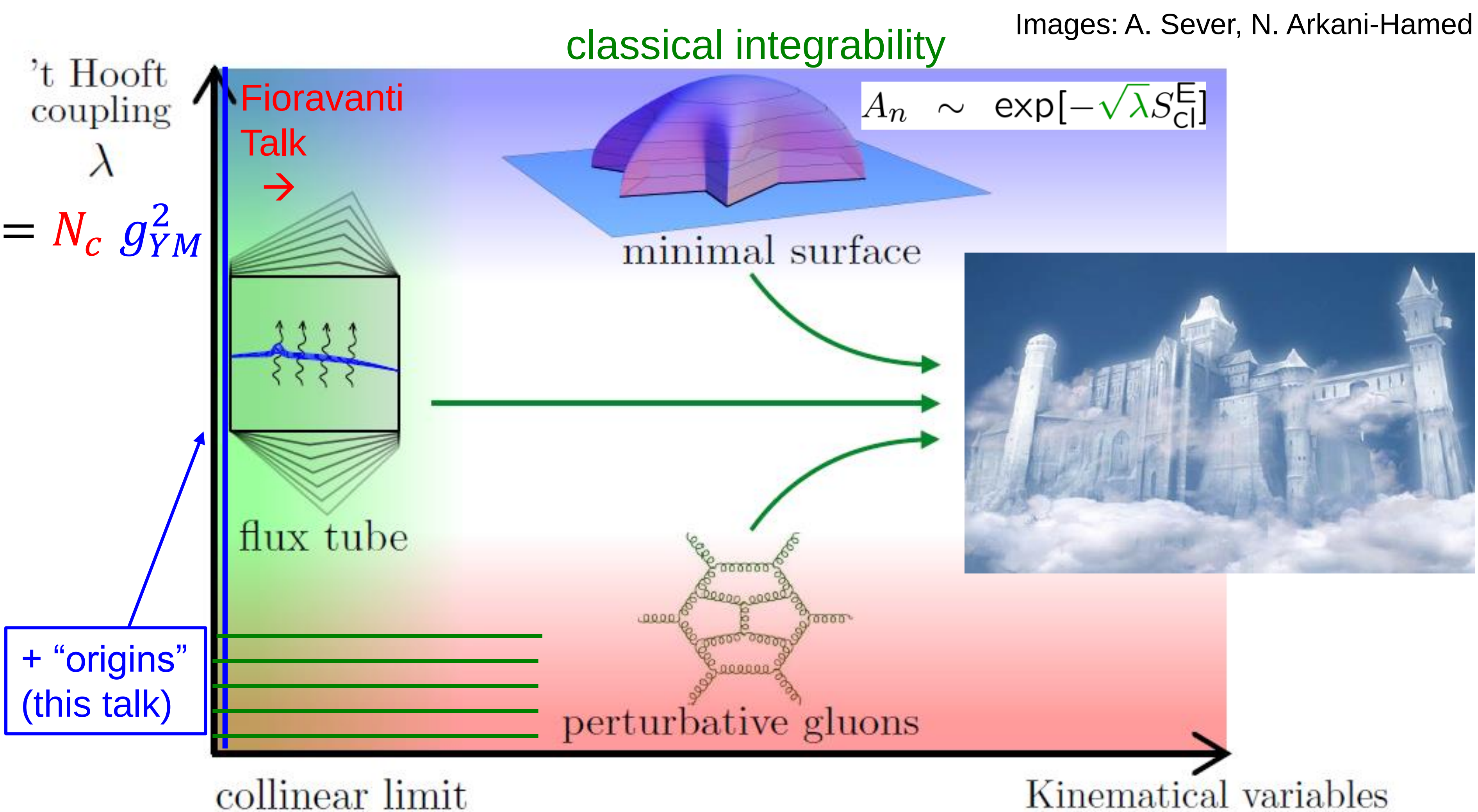
Minahan, Zarembo (2002);

Beisert, Kristjansen, Staudacher, hep-th/0303060;

Beisert, Staudacher (2003—2005);

Beisert, Eden, Staudacher, hep-th/0610251; …

Solving Planar N=4 SYM Scattering



“Origin” Proposal

Basso, LD, Liu, Papathanasiou, to appear

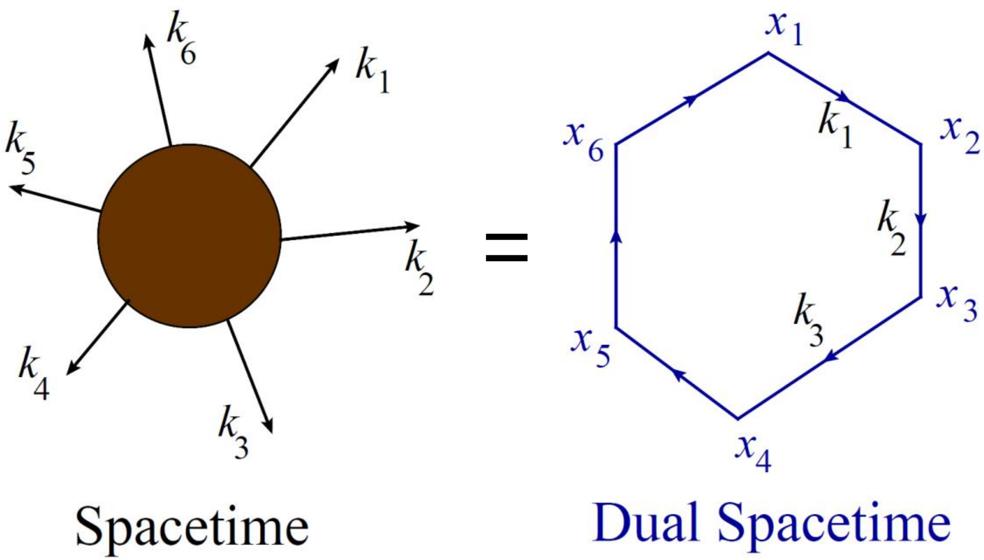
- In “suitable kinematics”, after subtracting a divergent part, the logarithm of the “MHV” n-gluon amplitude $\mathcal{A}_n$ is quadratic in the logarithms of small kinematic variables $\ell_i = \ln u_i$
- Quadratic terms given by the master formula contour integral,

$$\ln \mathcal{A}_n(g^2; \ell_i) = -\frac{1}{2} \oint_{\mathcal{C}} \frac{dz}{2\pi i z} \left(z - \frac{1}{z} \right) \mathcal{G}(g^2, z)\, \mathcal{S}_n(z, \ell_i)$$

“tilted” cusp anomalous dimension, contains all coupling dependence

classical string area density, quadratic in kinematic logs

Amplitudes = Wilson loops



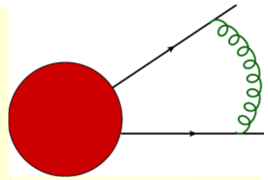
- Polygon vertices x_i are not positions but dual momenta, $x_i - x_{i+1} = k_i$
- Transform like positions under dual conformal symmetry

Alday, Maldacena, 0705.0303
Drummond, Korchemsky, Sokatchev, 0707.0243
Brandhuber, Heslop, Travaglini, 0707.1153
Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223, 0803.1466;
Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465

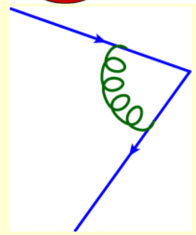
Duality verified to hold at weak coupling too!

Removing Divergences

- On-shell amplitudes IR divergent due to long-range gluons



- Polygonal Wilson loops UV divergent at cusps, anomalous dimension $\Gamma_{\rm cusp}$
  – known to all orders in planar N=4 SYM:
  Beisert, Eden, Staudacher, hep-th/0610251



- Both removed by dividing by BDS ansatz
  Bern, LD, Smirnov, hep-th/0505205

- Normalized [MHV] amplitude "remainder function" is finite, dual conformal invariant.

$$\lim_{\epsilon\to 0} \frac{\mathcal{A}_6(s_{i,i+1},\epsilon)}{\mathcal{A}_6^{\rm BDS}(s_{i,i+1},\epsilon)} = \exp[\mathcal{R}_6(u,v,w)]$$

Dual conformal invariance

- Wilson n-gon invariant under inversion: $x_i^\mu \to \frac{x_i^\mu}{x_i^2}, \quad x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}$

$$x_{ij}^2 = (k_i + k_{i+1} + \cdots + k_{j-1})^2 \equiv s_{i,i+1,\cdots,j-1}$$

- Fixed, up to functions of invariant cross ratios:

$$u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

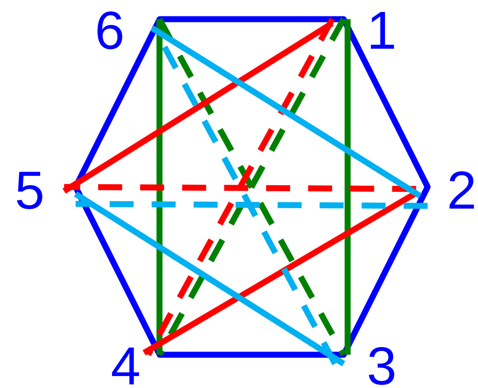
- $x_{i,i+1}^2 = k_i^2 = 0$ → no such variables for $n = 4,5$

$n = 6$ → precisely 3 ratios:

$$u_1 = u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}}$$

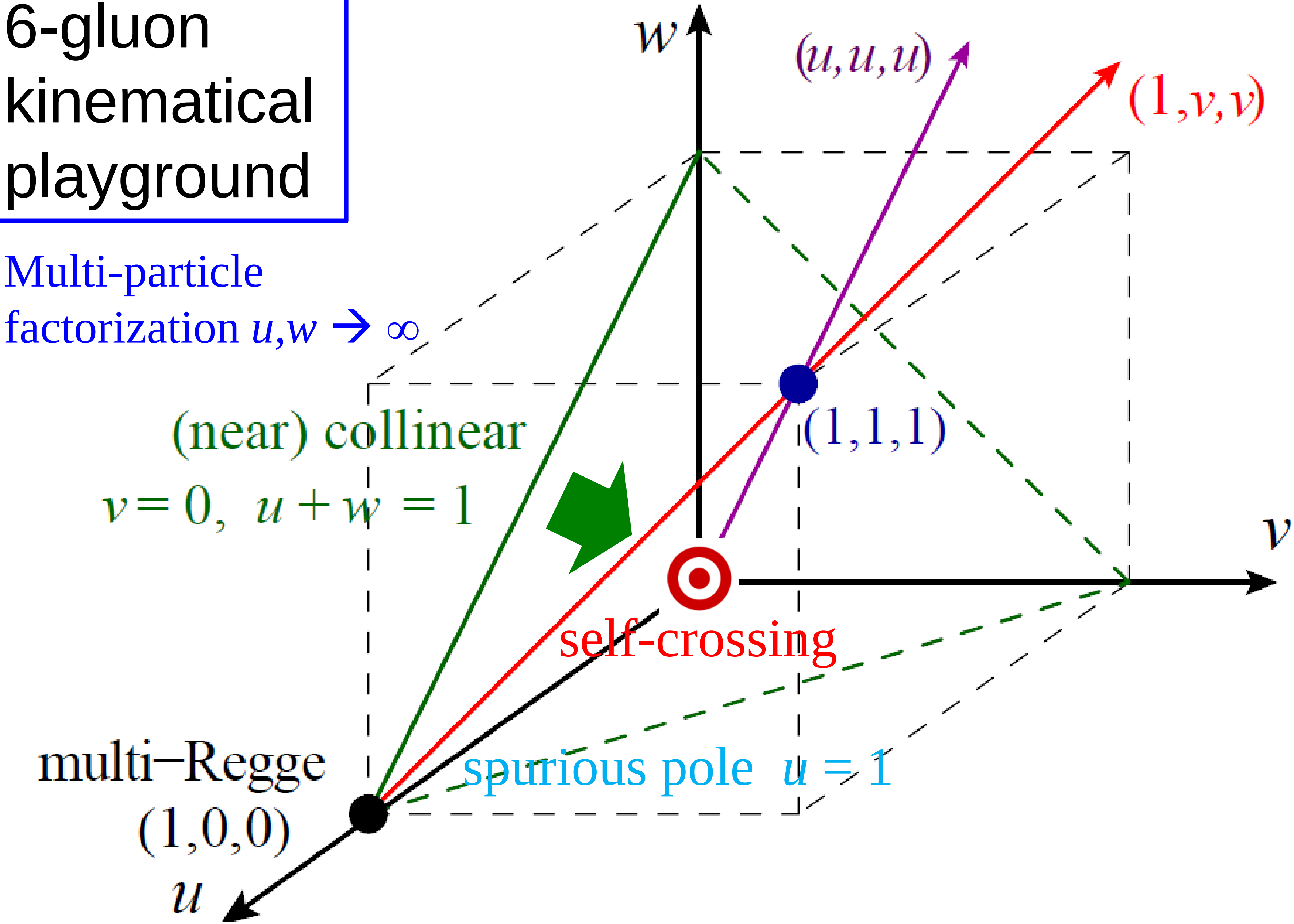
$$u_2 = v = \frac{s_{23}s_{56}}{s_{234}s_{123}}$$

$$u_3 = w = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$



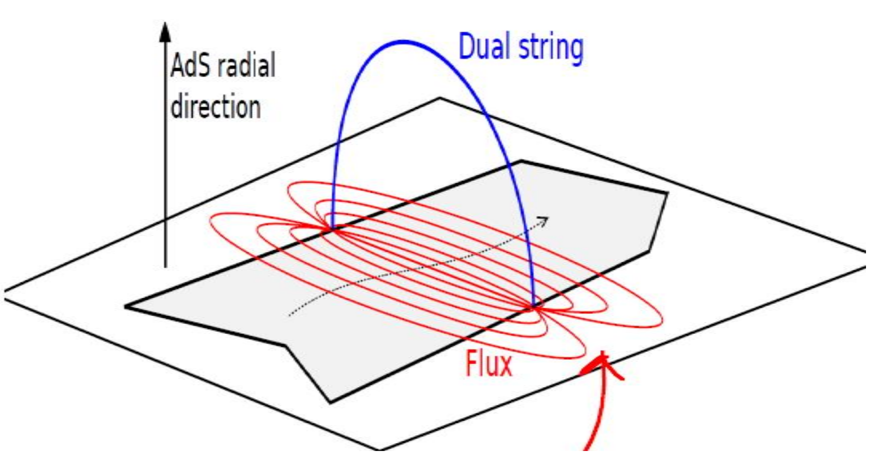
In general, $3n - 15$ independent variables

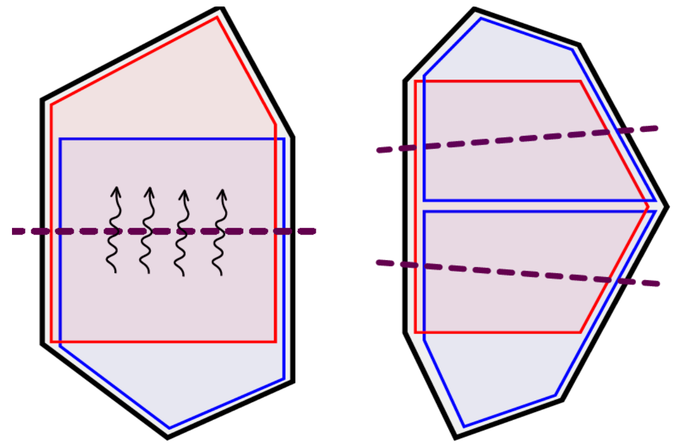
6-gluon kinematical playground



Flux tubes at finite coupling

Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788;
Basso, Sever, Vieira, 1303.1396, 1306.2058, 1402.3307, 1407.1736, 1508.03045
BSV+Caetano+Cordova, 1412.1132, 1508.02987; Fioravanti, Piscaglia, Rossi, 1508.08795





- Tile n-gon with pentagon transitions
- Quantum integrability → compute pentagons exactly in ’t Hooft coupling
- 4d S-matrix as expansion (OPE) in number of flux-tube excitations = expansion around near collinear limit

"Original" example

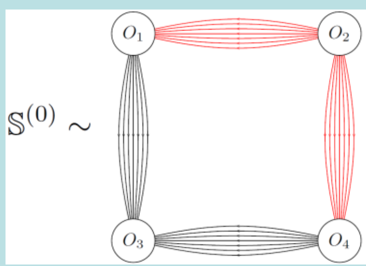
- Send $u_1, u_2, u_3 \to 0$
- Remarkably, $\ln \mathcal{A}_6$ is quadratic in logarithms through 7 loops
  Caron-Huot, LD, von Hippel, McLeod, Papathanasiou, 1903.10890
- Previously observed through 2 loops, and at strong coupling, on the diagonal $(u_1, u_1, u_1) \to 0$ Alday, Gaiotto, Maldacena, 0911.4708

$$\ln \mathcal{A}_6(u_i) \approx -\frac{\Gamma_{\rm oct}}{24} \ln^2(u_1 u_2 u_3) - \frac{\Gamma_{\rm hex}}{24} \sum_{i=1}^{3} \ln^2 \frac{u_i}{u_{i+1}} + C_0$$

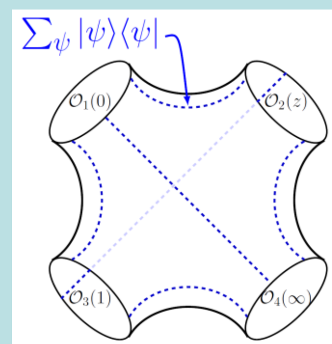
	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
$\Gamma_{\rm oct}$	4	$-16\zeta_2$	$256\zeta_4$	$-3264\zeta_6$	$\frac{126976}{3}\zeta_8$
$\Gamma_{\rm cusp}$	4	$-8\zeta_2$	$88\zeta_4$	$-876\zeta_6 - 32\zeta_3^2$	$\frac{28384}{3}\zeta_8 + 128\zeta_2\zeta_3^2 + 640\zeta_3\zeta_5$
$\Gamma_{\rm hex}$	4	$-4\zeta_2$	$34\zeta_4$	$-\frac{603}{2}\zeta_6 - 24\zeta_3^2$	$\frac{18287}{6}\zeta_8 + 48\zeta_2\zeta_3^2 + 480\zeta_3\zeta_5$
C_0	$-3\zeta_2$	$\frac{77}{4}\zeta_4$	$-\frac{4463}{24}\zeta_6 + 2\zeta_3^2$	$\frac{67645}{32}\zeta_8 + 6\zeta_2\zeta_3^2 - 40\zeta_3\zeta_5$	$-\frac{4184281}{160}\zeta_{10} - 65\zeta_4\zeta_3^2 - 120\zeta_2\zeta_3\zeta_5 + 228\zeta_5^2 + 420\zeta_3\zeta_7$

Mysterious octagon connection

- Remarkably, $\Gamma_{\rm oct} = \frac{2}{\pi^2} \ln\cosh(2\pi g)$
  recently appeared in light-like limit of correlator of 4 large R-charge operators, dubbed the octagon
  Coronado, 1811.00467, 1811.03282; Kostov, Petkova, Serban, 1903.05038; Belitsky, Korchemsky, 1907.13131, 2003.01121; Bargheer, Coronado, Vieira, 1904.00965, 1909.04077;…



$\sim$



- More recently, $\Gamma_{\rm oct}$ appeared in the FFOPE too (??)
  Sever, Tumanov, Wilhelm, 2112.10569

BES Kernel

Beisert, Eden, Staudacher, hep-th/0610251

- Plays a critical role in describing a spinning string, or equivalently, twist two operators in planar N=4 SYM.
  Basso, 1010.5237
- Integral equation for spin fluctuation density $\sigma(t)$ with magic kernel $K(t,t')$:

$$\frac{e^t - 1}{t}\sigma(t) = K(2gt, 0) - 4g^2 \int_0^\infty dt' K(2gt, 2gt')\sigma(t')$$

- Solution provides $\Gamma_{\text{cusp}}(g^2) = 8g^2\sigma(0)$
- Expanding in Bessel functions,

Benna, Benvenuti, Klebanov, Scardicchio, hep-th/0611135

equivalent to inverting a semi-infinite matrix,

$$\Gamma_{\text{cusp}}(g^2) = 4g^2\left[\frac{1}{1+\mathbb{K}}\right]_{11} \qquad \mathbb{K}_{ij} = 2j(-1)^{ij+j}\int_0^\infty \frac{dt}{t}\frac{J_i(2gt)J_j(2gt)}{e^t - 1}$$

Weak coupling expansion of $\mathbb{K}$

$$\mathbb{K}_{ij} = 2j(-1)^{ij+j}\int_0^\infty \frac{dt}{t}\frac{J_i(2gt)J_j(2gt)}{e^t-1}$$
$$= 2j(-1)^{j(i+1)}\sum_{k,l=0}^{\infty} g^{2(k+l)+i+j}\,(-1)^{k+l}$$
$$\times \frac{(2(k+l)+i+j-1)!}{k!l!(k+i)!(l+j)!}\zeta_{2(k+l)+i+j}$$

Belitsky, 1410.2534

- Write $\mathbb{K}_{ij}$ in 2 x 2 block form, according to whether i, j are odd/even:

$$\mathbb{K} = \begin{bmatrix} \mathbb{K}_{\circ\circ} & \mathbb{K}_{\circ\star} \\ \mathbb{K}_{\star\circ} & \mathbb{K}_{\star\star} \end{bmatrix}$$

- Odd zetas come from off-diagonal blocks

Tilted BES Proposal

B. Basso, LD, G. Papathanasiou, 2001.05460

$$\mathbb{K} = \begin{bmatrix} \mathbb{K}_{\circ\circ} & \mathbb{K}_{\circ\star} \\ \mathbb{K}_{\star\circ} & \mathbb{K}_{\star\star} \end{bmatrix}$$

- Introduce "tilt angle" $\alpha = 0, \quad \frac{\pi}{4}, \quad \frac{\pi}{3}$

  for oct, cusp, hex

- Then

$$\mathbb{K}(\alpha) = 2\cos\alpha \begin{bmatrix} \cos\alpha\, \mathbb{K}_{\circ\circ} & \sin\alpha\, \mathbb{K}_{\circ\star} \\ \sin\alpha\, \mathbb{K}_{\star\circ} & \cos\alpha\, \mathbb{K}_{\star\star} \end{bmatrix}$$

$$\Gamma_\alpha(g^2) = 4g^2 \left[\frac{1}{1 + \mathbb{K}(\alpha)}\right]_{11}$$

Weak coupling Γ_α

$$\Gamma_\alpha = 4g^2\{1 - 4\zeta_2 c^2 g^2 + 8\zeta_4 c^2(3 + 5c^2)g^4 - 8c^2[\zeta_6(25 + 42c^2 + 35c^4) + 4\zeta_3^2 s^2]g^6 + \cdots\}$$

$$c \equiv \cos\alpha\,, \qquad s \equiv \sin\alpha$$

- Can easily get to 20 loops.
- Finite radius of perturbative convergence same as for $\Gamma_{\pi/4} = \Gamma_{\text{cusp}}$
- Ratio of successive terms $\dfrac{\Gamma_{\text{cusp}}^{(L)}}{\Gamma_{\text{cusp}}^{(L-1)}} \to -16$

Constants are determinants

- We also find that

$$C_0 = -D\left(\frac{\pi}{3}\right) - \frac{1}{2}D(0) + D\left(\frac{\pi}{4}\right) - \frac{\zeta_2}{2}\Gamma_{\text{cusp}}$$

where

$$D(\alpha) = \ln\det[1 + \mathbb{K}(\alpha)]$$

- A number-theoretic "coaction principle" Schnetz, 1302.6445; Panzer, Schnetz, 1603.04289; Brown, 1512.06409

suggests a best ("cosmic") normalization for amplitude:

$\ln \mathcal{A}_6 \to \ln \mathcal{A}_6 - \ln \rho$, and through 7 loops [CDDvHMP, 1906.07116]

$$\ln \rho^{\text{new}} = D\left(\frac{\pi}{4}\right) - \frac{\zeta_2}{2}\Gamma_{\text{cusp}} = \ln \rho^{\text{old}} - \zeta_4 g^4 + \frac{50}{3}\zeta_6 g^6 - \frac{483}{2}\zeta_8 g^8 + \cdots$$

- In this normalization, only $\alpha = 0, \frac{\pi}{3}$ enter hexagon

Relation to master formula

$$\ln \mathcal{A}_n(g^2; \ell_i) = -\frac{1}{2}\oint_{\mathcal{C}} \frac{dz}{2\pi i z}\left(z - \frac{1}{z}\right)\mathcal{G}(g^2, z)\, \mathcal{S}_n(z, \ell_i)$$

$$\mathcal{G}\left(g^2, z = -e^{2i\alpha}\right) = \Gamma_\alpha(g^2),$$

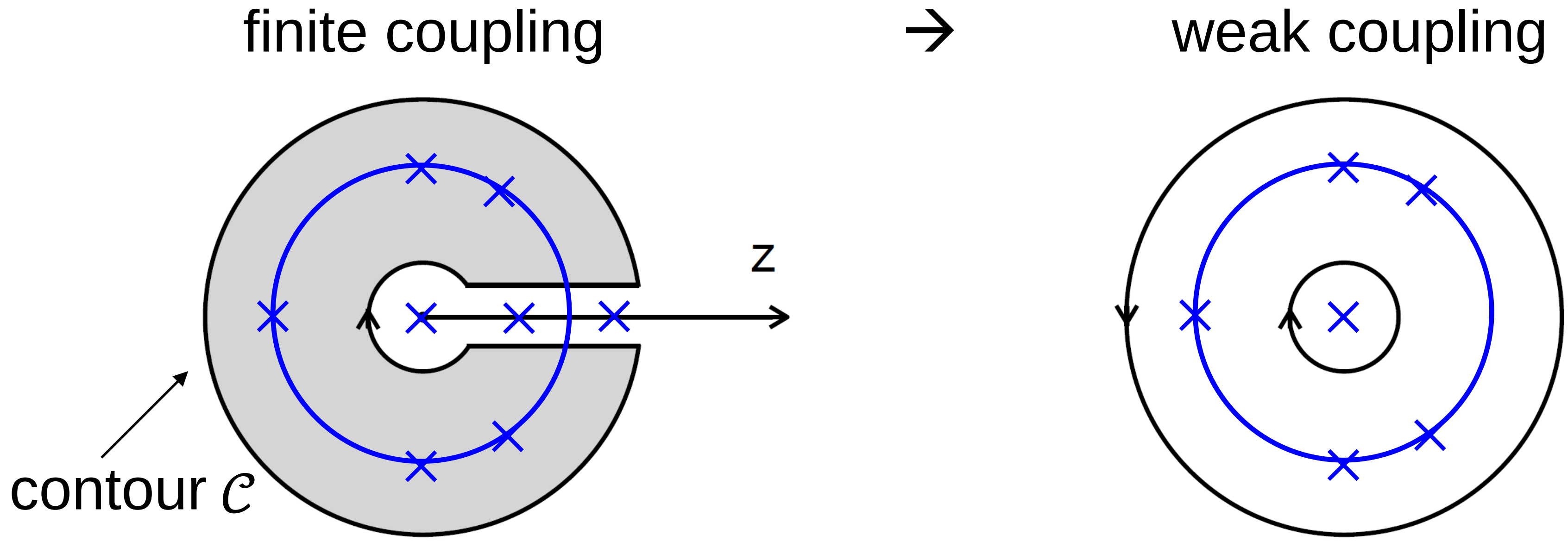
$$\mathcal{S}_n(z, \ell_i) = \frac{z(1-z^3)}{4(1+z)(1+z^2)\underbrace{(1-z^{3(n-4)})}_{\text{the important poles on unit circle}}}\,\mathcal{P}_n(z, \ell_i)$$

(generically)

$\mathcal{P}_n$ polynomial in $z, \frac{1}{z}$ $\Rightarrow$ no poles on unit circle

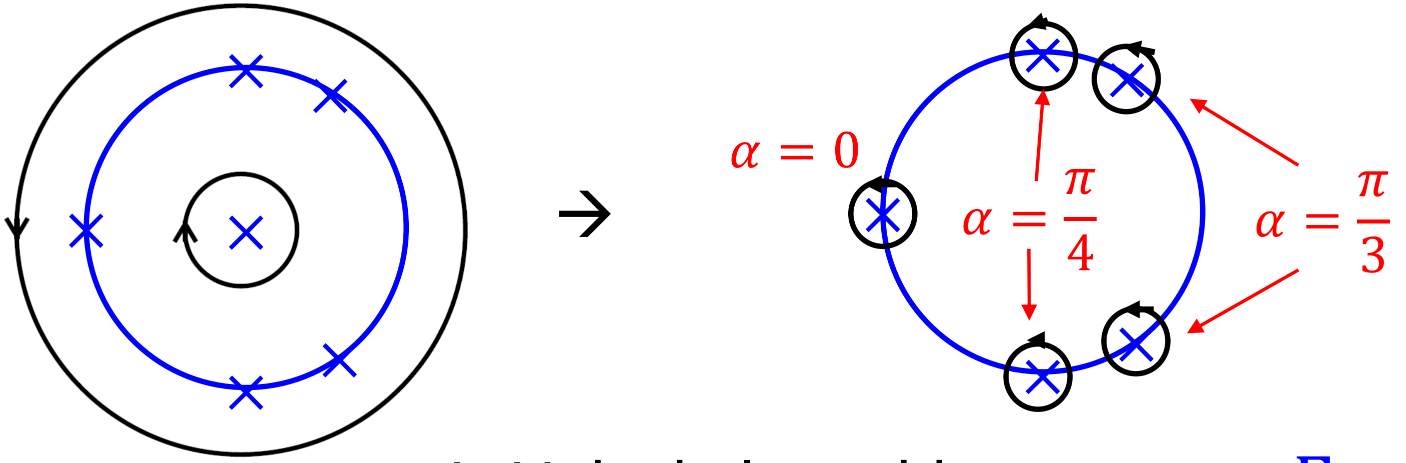
$$\mathcal{P}_6(z, \ell_i) = (\ell_1 - z\,\ell_2 + z^2\ell_3)(\ell_1 - \frac{1}{z}\,\ell_2 + \frac{1}{z^2}\ell_3)$$

Contour of integration



- Poles on real axis are for finite coupling $\mathcal{G}(g^2, z)$.
- Perturbatively: $\mathcal{G}$ is polynomial in $\sin\alpha$, $\cos\alpha$ $(z, \frac{1}{z})$
  → only unit circle poles from prefactor within $\mathcal{S}_n$
- Manifest symmetry under $z \leftrightarrow \frac{1}{z}$

Two weak coupling evaluations



1. Unit circle residues expose Γ_α representation but lead to ugly trigonometric factors (for $n > 6$)

↓

2. $z = 0$ and $z = \infty$ residues manifestly have only rational multiples of the ζ_k values in the coefficients of $\ell_i \ell_j$

Example:
Line between two $n = 7$ origins

$$\mathcal{R}_7(u_{2,3,4,5,6} \ll 1;\ u_7 + u_1 = 1) = \sum_{i=1}^{3} c_i P_i$$

$$P_1 = \sum_{i=1}^{7} \ell_i \ell_{i+2} - \sum_{i=1}^{3} \ell_i \ell_{i+4}$$

$$P_2 = -\ell_7 \ell_1 + \sum_{i=1}^{7} \ell_i^2 + \sum_{i=1}^{4} \ell_i \ell_{i+3}$$

$$P_3 = \sum_{i=1}^{6} \ell_i \ell_{i+1} + \sum_{i=1}^{5} \ell_i \ell_{i+2}$$

$$c_1 = -\frac{35}{8}\zeta_6 g^8 + \cdots$$

$$c_2 = -\frac{5}{2}\zeta_4 g^6 + \left(\frac{413}{8}\zeta_6 - 2\zeta_3^2\right) g^8 + \cdots$$

$$c_3 = \zeta_2 g^4 - \frac{37}{2}\zeta_4 g^6 + \left(\frac{1975}{8}\zeta_6 - 2\zeta_3^2\right) g^8 + \cdots$$

The 8-point origin universe

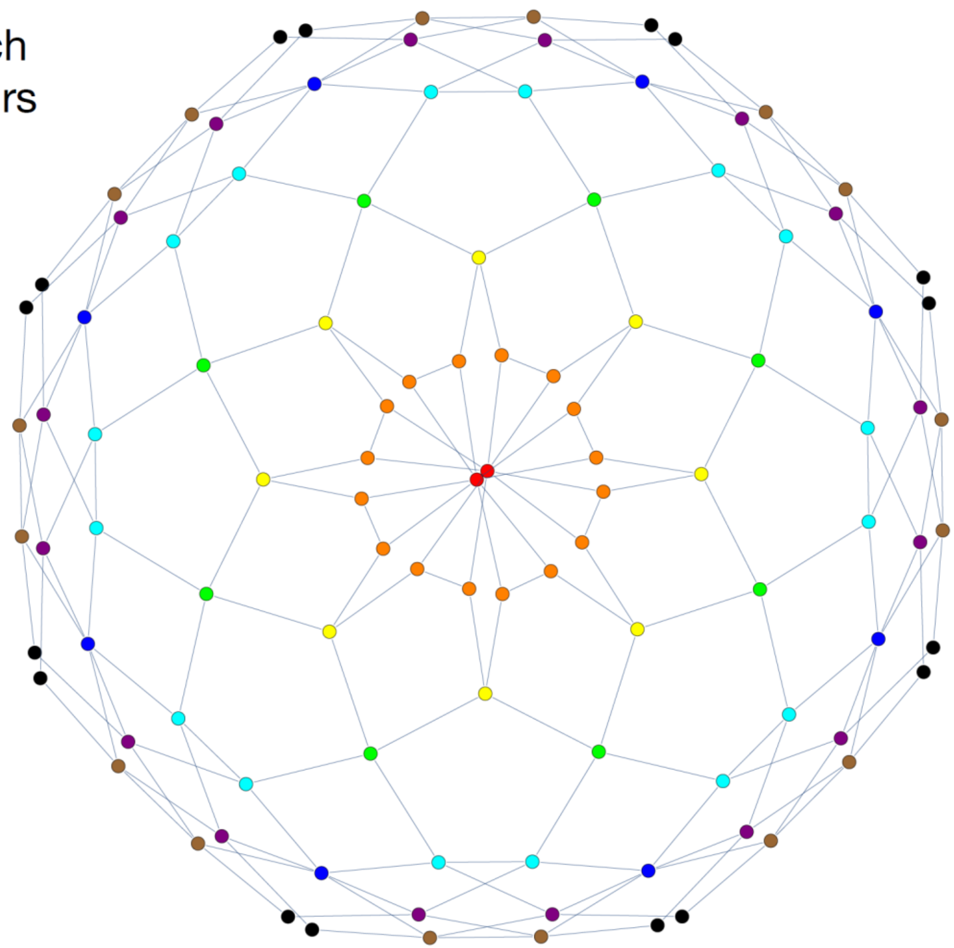
Origin Class	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	v_1	v_2	v_3	v_4
O_1	0	0	0	0	0	0	0	0	0	1	0	1
O_2	0	0	0	0	0	0	0	1	0	1	0	1
O_3	0	0	0	0	0	0	0	1	0	0	1	1
O_4	0	0	0	0	0	0	1	1	0	0	1	0
O_5	0	0	0	0	0	1	0	1	0	0	1	0
O_6	0	0	0	0	1	0	0	1	0	1	0	0
O_7	0	0	0	0	1	0	0	1	0	0	1	0
O_8	0	0	0	1	0	0	0	1	0	1	0	0
O_9	0	0	0	1	0	0	0	1	0	0	0	1
O_{10}	0	0	1	0	0	1	0	1	0	0	0	0

cluster mutations, other parametric scalings in "positive region" allow us to study all these

outlier??

And many lines, surfaces and volumes between them!

Origins form web with links along which double-log behaviours extend



O1 red

O2 orange
O3 yellow
O4 green
O5 cyan
O6 blue
O7 purple
O8 brown
O9 black

O10 missing

Observation: double-log behaviour extends to domains connecting origins with same # of vanishing cross ratios

For example, a 3d cube with 8 corners, O6 (x2), O7 (x2), O8 (x3) and O9, exhibits quadratic log behavior throughout its volume!

Checks of master formula

- $n = 6, \ (u_1, u_2, u_3) \to (0,0,0)$

Checked to 7 loops originally CDvHMP, 1903.10890, now to 8 loops LD, Liu, to appear

- $n = 7, \ (u_1, u_2, u_3, u_4, u_5, u_6, u_7) \to (0,0,0,0,0,0,1)$

and lines between. Checked to 4 loops LD, Liu, 2007.12966

- $n = 8, \ (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, v_1, v_2, v_3, v_4) \to$ many different limits

Checked to 2 loops Basso, LD, Liu, Papathanasiou, to appear

All loop structural check

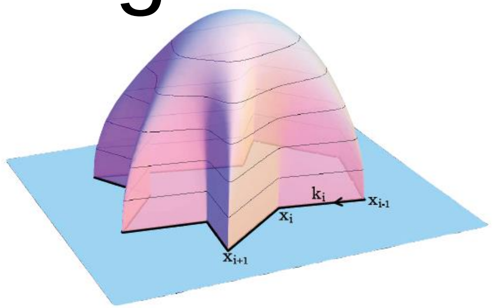
Make a “baby bootstrap”, assume:

1. Piecewise quadratic log behavior
2. Residual dihedral symmetry
3. MHV final entry condition
4. Continuity at junctions between domains

The number of independent solutions matches the number of roots in $\mathcal{S}_n$

Formula inspired by strong coupling

AdS



Alday, Maldacena 0705.0303

- $\ln \mathcal{A}_n \sim \sqrt{\lambda} \times$ [area of worldsheet in AdS_5]

- Minimal area problem classically integrable, TBA
  Alday, Gaiotto, Maldacena, Sever, Vieira, 0911.4708, 1010.5009, 1102.0062

$$\log Y_{2,s}(\theta) = -|m_s|\sqrt{2}\cosh(\theta - i\phi_s) - K_2 \star \alpha_s - K_1 \star \beta_s$$

$$\log Y_{1,s}(\theta) = -|m_s|\cosh(\theta - i\phi_s) - C_s - \tfrac{1}{2}K_2 \star \beta_s - K_1 \star \alpha_s - \tfrac{1}{2}K_3 \star \gamma_s$$

$$\log Y_{3,s}(\theta) = -|m_s|\cosh(\theta - i\phi_s) + C_s - \tfrac{1}{2}K_2 \star \beta_s - K_1 \star \alpha_s + \tfrac{1}{2}K_3 \star \gamma_s$$

$$K_1(\theta) = \frac{1}{2\pi\cosh\theta} \;,\; K_2(\theta) = \frac{\sqrt{2}\cosh\theta}{\pi\cosh 2\theta} \;,\; K_3(\theta) = \frac{i}{\pi}\tanh 2\theta$$

$$\alpha_s \equiv \log\frac{(1+Y_{1,s})(1+Y_{3,s})}{(1+Y_{2,s-1})(1+Y_{2,s+1})} ,$$

- For generic kinematics, solve numerically, iteratively

Area formula (cont.)

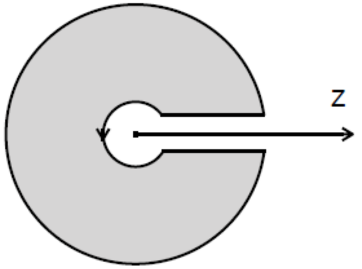
- In origin limit, TBA simplifies drastically, especially after Fourier transforming to simplify convolutions

Ito, Satoh, Suzuki, 1805.07556

$$n = 6: \qquad \hat{K}(s) = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} e^{i\theta s} K(\theta) = \frac{1}{2\cosh\frac{\pi s}{2}}$$

$$\text{Area} = \int_{-\infty}^{\infty} ds\, \hat{I}(-s) \hat{f}(s), \qquad \hat{f}(s) = \frac{1}{1-\hat{K}(s)} \hat{I}(s)$$

- $\hat{I}(s) = -\dfrac{2\{\ell_1 \exp[-\frac{\pi s}{2}] - \ell_2 + \ell_3 \exp[\frac{\pi s}{2}]\}}{\cosh\frac{\pi s}{4} \cosh\frac{\pi s}{2}}$
- Matrix for $n > 6$.
- Change integration variable to $z = \exp[\frac{\pi s}{2}]$
- Insert $\ln(z)$ by switch of contour from $\mathbb{R}^+$ to $\mathcal{C}$



Promote to exact coupling proposal

- Insert $\dfrac{\mathcal{G}(g^2,z)}{\mathcal{G}(g^2,z=-e^{2i\alpha})|_{g\to\infty}}$ into Area formula

where

$$\mathcal{G}\left(g^2, z=-e^{2i\alpha}\right)|_{g\to\infty} = \frac{8\alpha g}{\pi \sin 2\alpha} \propto \sqrt{\lambda}\ln z \frac{1}{z-1/z}$$

$$\Rightarrow \quad \ln \mathcal{A}_n(g^2;\ell_i) = -\frac{1}{2}\oint_{\mathcal{C}} \frac{dz}{2\pi i z}\left(z-\frac{1}{z}\right)\mathcal{G}(\, g^2, z)\, \mathcal{S}_n(z,\ell_i)$$

$$\mathcal{S}_n(z,\ell_i) = \frac{z(1-z^3)}{4(1+z)(1+z^2)(1-z^{3(n-4)})}\,\mathcal{P}_n(z,\ell_i)$$

(generically)

$$\mathcal{P}_6(z,\ell_i) = (\ell_1 - z\,\ell_2 + z^2\ell_3)(\ell_1 - \tfrac{1}{z}\,\ell_2 + \tfrac{1}{z^2}\,\ell_3)$$

Properties of Γ_{α}

General Properties of Γ_α

- Finite radius of convergence at weak coupling → $g < \frac{1}{4}$
- Obtain to much larger values of g by truncating

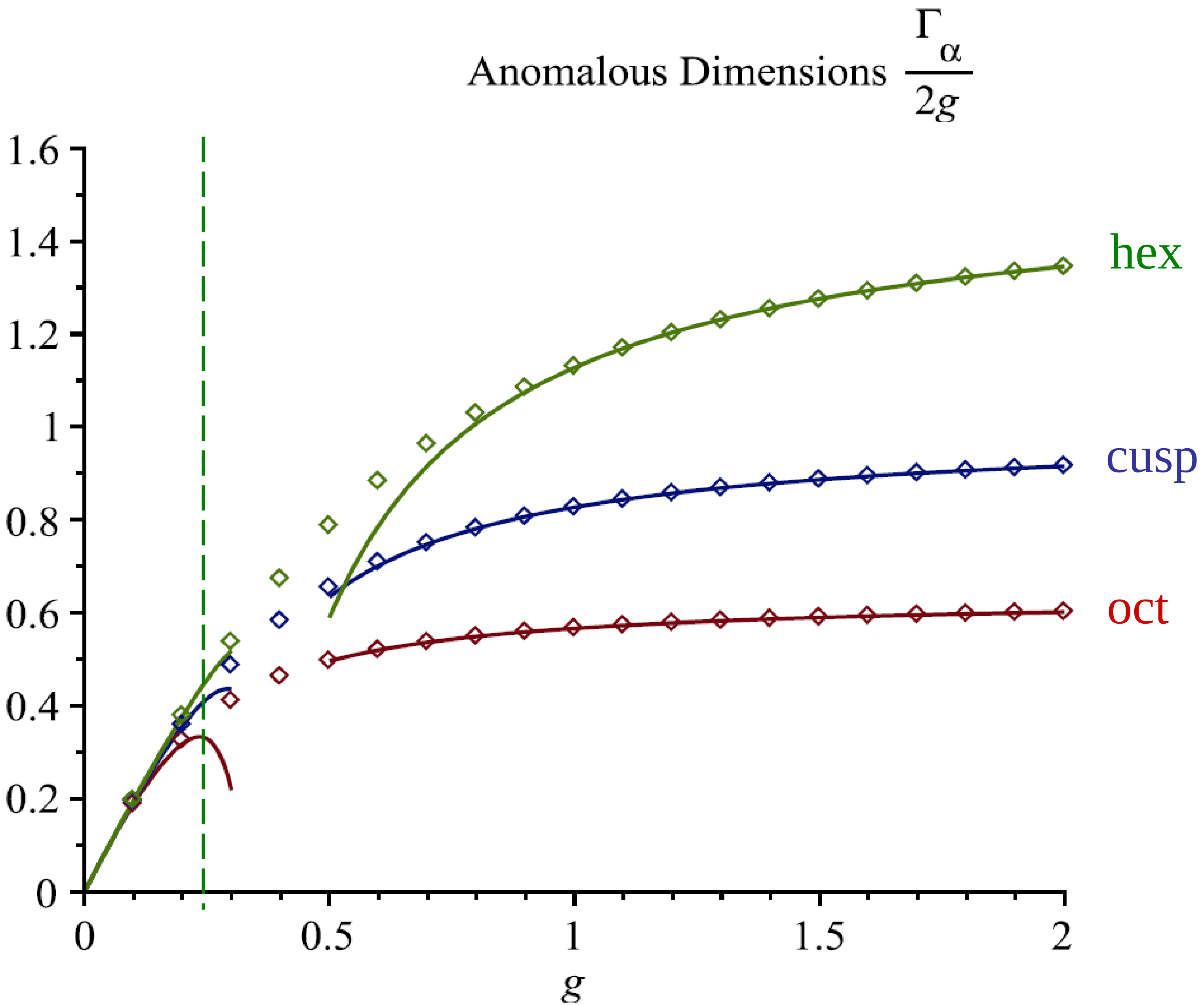
$$\mathbb{K}_{ij} = 2j(-1)^{ij+j} \int_0^\infty \frac{dt}{t} \frac{J_i(2gt) J_j(2gt)}{e^t - 1}$$

Benna, Benvenuti, Klebanov, Scardicchio, hep-th/0611135

to finite size, evaluate Bessel integrals numerically

- Strong coupling expansion is asymptotic, and not Borel resummable Basso, Korchemsky, Kotanski, 0708.3933

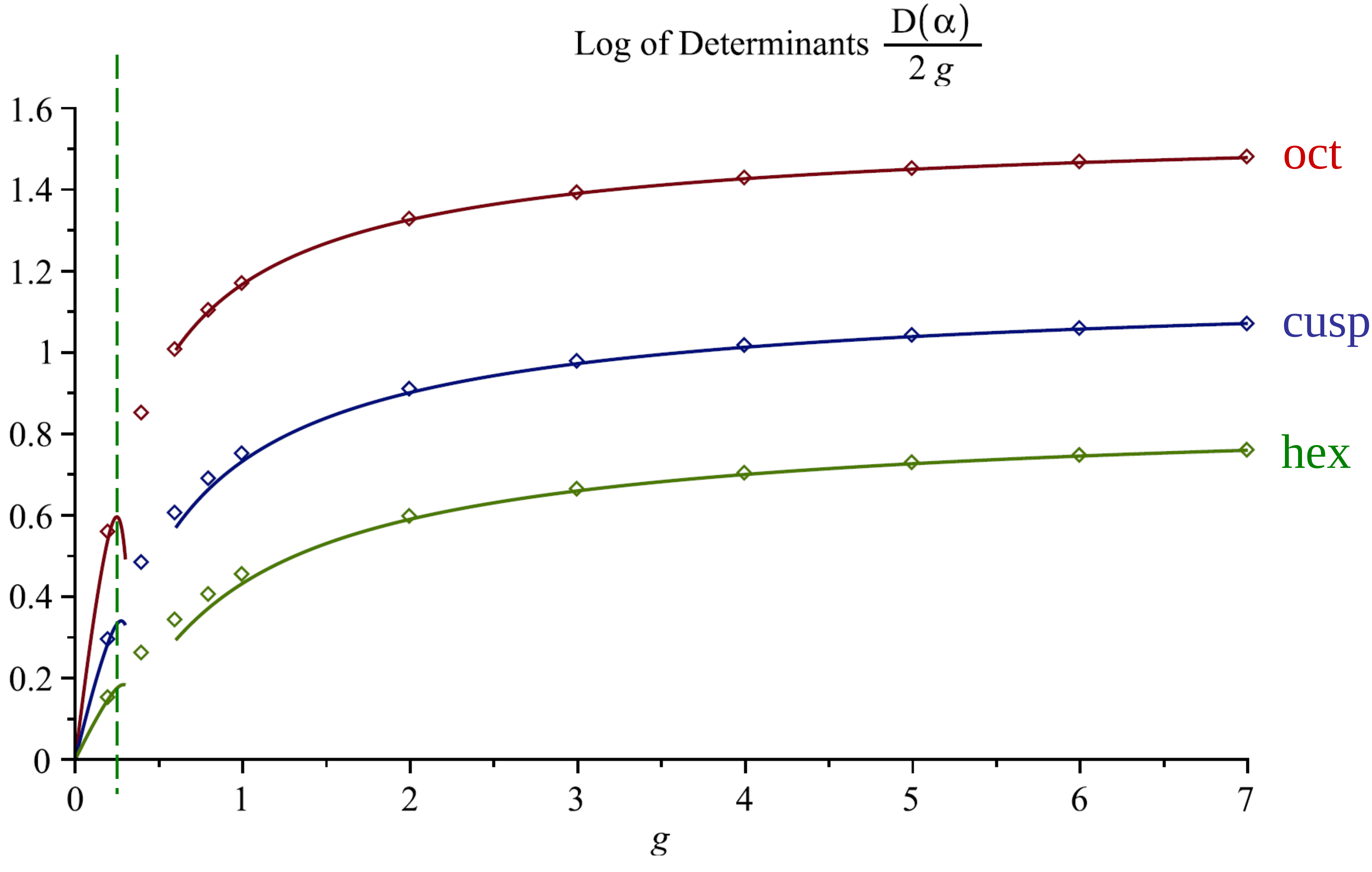
location of poles in Borel plane can be determined from high order expansion Basso, LD,Dunne, Liu, to appear



$\alpha = \frac{\pi}{3}$

$\alpha = \frac{\pi}{4}$

$\alpha = 0$



Log of Determinants $\frac{D(\alpha)}{2g}$
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6
7
g
oct
cusp
hex

Conclusions & Outlook

- "Origins" are kinematic regions where scattering amplitudes simplify dramatically in planar N=4 SYM
- Rich kinematical structure starting at $n = 8$ gluons
- Results controlled by tilted cusp anomalous dimension Γ_α which is determined by a simple modification to the BES kernel
- Can perform the same weak, finite and strong coupling analyses as for the usual cusp anomalous dimension
- Some challenges:

  - Prove the master formula; establish its range of validity
  - interpolate between origin and other limits?
  - NMHV amplitudes at origin?
  - strong coupling, resurgence, …

Extra Slides

Strong Coupling Validation

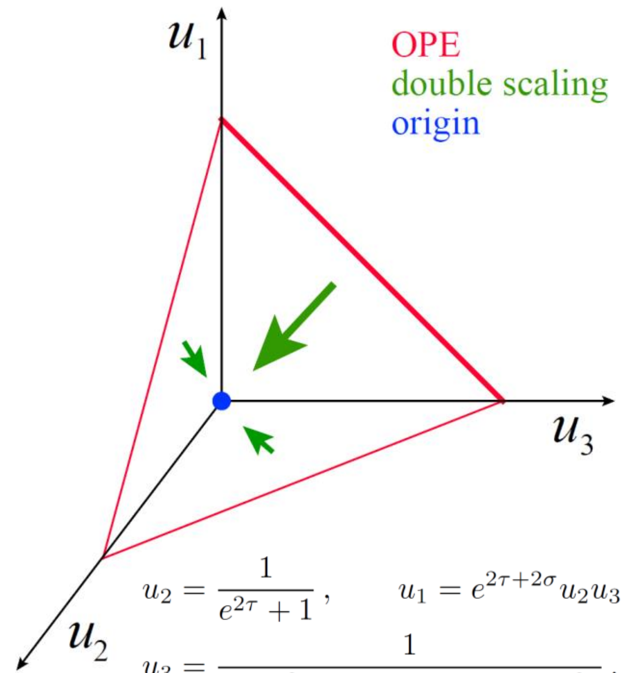
- Strong coupling limit tested against string theory: area of minimal surface Alday, Maldacena, 0705.0303 plus constant from determinant of scalars for S^5 in $\mathrm{AdS}_5 \times \mathrm{S}^5$ Basso, Sever, Vieira, 1405.6350
- On diagonal, area can be computed analytically:

Alday, Gaiotto, Maldacena, 0911.4708

$$\left.\frac{\ln \mathcal{E}(u,u,u)}{\Gamma_{\mathrm{cusp}}}\right|_{g\to\infty} = -\frac{3}{4\pi}\ln^2 u - \frac{\pi^2}{12} - \frac{\pi}{6} + \frac{\pi}{72}$$

- Agrees perfectly with strong coupling limit of C_0

Origin of the results



$$u_2 = \frac{1}{e^{2\tau}+1}, \qquad u_1 = e^{2\tau+2\sigma} u_2 u_3$$

$$u_3 = \frac{1}{1+e^{2\sigma}+2e^{\sigma-\tau}\cos\phi + e^{-2\tau}}.$$

- To approach origin via pentagon OPE, must sum over large number N of large helicity a_k gluonic bound state flux tube excitations.
- Framed Wilson loop:

$$\tau \to \infty$$

$$\varphi \equiv i\phi \to \infty$$

$$\mathcal{W}_6 = \mathcal{E} \times \exp\left[\frac{\Gamma_{\text{cusp}}}{2}\left(\sigma^2 + \tau^2 + \zeta_2\right)\right]$$

gluonic contribution:

rapidity, energy, momentum, measure

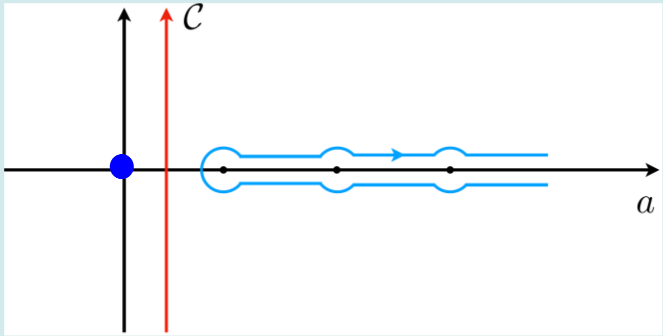
$$\mathcal{W}_6 = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\mathbf{a}} e^{i\phi \sum_{k=1}^{N} a_k} \int \frac{d\mathbf{u}}{(2\pi)^N} \frac{e^{-\tau E + i\sigma P} \prod_k \mu_k}{\prod_{k<l} P_{kl} P_{lk}}$$

pentagon transition

Weak coupling

- Expand E, P, μ_k, P_{kl} in g.
- N excitation contribution only starts at N^2 loops, so can get to 8 loops (9 loops for log) with only 2 excitations.
- Large a_k → Sommerfeld-Watson transform

$$\sum_{a\geqslant 1}(-1)^a f(a) \to \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{i\,f(a)\,da}{2\sin(\pi a)}$$



- Deform a integral to $a = 0$ residue (after doing u integrals)
- Agrees with full amplitude limit, goes to 8 (9) loops

Finite coupling

- To simplify E, p, μ_k, P_{kl} , analytically continue $\mathbf{u}$ to "Goldstone sheet" Basso, Sever, Vieira, 1407.1736

… → $$\mathcal{E} = \mathcal{N} \int \prod_{i=1}^{\infty} d\xi_i^+ d\xi_i^- F_\varphi(\vec{\xi})\, e^{-\vec{\xi}\cdot M \cdot \vec{\xi}}$$ $M \sim 1 + \mathbb{K}$

- $\vec{\xi}$ is conjugate to charges $\vec{Q} = \sum_{k=1}^{N} \vec{q}\,(u_k, a_k)$
- F_φ is Fredholm determinant,

$$\ln F_\varphi = -\sum_{N\geqslant 1} \frac{1}{N} \sum_{\mathbf{a}} \oint \frac{d\mathbf{u}}{(2\pi)^N} \prod_{k=1}^{N} \frac{\hat{\mu}_k e^{\varphi a_k}}{x_k^+ - x_{k+1}^-}\, e^{2i\vec{Q}\cdot\vec{\xi}}$$

A Secretly Gaussian Integral

- At weak coupling, $Q_i \sim g^i$, and can expand

$$\ln F_\varphi(\vec{\xi}) = \langle 1\rangle + 2i\langle Q_i^m\rangle \xi_i^m - 2\langle Q_i^m Q_j^n\rangle \xi_i^m \xi_j^n + \ldots$$

- All moments > 2 vanish as $\varphi \to \infty$ (!):

$$\lim_{\varphi\to\infty} \langle Q_i^m Q_j^n\, Q_k^p \ldots \rangle = 0$$

- Also compute $\langle 1\rangle$, $\langle \vec{Q}\rangle$, $\langle \overleftrightarrow{QQ}\rangle$ explicitly through 4 loops, extrapolate by writing in terms of $\mathbb{K}(\alpha)$
- Leads to our finite-coupling proposals.

Planar N=4 SYM amplitudes

- Amplitudes are very rigid:

$$\text{Amplitudes} = \sum_i rational_i \times transcendental_i$$

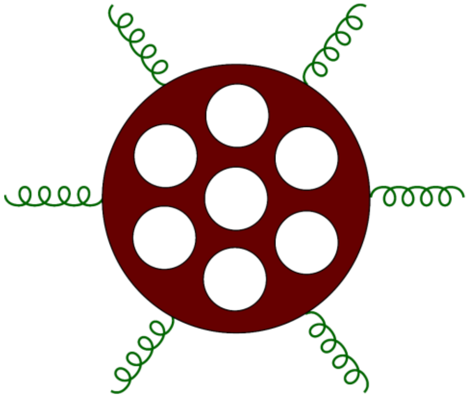
- For planar N=4 SYM, we understand rational structure quite well:
- $rational_i$ is trivial for MHV, dual superconformal invariants for NMHV
- Can focus on the transcendental functions.
- Space of functions is so restrictive, and physical constraints are so powerful, one can write (MHV) L loop answer as linear combination of known weight $2L$ polylogarithms.
- Unknown coefficients found by solving (a large number of) linear constraints

Hexagon function bootstrap

Loops

Loops	
3	LD, Drummond, Henn, 1108.4461, 1111.1704;
4,5	Caron-Huot, LD, Drummond, Duhr, von Hippel, McLeod, Pennington, 1308.2276, 1402.3300, 1408.1505, 1509.08127; 1609.00669;
6,7	Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1903.10890, 1906.07116; LD, Dulat, 20mm.nnnnn (NMHV 7 loop)

- Use analytical properties of perturbative (six) point amplitudes in planar N=4 SYM to determine them directly, without ever peeking inside the loops
- Step toward doing this nonperturbatively (no loops to peek inside) for general kinematics

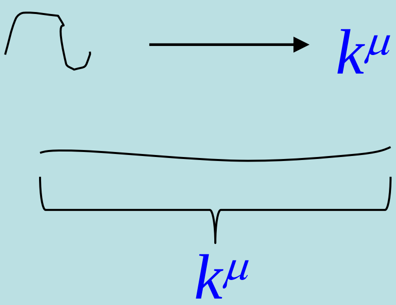


T-duality symmetry of string theory

Alday, Maldacena, 0705.0303

- Exchanges string world-sheet variables σ, τ
- $X^\mu(\tau,\sigma) = x^\mu + k^\mu \tau$ + oscillators

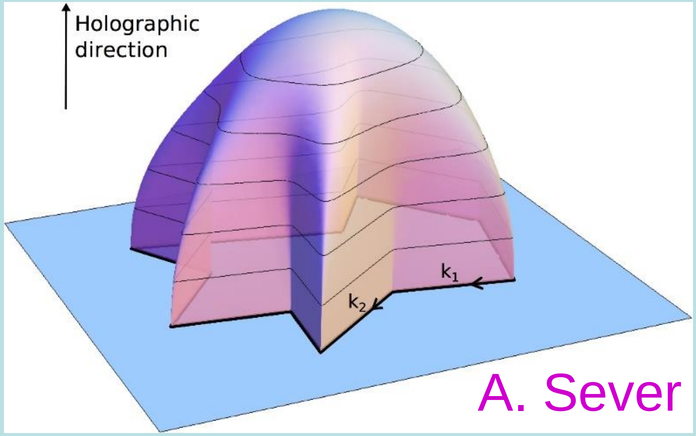
→ $X^\mu(\tau,\sigma) = x^\mu + k^\mu \sigma$ + oscillators



- Strong coupling limit of planar gauge theory

is semi-classical limit of string theory: world-sheet stretches tight around minimal area surface in AdS.

- Boundary determined by momenta of external states: light-like polygon with null edges = momenta k^μ



A. Sever

BDS-like ansatz

$$\frac{\mathcal{A}_6^{\text{BDS-like}}}{\mathcal{A}_6^{\text{MHV}\,(0)}} = \exp\left[\sum_{L=1}^{\infty} a^L \left(f^{(L)}(\epsilon)\frac{1}{2}\hat{M}_6(L\epsilon) + C^{(L)} \right)\right]$$

where $f^{(L)}(\epsilon) = \frac{1}{4}\gamma_K^{(L)} + \epsilon\,\frac{L}{2}\mathcal{G}_0^{(L)} + \epsilon^2\, f_2^{(L)}$ are constants, and

$$\begin{aligned}\hat{M}_6(\epsilon) &= M_6^{1-\text{loop}}(\epsilon) + Y(u,v,w) \\ &= \sum_{i=1}^{6}\left[-\frac{1}{\epsilon^2}(1-\epsilon \ln s_{i,i+1}) - \ln s_{i,i+1}\ln s_{i+1,i+2} + \frac{1}{2}\ln s_{i,i+1}\ln s_{i+3,i+4}\right] + 6\zeta_2\end{aligned}$$

- Y is dual conformally invariant part of one-loop amplitude $M_6^{1-\text{loop}}$ containing all 3-particle invariants:

$$Y(u,v,w) = -\mathcal{E}^{(1)} = -\text{Li}_2\left(1-\frac{1}{u}\right) - \text{Li}_2\left(1-\frac{1}{v}\right) - \text{Li}_2\left(1-\frac{1}{w}\right)$$

- More minimal BDS-like ansatz contains all IR poles, but **no 3-particle invariants**.

At $(u,v,w) = (1,1,1)$, amplitude →MZVs

Allowed MZV's obey a Galois "co-action" principle, restricting the combinations that can appear
Brown, Panzer, Schnetz

MHV

$$\mathcal{E}^{(1)}(1,1,1) = 0\,,$$
$$\mathcal{E}^{(2)}(1,1,1) = -10\,\zeta_4\,,$$
$$\mathcal{E}^{(3)}(1,1,1) = \frac{413}{3}\,\zeta_6\,,$$
$$\mathcal{E}^{(4)}(1,1,1) = -\frac{5477}{3}\,\zeta_8 + 24\left[\zeta_{5,3} + 5\,\zeta_3\,\zeta_5 - \zeta_2\,(\zeta_3)^2\right],$$
$$\mathcal{E}^{(5)}(1,1,1) = \frac{379957}{15}\,\zeta_{10} - 12\left[4\,\zeta_2\,\zeta_{5,3} + 25\,(\zeta_5)^2\right] - 96\left[2\,\zeta_{7,3} + 28\,\zeta_3\,\zeta_7 + 11\,(\zeta_5)^2 - 4\,\zeta_2\,\zeta_3\,\zeta_5 - 6\,\zeta_4\,(\zeta_3)^2\right]$$

NMHV

$$E^{(1)}(1,1,1) = -2\,\zeta_2\,,$$
$$E^{(2)}(1,1,1) = 26\,\zeta_4\,,$$
$$E^{(3)}(1,1,1) = -\frac{940}{3}\,\zeta_6\,,$$
$$E^{(4)}(1,1,1) = -\frac{36271}{9}\,\zeta_8 - 24\left[\zeta_{5,3} + 5\,\zeta_3\,\zeta_5 - \zeta_2\,(\zeta_3)^2\right],$$
$$E^{(5)}(1,1,1) = -\frac{1666501}{30}\,\zeta_{10} + 12\left[4\,\zeta_2\,\zeta_{5,3} + 25\,(\zeta_5)^2\right] + 132\left[2\,\zeta_{7,3} + 28\,\zeta_3\,\zeta_7 + 11\,(\zeta_5)^2 - 4\,\zeta_2\,\zeta_3\,\zeta_5 - 6\,\zeta_4\,(\zeta_3)^2\right]$$

Cosmic normalization

- To fit amplitudes into the minimal space of functions requires, starting at 3 loops, redefining the BDS-like ansatz, by a multi-loop constant ρ :

$$\mathcal{A}_6^{\mathrm{BDS-like}'} = \mathcal{A}_6^{\mathrm{BDS-like}} \times \rho$$

$$\begin{aligned}\rho(g^2) = 1 + 8(\zeta_3)^2 g^6 - 160\zeta_3\zeta_5\, g^8 + \Big[1680\zeta_3\zeta_7 + 912(\zeta_5)^2 - 32\zeta_4(\zeta_3)^2\Big]\, g^{10} \\ - \Big[18816\zeta_3\zeta_9 + 20832\zeta_5\zeta_7 - 448\zeta_4\zeta_3\zeta_5 - 400\zeta_6(\zeta_3)^2\Big]\, g^{12} \\ + \Big[221760\zeta_3\zeta_{11} + 247296\zeta_5\zeta_9 + 126240(\zeta_7)^2 - 3360\zeta_4\zeta_3\zeta_7 - 1824\zeta_4(\zeta_5)^2 \\ - 5440\zeta_6\zeta_3\zeta_5 - 4480\zeta_8(\zeta_3)^2\Big]\, g^{14} \; + \; \mathcal{O}(g^{16}).\end{aligned}$$

- Now we have a flux tube interpretation for ρ !