Correspondence between **classical** and **quantum** integrable theories: applications and origin

"Talking Integrability: Spin, Fields and Strings", 31-8-2022, KAVLI Institute, UC Santa Barbara

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- Sketch of a PLAN:

Leitmotiv

- 1)Motivations: in many research topics **Thermodynamic Bethe Ansatz** appears with different physical meanings. Little in gauge theory, not in **General Relativity**, **BH** physics: **APPLICATIONS**.
- 2) **Traditional TBA**: particle scattering in 2D QFT, **I way**
- 3)AdS/CFT: gauge theory Operator Product Expansion→**Form factor series** for null polygonal WLs re-sums to TBA at strong coupling: **II way**.

Gauge th. or BH physics 2D CFT

- 4) **Ordinary Differential Eq.**→ Integrable Models **ODE/IM correspondence**: functional, integral eqs→STRUCTURE. **III way**.
- 5) **PDE**/IM with **masses** enlarge the view, then **why ODE/IM?**

Classical Integrable system=classical Lax pair

ORIGIN:←

Ubiquitus TBA

many talks here: **I way**

- We know TBA for computing **energy**, less its **discontinuity formulae**, Kontsevich-Soibelman (Donaldson-Thomas invariants): **WKB** (Dalabaere-Pham), **resurgence**, (compactified) **susy gauge theories** wall-crossing of BPS states entails

Gaiotto-Moore-Neitzke

- $$\mathcal{X}_\gamma(\zeta) = \mathcal{X}^{\rm sf}_\gamma(\zeta)\exp\left[-\frac{1}{4\pi i}\sum_{\gamma'}\Omega(\gamma';u)\langle\gamma,\gamma'\rangle\int_{\ell_{\gamma'}}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log(1-\sigma(\gamma')\mathcal{X}_{\gamma'}(\zeta'))\right]. \tag{5.13}$$

- which are nothing but **TBA EQS**, but **no scattering**

- *Note added Nov. 20, 2009*:

  It was pointed out to us some time ago by A. Zamolodchikov that one of the central results of this paper, equation (5.13), is in fact a version of the Thermodynamic Bethe Ansatz [45]. In this appendix we explain that remark. Another relation between four-dimensional super Yang-Mills theory and the TBA has recently been discussed by Nekrasov and Shatashvili [46].

  The TBA equations for an integrable system of particles a with masses m_a, at inverse temperature β, with integrable scattering matrix $S_{ab}(\theta-\theta')$, where θ is the rapidity, are

  $$\epsilon_a(\theta) = m_a\beta\cosh\theta - \sum_b\int_{-\infty}^{+\infty}\frac{d\theta'}{2\pi}\phi_{ab}(\theta-\theta')\log(1+e^{\beta\mu_b-\epsilon_b(\theta')}) \tag{E.1}$$

  where $\phi_{ab}(\theta) = -i\frac{\partial}{\partial\theta}\log S_{ab}(\theta)$. Here the scattering matrix is diagonal, that is, the soliton creation operators obey $\Phi_a(\theta)\Phi_b(\theta') = S_{ab}(\theta-\theta')\Phi_b(\theta')\Phi_a(\theta)$.

circumference R

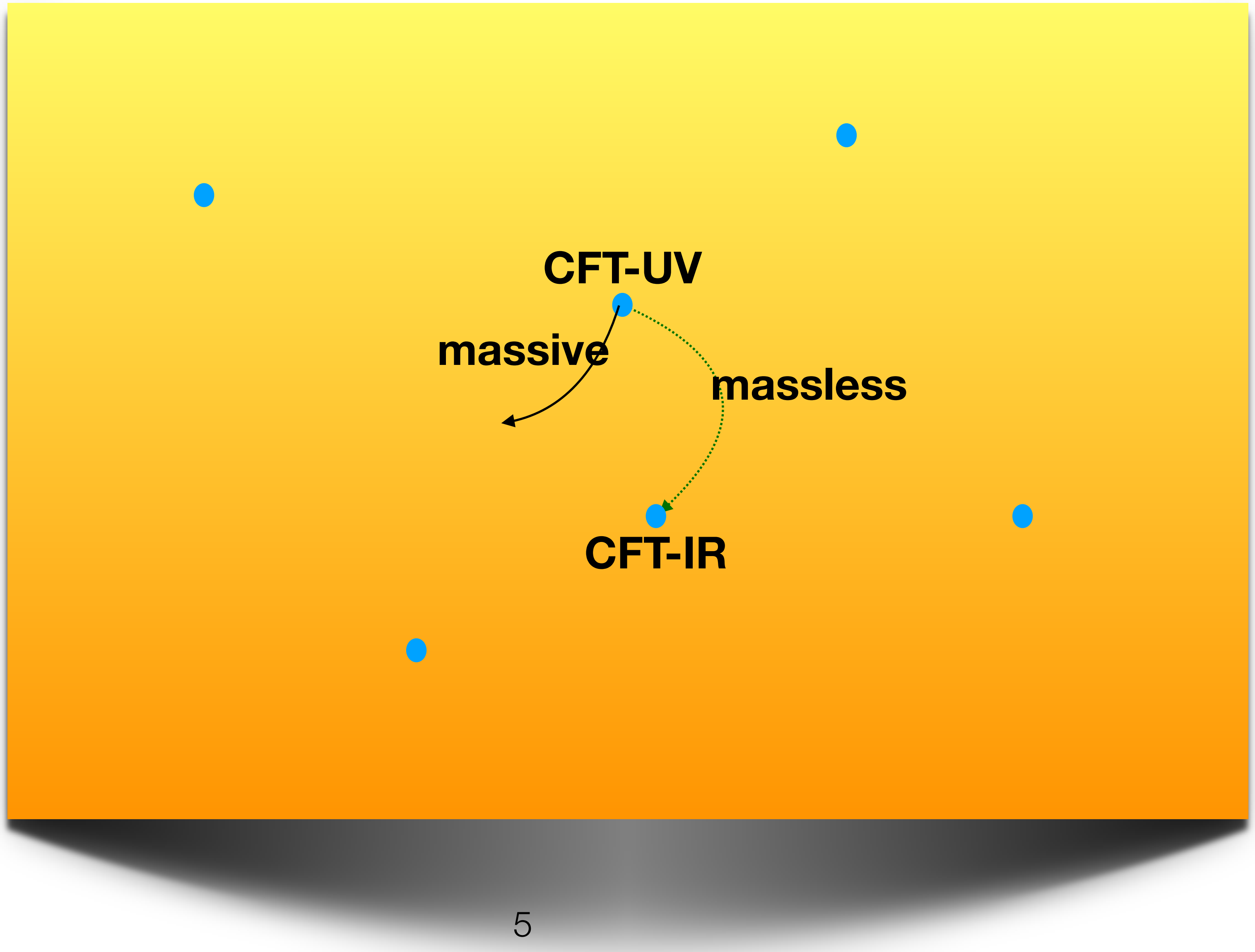
more general

Ubiquitus TBA for different communities

- The same mathematical problem for very different physical problem: gluon **scattering amplitudes/Wilson loops** (null polygon) in N=4 **STRONG gauge theory** dual to **minimal string area** (thanks to **AdS/CFT**): same TBA.
- We re-summed the OPE (Form-Factor) series of Wl (collinear limit) to TBA.
- The **general phenomenon on the background** is the so-called linear **Ordinary Differential Equation→Integrable Model (ODE→IM) correspondence** (2D CFTs), possibly extended to linear **PDE (Massive QFTs)**. The II way to TBA.
- Recently we proposed **an advance** (different ODE) which identifies **NS** (**SW** with one Omega background) **periods** with integrable quantities **T,Q**: **connexion with SW geometry**. **Gauge motivation. And origin (?)**
- **ORIGIN: IM→ODE**. TBA is part of **general integrability structure**: QQ, TQ, TT, Y system and ONE single (complex) Non Linear Integral Equation (TBA eqs. re-sum to it), QSC: general benefit. In common with **SPIN CHAINS**.

Motivation: finite size spectrum and the space of theories:

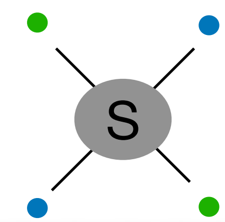
flows from conformal theories ↔ non perturbative physics



(Thermodynamic) Bethe Ansatz

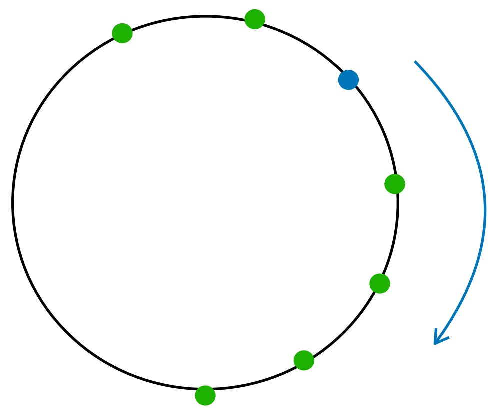
(Yang-Yang) Zamolodchikov;.......;...

- Physics Bethe eqs: propagation of a test (blu) particle and its scattering on the others (green)= 1



$$e.g. = \frac{p_l - p_m - i}{p_l - p_m + i}$$

Spin chain lenght L



$$e^{ip_l L} \prod_{m=1, m\neq l}^{L} S(p_l, p_m) = 1$$

free phase scattering phase

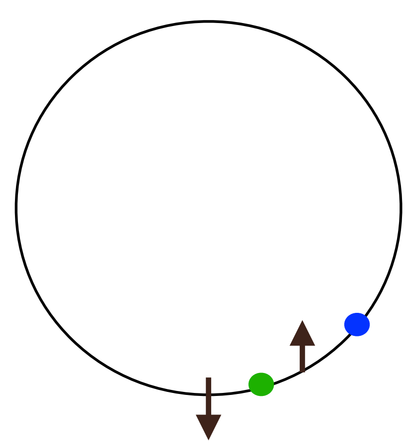
- This implies a sort of microscopic S matrix, valid at finite size (periodic) for **spin chains. Not for field theories!**

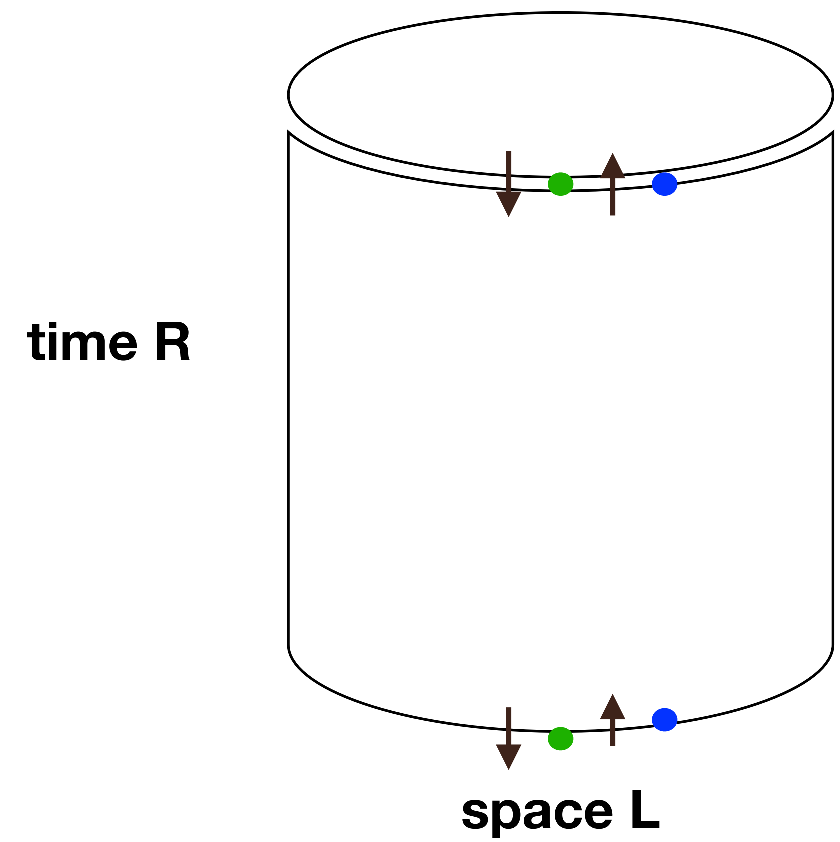
- In field theories, same form eqs. with S-matrix, valid only at infinite size. For finite size, **not** exact energy.

Computation of the LxR torus partition in two ways

finite L $\simeq$

large $L \to \infty$ $\quad e^{ip_l L} \prod_{m=1, m\neq l}^{L} S(p_l, p_m) = 1$



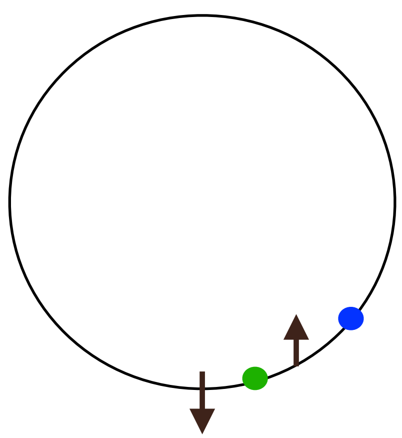


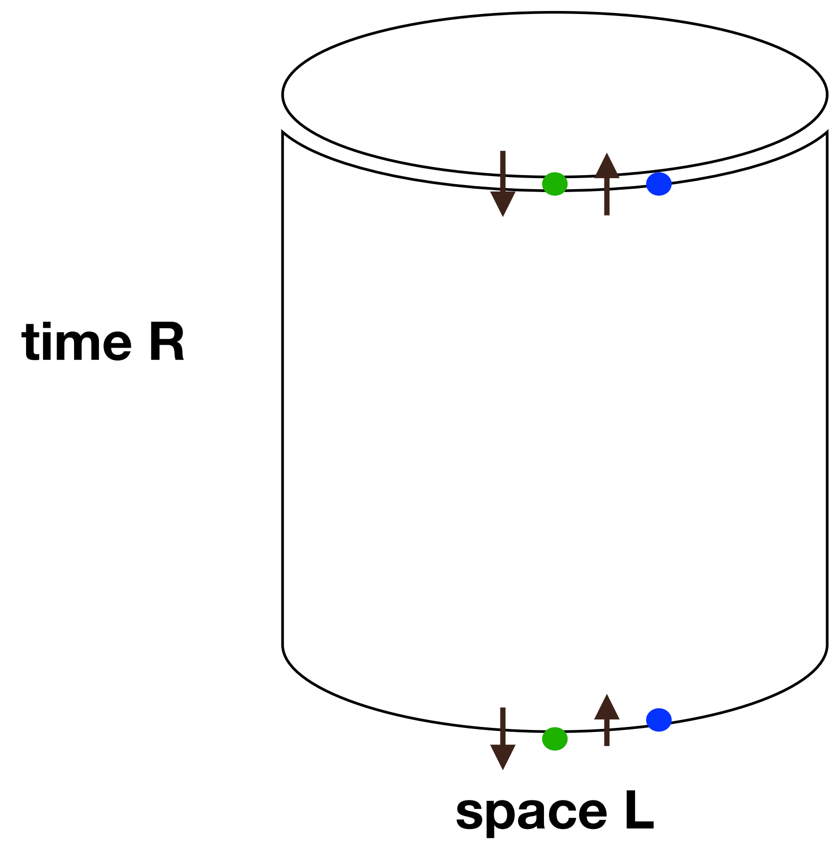
1) Direct theory: ground state energy/anomalous dimension (gauge theory) $E_0 = \Delta_0$ cannot be computed exactly at finte L: $\simeq$

Computation of the LxR torus partition in two ways

finite L $\simeq$

large $L \to \infty$ $\quad e^{ip_l L} \prod_{m=1, m\neq l}^{L} S(p_l, p_m) = 1$



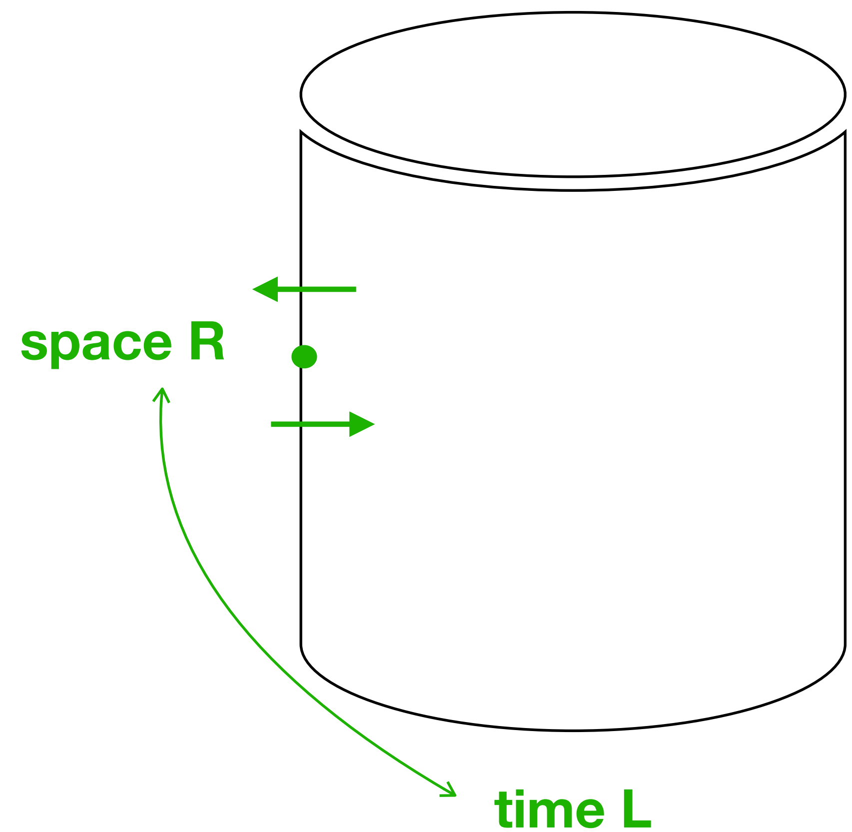


1) Direct theory: ground state **exact** energy dominates partition function as $R \to \infty$:

$$Z = \exp[-R\Delta_0] + \dots \qquad \Delta_0 = E_0$$

Computation of the LxR torus partition in two ways

2) Mirror theory (**space**↔**time**):
Bethe eqs. (derived from those 1) in space $R \to \infty$ become exact!



Finite L not longer a problem: Thermodynamics/statistical mechanics at temperature T=1/L (Yang-Yang) gives the minimal **free energy** $f_{min}(L)$:

$$Z = \exp[-RLf_{min}(L)] + \dots$$

$$\exp[-R\Delta_0(L)] = \exp[-RLf_{min}(L)]$$

- ground state energy/anomalous dimension in 1) **given by thermodynamic free energy computed in the mirror theory 2).**

- Minimising a functional $\Longrightarrow$ non linear integral eq., whose solution furnishes energies/ dimensions (as integrals on it):

known

$$\ln Q(\theta) = \tilde{E}(\theta) + \int_{-\infty}^{\infty} \frac{1}{\cosh(\theta - \theta')} \ln\left[1 + Q^2(\theta')\}\right] d\theta' \quad \Longrightarrow \quad \boxed{\Delta \sim \int_{-\infty}^{\infty} \frac{d\tilde{p}}{d\theta} \ln\left[1 + Q^2(\theta)\}\right] d\theta}$$

$Q^2(\theta) = e^{-\epsilon(\theta)} = Y(\theta)$ pseudoenergy so that $\epsilon(\theta) = m\cosh\theta - \int_{-\infty}^{+\infty} \frac{d\theta'}{2} \frac{1}{\cosh(\theta - \theta')} \ln[1 + e^{-\epsilon(\theta')}]$

- **Other states/operators $\Longleftarrow$ excited states**: analytic continuation of the solution or poles in NLIE which only modifies the driving term $\tilde{E}(\theta \,|\, \theta_i)$

Vacuum/Excited states Thermodynamic Bethe Ansatz

- Vacuum equations of the form

$$\epsilon_a(u) = \mu_a + \tilde{e}_a(u) - \sum_b \int dv \;\; K_{a,b}(u,v) \ln(1 + e^{-\epsilon_b(v)})$$

  with mirror energy $\tilde{e}_a(u)$ as driving term and scattering factors

$$K_{a,b}(u,v) \propto \partial_v \ln S_{a,b}(u,v)$$

- Excited states $E(L)$ are connected to the vacuum by analytic continuation in some parameter (*e.g.* μ_a and L) $\Rightarrow$ additional inhomogeneous terms in the equations $\sum_i \ln S_{a,b}(u, u_i)$ depending on TBA complex singularities u_i:

$$e^{-\epsilon_a(u_i)} = -1$$

  these are the exact Bethe roots (with wrapping).
- $\Rightarrow$ Delicate and massive numerical work for analytic continuation.

- From the vacuum TBA to **Y-system** e.g.
- $$Y_n(\omega E) Y_n(\omega^{-1} E) = (1 + Y_{n+1}(E))(1 + Y_{n-1}(E))$$
- $Y_n(E) = e^{-\epsilon_n(u)}$, $E = e^{2u}$, upon inverting universal kernel 1/cosh into the shift operator on the l.h.s..Subtlety: from physical to universal kernels.
- **Excited Ys** must satisfy the **same Y-system**.

A second way to TBA: the OPE for null polygonal WLs

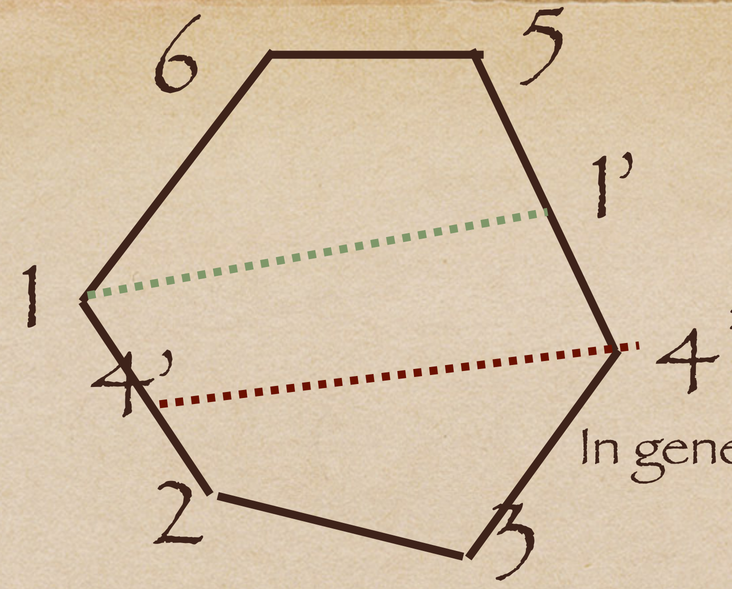
- Theory: N=4 SYM in planar limit $\lambda = N_c g_{YM}^2, \; N_c \to \infty$
- Exponential of circulation of the gauge field $\overset{\text{dual}}{=}$ quantum area of II B string theory on $AdS_5 \times S^5$ (Alday,Maldacena; Korchemky,Sokachev,…Bern,Dixon,…)
- Light-like polygons can be decomposed into light-like Pentagons (and Squares): an Operator Product Expansion (OPE)

  Alday,Gaiotto,Maldacena;Basso,Sever,Vieira; Belitski; DF,Piscaglia,Rossi;
- Simplest example: Hexagon into two Pentagons P→String Flux·tube
- The same as **two**-point correlation function <PP> into **Form-Factors** in quantum integrable 2D field theories

  Dixon, Friedan,Martinec,Shenker;Knizhnik,… Cardy; Castro-Alvaredo,Doyon, DF….

- In a picture:

hexagon



=P(12341') P(14'456)

In general: E-5 shared squares, E-4 pentagons

- Which mathematically means:

    - $W=\Sigma \exp(-rE)\langle 0|P|n\rangle\langle n|P|0\rangle$

Multi-P correlation function:general m,n transition

- =<PP>: the same as <u>2D Form Factor (FF) decomposition</u>

- Form-Factors obey axioms with the S-matrix: 1) Watson eqs., 2) <u>Monodromy (q-KZ)</u>, 3) <u>Kinematic Poles</u>, 4) Bound-state eqs. etc.

- We have to modify the 2) (and 3)) (for conical twist fields)

- Eigen-states $|n\rangle$? 2D excitations over the GKP folded string (of length=2 ln s) which stretches from boundary to boundary (for large s) of AdS:<u>S-matrix</u> Basso

DF,Piscaglia,Rossi

FFs series summing to TBA (DF, Piscaglia, Rossi)

- Quite unique example of Form-Factor series re-summation. Result: thermodynamic bubble Ansatz of string minimal area at strong coupling (Alday-Gaiotto-Maldacena)
- The key idea: Hubbard-Stratonovich transformation replaces the infinite sums with a path integral

$$W_{hex}^{(g)} = Z^{(g)}[X^g] = \int \mathcal{D}X^g e^{-S^{(g)}[X^g]}$$

$$S^{(g)}[X^g] = \frac{1}{2}\int d\theta\, d\theta'\, X^g(\theta) T^g(\theta,\theta') X^g(\theta') + \\ + \int \frac{d\theta'}{2\pi}\, \mu^g(\theta') \left[\mathrm{Li}_2(-e^{-E(\theta')+i\phi}\, e^{X^g(\theta')}) + \mathrm{Li}_2(-e^{-E(\theta')-i\phi}\, e^{X^g(\theta')})\right]$$

$S^{(g)}[X^g] \sim \sqrt{\lambda} \to \infty$: saddle point eqs. are TBA eqs.

$$X^g(\theta) - \int \frac{d\theta'}{2\pi} G^g(\theta,\theta')\mu^g(\theta') \log\left[(1+e^{X^g(\theta')}e^{-E(\theta')+i\phi})(1+e^{X^g(\theta')}e^{-E(\theta')-i\phi})\right] = 0$$

$$\int d\theta'\, G^g(\theta,\theta') T^g(\theta',\theta'') = \delta(\theta-\theta'')$$

IMPORTANT LESSON

- Classical string equations (strong coupling), i.e. classical (Lax) integrable system is solved by quantum TBA
- Surprise: yes, because classical string dynamics; no, because we knew classical static potentials leading to TBA (ODE/IM,next).
- Mystery: string (second) quantise TBA?

ODE/IM Correspondence (**III way**)

(Dorey,Tateo,BLZ,DF, Dunning,Suzuki,Frenkel,Bender,Masoero,….. >1998)

- Simplest example: Schrödinger eq. on the half line $(0,\infty)$ (Stokes line)

$$\left(-\frac{d^2}{dx^2}+x^{2M}+\frac{l(l+1)}{x^2}\right)\psi(x)=E\psi(x) \qquad \text{important } M>0$$

- we fix the subdominant solution such that at complex infinity

$$y \sim x^{-M/2}\exp\left(-\frac{1}{M+1}x^{M+1}\right),$$

$$y' \sim -x^{M/2}\exp\left(-\frac{1}{M+1}x^{M+1}\right)$$

$$|\arg x| < \frac{3\pi}{2M+2}$$

Changes sign

- Changing anti-Stokes sector $\mathcal{S}_k = \left|\arg x - \frac{2k\pi}{2M+2}\right| < \frac{\pi}{2M+2}$ this solution becomes dominant

Discrete Symmetry Breaking

- Omega symmetry of the eq. not of the solution which rotates by $\omega = \exp(\pi i/(M+1)) = q$ (quantum group)
- $\hat{\Omega} : x \to qx\,,\quad E \to q^{-2}E\,,\quad l \to l$ $\qquad y_k \equiv y_k(x, E, l) = \omega^{k/2} y(\omega^{-k} x, \omega^{2k} E, l)$
- y_k subdominant in $\mathcal{S}_k$ and dominant in $\mathcal{S}_{k\pm 1}$.
- About x=infinity, irregular singularity.
- Lambda symmetry, about x=0, regular singularity:
- $\hat{\Lambda} : x \to x\,,\quad E \to E\,,\quad l \to -1-l$ $\qquad \hat{\Lambda}\psi^{\pm} = \psi^{\mp}$ $\qquad$ around $x = 0$ $\quad \psi^{+} \simeq x^{l+1} \quad \psi^{-} \simeq x^{-l}$

$l(l+1)$ invariant

Transfer matrix T, Q and various functional equations

- Stokes multipliers
- $$y_{k-1}(x,E,l) = C_k(E,l)\, y_k(x,E,l) + \tilde{C}_k(E,l)\, y_{k+1}(x,E,l)$$
- All the C_k and $\tilde{C}_k$ in terms of Wronskians, e.g. $k = 0$ $(y_{-1} = C y_0 + \tilde{C} y_1)$
- $$C = \frac{W_{-1,1}}{W_{0,1}}, \qquad \tilde{C} = -\frac{W_{-1,0}}{W_{0,1}}$$
- By using the leading asymptotics
- all of the $\tilde{C}_k$ are identically equal to -1 $\quad C(E,l) = \frac{1}{2i} W_{-1,1}(E,l)$
- $$C(E,l)\, y(x,E,l) = \omega^{-1/2} y(\omega x, \omega^{-2}E, l) + \omega^{1/2} y(\omega^{-1}x, \omega^{2}E, l)$$

- If l=0, no singularity in x=0, then Baxter TQ-relation
- $$\mathbf{T}(\lambda)\mathbf{Q}_{\pm}(\lambda) = \mathbf{Q}_{\pm}(q^{-1}\lambda) + \mathbf{Q}_{\pm}(q\lambda)$$
- but keeping $l \neq 0$, it still works
- $$C(E,l)\,D^{\mp}(E,l) = \omega^{\mp(1/2+l)}D^{\mp}(\omega^{-2}E,l) + \omega^{\pm(1/2+l)}D^{\mp}(\omega^{2}E,l)$$
- In fact **'Scattering Coefficients' = spectral determinants**
- $$D^{\mp}(E,l) \equiv W[\,y(x,E,l),\psi^{\pm}(x,E,l)]$$
- are projections on the $\psi^{\pm}$: zeroes $\mathbf{E_n}$ = **Bethe roots (bound state)**
- 2D physics: The original transfer matrix **T** and **Q** are **operators,** in **Statistical Field Theory** or **Spin Chain**, here eigenvalues, i.e. functions.

- From the TQ relation or the QQ-system (more fundamental), n=0 (n=1 definition of T) of

$$(4l+2)\,i\,C^{(n)}(E) = \omega^{(n+1)(l+1/2)}D^{-}(\omega^{n+1}E,l)\,D^{+}(\omega^{-n-1}E,l) - \omega^{-(n+1)(l+1/2)}D^{-}(\omega^{-n-1}E,l)\,D^{+}(\omega^{n+1}E,l)$$

- **ODE develops functional equations** with $T_{n/2}(\nu E^{1/2}) = C^{(n)}(E) = \frac{1}{2i}W_{-1,n}(\omega^{-n+1}E)$.
- Fused T relations

$$\mathbf{T}(\lambda)\mathbf{T}_j(q^{j+1/2}\lambda) = \mathbf{T}_{j-1/2}(q^{j+1}\lambda) + \mathbf{T}_{j+1/2}(q^{j}\lambda)$$

or

$$\mathbf{T}(\lambda)\mathbf{T}_j(q^{-j-1/2}\lambda) = \mathbf{T}_{j-1/2}(q^{-j-1}\lambda) + \mathbf{T}_{j+1/2}(q^{-j}\lambda)$$

- which brings the TT-system or discrete Hirota eq.

$$\boxed{\mathbf{T}_j(q^{-1/2}\lambda)\mathbf{T}_j(q^{1/2}\lambda) = \mathbf{1} + \mathbf{T}_{j+1/2}(\lambda)\mathbf{T}_{j-1/2}(\lambda)}$$

- Finally the Y-system for the invariant quantity
- $$Y_n(E) = C^{n+1}(E)C^{n-1}(E)$$
- which easily brings the T-system into the **Y-system** form
- $$Y_n(\omega E)Y_n(\omega^{-1}E) = (1 + Y_{n+1}(E))(1 + Y_{n-1}(E))$$
- Upon taking the log, inverting the shift operator on the l.h.s., and using a suitable asymptotic as zero-mode, we can obtain universal kernel 1/cosh **TBA**, equivalent to physical TBA eqs. $Y_n(E) = e^{-\epsilon_n(\theta)}, \; E = e^{2\theta}$: solution up to quadratures.

2D CFT dictionary

- Eigenvalues of statistical mechanics operators Q and T on the <u>conformal primary</u> (dimension)

$$\Delta = \left(\frac{p}{\beta}\right)^2 + \frac{c-1}{24}. \qquad p = \frac{2l+1}{4M+4}$$

- with Minimal Model central charge

$$c = 13 - 6\left(\beta^2 + \beta^{-2}\right) \qquad \beta^2 = \frac{1}{M+1}$$

Sine-Gordon coupling

$$q = e^{i\pi\beta^2}$$

Descendent/excited states

BLZ; DF

- The potential acquires an extra piece $\sum_j \frac{2}{(x-x_j)^2}$ with double poles
- They satisfy algebraic equations, similar to Bethe's: trivial monodromy or Vir symmetry.

D3 brane BH and ODE/IM correspondence

…Gubser,Hashimoto;Bianchi,Consoli,Grillo,Morales;Di Russo…

- The ODE describing the scalar perturbation of **Black Hole**

$$\frac{d^2\phi}{dr^2} + \left[\omega^2\left(1+\frac{L^4}{r^4}\right) - \frac{(l+2)^2-\frac{1}{4}}{r^2}\right]\phi = 0$$

- Change of variables $r = Le^{\frac{y}{2}}$ $\omega L = -2ie^{\theta}$ $P = \frac{1}{2}(l+2)$

to bring it into the integrability form $\phi = e^{\frac{y}{4}}\psi$

$$-\frac{d^2}{dy^2}\psi + \left[e^{2\theta}(e^{y}+e^{-y}) + P^2\right]\psi = 0$$

DF,Gregori

- Basis of solutions going to zero (subdominant)→**BH b.cs.**

$$U_0(y) \simeq \frac{1}{\sqrt{2}}\exp\left\{-\theta/2 - y/4\right\}\exp\left\{-2e^{\theta+y/2}\right\}, \Re y \to +\infty\,;\; V_0(y) \simeq \frac{1}{\sqrt{2}}\exp\left\{-\theta/2 + y/4\right\}\exp\left\{-2e^{\theta-y/2}\right\}, \Re y \to -\infty$$

Discrete Symmetry Breaking

- Lambda and Omega symmetries of the ODE **not** of the solutions which rotate as
  $$\Lambda : \theta \to \theta + i\frac{\pi}{2} \quad y \to y + \pi i\,, \quad \Omega : \theta \to \theta + i\frac{\pi}{2} \quad y \to y - \pi i$$
  and generate the dominant (big) solutions:
  $U_k = \Lambda^k U_0 \quad V_k = \Omega^k V_0$ by repeated action.

- ODE/IM fundamental Wronskian is the same as the **gravitational** one $Q(\theta, P^2) = W[U_0, V_0]$

Quasinormal modes=Bethe roots

- Proper eigen-frequencies of the back hole
$$Q(\theta_n) = 0$$
- We can compute them playing with Wronskian:
$$iV_0(y) = Q(\theta + i\pi/2)U_0(y) - Q(\theta)U_1(y)$$
$$iV_1(y) = Q(\theta + i\pi)U_0(y) - Q(\theta + i\pi/2)U_1(y)$$
- Eventually taking the Wronskian $W[V_0, V_1] = i$, as in scattering theory, the **QQ-system**

Unitarity
$$Q(\theta + i\pi/2)Q(\theta - i\pi/2) = 1 + Q(\theta)^2$$

- Upon taking the log and inverting the shift operator $s * l = l(\theta + i\pi/2) + l(\theta - i\pi/2) \Rightarrow s^{-1} \sim \frac{1}{\cosh}$ we obtain the **Thermodynamic Bethe Ansatz equation**

$$\ln Q(\theta) = -\frac{8\sqrt{\pi^3}}{\Gamma^2(\frac{1}{4})} e^{\theta} + \int_{-\infty}^{\infty} \frac{\ln\left[1 + Q^2(\theta')\right]}{\cosh(\theta - \theta')} \frac{d\theta'}{2\pi}$$

- Sort of solution up to quadratures. Important: **Q** is the **spectral determinant.**

T, Q and the SW-NS periods

(DF, D. Gregori; Grassi, Marino, Gu; He,…)

- We **quantise/deform the quadratic SW differential:** the Mathieu eq. (AGT correspondence: level 2 null vector eq.)

$$-\frac{\hbar^2}{2}\frac{d^2}{dz^2}\psi(z) + [\Lambda^2 \cos z - u]\psi(z) = 0$$

- Namely, quantum SW differential $\mathcal{P}(z) = -i\frac{d}{dx}\ln\psi(z)$ and periods

$$a(\hbar,u,\Lambda) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{P}(z;\hbar,u,\Lambda)\,dz \;, \qquad a_D(\hbar,u,\Lambda) = \frac{1}{2\pi}\int_{-\arccos(u/\Lambda^2)-i0}^{\arccos(u/\Lambda^2)-i0}\mathcal{P}(z;\hbar,u,\Lambda)\,dz$$

- ODE/IM uses its non-compact (modified) version: **two irregular singularities** (M=-2)

$$\left\{-\frac{d^2}{dy^2} + 2e^{2\theta}\cosh y + P^2\right\}\psi(y) = 0 \qquad z = -iy - \pi$$

- upon **gauge/integrability** change of variable

$$\frac{\hbar}{\Lambda} = e^{-\theta}\,, \qquad \frac{u}{\Lambda^2} = \frac{P^2}{2e^{2\theta}}$$

- **Integrability/gauge identification**

integrability gauge

$$T(\hbar, u, \Lambda) \equiv T(\theta, P^2) = iW[V_1, V_{-1}] = 2\cos\left\{2\pi a(\hbar, u, \Lambda)\right\}$$

$$Q(\hbar, u, \Lambda) \equiv Q(\theta, P^2) = \exp\{2\pi i a_D(\hbar, u, \Lambda)\}$$

- The fundamental relation of the theory: **QQ-SYSTEM**

integrability gauge

$$1+Q^2(\theta, P^2) = Q(\theta - i\pi/2, P^2)Q(\theta + i\pi/2, P^2)\,, \qquad 1+Q^2(\theta, u) = Q(\theta - i\pi/2, -u)Q(\theta + i\pi/2, -u)$$

- which gives with $Y = Q^2 = e^{-\varepsilon}$ the **Y-system** (very simple case!) from which the **gauge TBA eqs.**

monopole

$$\varepsilon(\theta, u, \Lambda) = -4\pi i a_D^{(0)}(u, \Lambda)\frac{e^\theta}{\Lambda} - 2\int_{-\infty}^{\infty} \frac{\ln\left[1+\exp\{-\varepsilon(\theta', -u, \Lambda)\}\right]}{\cosh(\theta - \theta')}\frac{d\theta'}{2\pi}$$

$$\varepsilon(\theta, -u, \Lambda) = -4\pi i a_D^{(0)}(-u, \Lambda)\frac{e^\theta}{\Lambda} - 2\int_{-\infty}^{\infty} \frac{\ln\left[1+\exp\{-\varepsilon(\theta', u, \Lambda)\}\right]}{\cosh(\theta - \theta')}\frac{d\theta'}{2\pi}$$

dyon, i.e. **strong coupling spectrum**

- Quantum integrability tells more: e.g. T→weak coupling spectrum (a electric), inside **TQ-system**

$$T(\theta)Q(\theta) = Q(\theta - i\pi/2) + Q(\theta + i\pi/2)$$

$2\cos\{2\pi a\}$ $\exp\{2\pi i a_D\}$

- +periodicity, gauge interpretation: **quantum Bilal-Ferrari relations**. T and Q are generating functions for **conserved charges**: small $\hbar/\Lambda = e^{-\theta} \ll 1$ asymptotic expansion. Also for **quantum periods** (zero order=Seiberg-Witten).

- Connexion with SW geometry: an origin of ODE/IM

Explanation of gauge quantisation for BH

Aminov,Grassi,Hatsuda

- Connexion formula of **spectral determinant** with **instanton prepotential** $A_D = \partial \mathcal{F} / \partial a$

2D solitons vs instantons 4D

$$Q(a, \Lambda, \hbar) = i \frac{\sinh A_D}{\sinh 2\pi i a}$$

- quantisation easily follows from

$$Q = 0 \Rightarrow A_D(a, \Lambda_n, \hbar) = i\pi n \,, \;\; n \in \mathbb{Z}$$

DF,Gregori Aminov,Grassi,Hatsuda;Bonelli,Iossa,Lichtig,Panea,Tanzini

- Surprise: previous eq. is the $b = 1$ case of

Generalised Mathieu eq. $$\left\{-\frac{d^2}{dy^2} + e^{2\theta}(e^{y/b} + e^{-yb}) + P^2\right\}\psi(y) = 0$$

- which describes **Liouville field theory vacua**

$$\Delta = (c-1)/24 - P^2 \qquad c = 1 + 6(b + b^{-1})^2$$

Zamolodchikov; DF,Gregori

- Self-dual point of the symmetry $b \to 1/b$! Meaning? And somehow previous $\beta = ib$ or $M < -1$.

- NO AGT Liou.: correspondence of correspondences

Generalisations

DF,Gregori

- Intersection of four stacks of D3 branes (extremal **Kerr BH**; equal charges: **Reissner-Nöstrom BH**)

$$\frac{d^2\phi}{dr^2} + \left[-\frac{(l+\frac{1}{2})^2 - \frac{1}{4}}{r^2} + \omega^2 \sum_{k=0}^{4} \frac{\Sigma_k}{r^k} \right] \phi = 0$$

- which becomes in integrability form

$$-\frac{d^2}{dy^2}\psi + \left[e^{2\theta}(e^{2y} + e^{-2y}) + 2e^{\theta}(M_1 e^{y} + M_2 e^{-y}) + P^2 \right] \psi = 0$$

- $N_f = 2$, but results on Schwarzshild, Kerr ($N_f = 3$)…..

The ODE→IM correspondence

ODE

Integrable Model

Schrödinger equation → **Scattering data**

On this side we can use:

WKB and other ODE powerful techniques

ODE may be simpler!!

On this side we find:

$Q(\theta_n) = 0$, **Energies=QNMs**

θ_n **Bethe roots**

From spectral determinant → **wave function**

Love number, echoes,….

Gravitation waves

The ODE←IM correspondence?

- Inverting the arrow means reconstructing **from a (given) Quantum Integrable System** the Ordinary Differential Equation (or something similar) which gives it.

- In other words, understanding the the **origin** of the correspondence (and maybe better integrability). And **ODE is simpler, clearer and more treatable**.

- But **possible only with masses**.

PDE→Massive integrable theories

(GMN, LZ, DF-Rossi-Shu)

- The **2D conformal potential is a static limit**, instead in presence of mass it is given by a **flow**: solution of classical sinh-Gordon equation

$$\frac{\partial^2}{\partial w \partial \bar{w}} \hat{\eta} = 2 \sinh 2\hat{\eta} \,,$$

$$u_{\pm}(w', \bar{w}') = \pm \frac{\partial^2}{\partial w^2} \hat{\eta}(w, \bar{w}) - \left(\frac{\partial}{\partial w} \hat{\eta}(w, \bar{w}) \right)^2 \,, \quad w' = -iw, \;\; \bar{w}' = i\bar{w}$$

$$\frac{\partial^2}{\partial w'^2} \psi_{\pm}(w', \bar{w}'|\lambda) + \lambda^2 \psi_{\pm}(w', \bar{w}'|\lambda) = u_{\pm}(w'; \bar{w}') \psi_{\pm}(w', \bar{w}'|\lambda) \,,$$

- Introduce **moduli** c_n: change of variable $dw = \sqrt{p(z)}dz$, polynomial $p(z, \overrightarrow{c}) = z^{2N} + \sum_{n=0}^{2N-1} c_n z^n$.

- **Conformal limit** $\bar{z} = 0, z \to 0$: **polynomial potential** (scaling $ze^{\theta/(1+N)} = x$ and $c_n e^{\theta(2N-n)/(1+N)} = c_n^{cft}$ fixed when $\theta \to +\infty$) $\eta \simeq l \ln(z\bar{z}) \Rightarrow$

- $$-\frac{d^2}{dx^2}\psi_{cft} + \left(p(x, \overrightarrow{c}^{cft}) + \frac{l(l+1)}{x^2} \right) \psi_{cft} = 0 \,, \quad p(x, \overrightarrow{c}^{cft}) = x^{2N} + \sum_{n=0}^{2N-1} c_n^{cft} x^n$$

moduli (→ c_n^{cft})

- All the integrable structures (NOT only TBA) can be derived in this full generalisation because of the discrete broken symmetries (DF-Rossi):

$$\hat{\Omega}: \quad z \to z e^{\frac{i\pi}{N}}, \quad \theta \to \theta - \frac{i\pi}{N}, \quad \vec{c} \to \vec{c}^R, \quad \vec{c}^R = (c_0, c_1 e^{-\frac{i\pi}{N}}, ..., c_n e^{-\frac{i\pi n}{N}}, ..., c_{2N-2} e^{\frac{2i\pi}{N}})$$

Dorey-Tateo, BLZ,DF….

- and in addition

$\hat{\Pi}: \theta \to \theta - i\pi$, change of the sign of momentum k (GMN)

AdS_3 **Wilson loops**+twist l

- so that the QQ-SYSTEM originates: **quantum (homogeneous) sine-Gordon**

$$Q_+\left(\theta + \frac{i\pi}{2N}, \vec{c}\right) Q_-\left(\theta - \frac{i\pi}{2N}, \vec{c}^R\right) - Q_+\left(\theta - \frac{i\pi}{2N}, \vec{c}^R\right) Q_-\left(\theta + \frac{i\pi}{2N}, \vec{c}\right) = -2i\cos\pi l$$

- which generates **<u>all the other integrability functional and integral equations</u>**: e.g. the <u>**N**on **L**inear **I**ntegral **E**quation</u> (which sum up many TBA eqs.),

Baxter: $Q_\pm\left(\theta + i\tau - i\pi, \vec{c}^{R^{-1}}\right) + Q_\pm\left(\theta - i\tau + i\pi, \vec{c}^R\right) = T(\theta, \vec{c}) Q_\pm(\theta, \vec{c})$ period $\tau = \pi + \pi/N$

$$Q_\pm\left(\theta - i\tau, \vec{c}^R\right) = e^{\mp i\pi\left(l + \frac{1}{2}\right)} Q_\pm(\theta, \vec{c})$$

Universal $(\hat{\Pi})$:
no rotation

$$T(\theta, \vec{c}) Q_\pm(\theta, \vec{c}) = e^{\mp i\pi\left(l+\frac{1}{2}\right)} Q_\pm(\theta + i\pi, \vec{c}) + e^{\pm i\pi\left(l+\frac{1}{2}\right)} Q_\pm(\theta - i\pi, \vec{c})$$

Summary so far

- From a (classical) differential operator (Schrödinger) or better a Lax pair $\longrightarrow$ Quantum integrable system (field theory)
- No quantisation but equivalence
- Spin chain? Lattice models?
- Serendipity: <u>how to invert the arrow</u>?

From quantum integrable theory → classical Lax

- How to define a quantum integrable system? For us it is fine encoding the **conserved charges** into Q which satisfy QQ-system:

$$Q_+\left(\theta+\frac{i\pi}{2N},\vec{c}\right)Q_-\left(\theta-\frac{i\pi}{2N},\vec{c}^R\right)-Q_+\left(\theta-\frac{i\pi}{2N},\vec{c}^R\right)Q_-\left(\theta+\frac{i\pi}{2N},\vec{c}\right)=-2i\cos\pi l$$

- and being a Bloch-Floquet solution

$$Q_\pm\left(\theta-i\tau,\vec{c}^R\right)=e^{\mp i\pi\left(l+\frac{1}{2}\right)}Q_\pm(\theta,\vec{c})$$

$$e^{i\pi l}Q_+(\theta,\vec{c})Q_-(\theta+i\pi,\vec{c})+e^{-i\pi l}Q_-(\theta,\vec{c})Q_+(\theta+i\pi,\vec{c})=-2\cos\pi l$$

- From which we derive the universal TQ (key eq.!)

$$e^{\mp i\pi\left(l+\frac{1}{2}\right)}Q_\pm(\theta+i\pi,\vec{c})+e^{\pm i\pi\left(l+\frac{1}{2}\right)}Q_\pm(\theta-i\pi,\vec{c})=T(\theta,\vec{c})Q_\pm(\theta,\vec{c})$$

- II order finite difference eq. for Q with ‘potential’ T. We invert the shift operator in the l.h.s. $\lim_{\epsilon\to 0^+}\left[\tanh\left(x+\frac{i\pi}{2}-i\epsilon\right)-\tanh\left(x-\frac{i\pi}{2}+i\epsilon\right)\right]=2\pi i\delta(x)$

$$Q_\pm\left(\theta+i\frac{\tau}{2},\vec{c}\right)=q(\theta,\vec{c})\pm$$
$$\pm\int_{-\infty}^{+\infty}\frac{d\theta'}{4\pi}\tanh\frac{\theta-\theta'}{2}T\left(\theta'+i\frac{\tau}{2},\vec{c}\right)e^{-w_0(\vec{c})(e^{\theta}+e^{\theta'})-\bar{w}_0(\vec{c})(e^{-\theta}+e^{-\theta'})}e^{\pm(\theta-\theta')l}Q_\pm\left(\theta'+i\frac{\tau}{2},\vec{c}\right)$$

- Fix: field theory, ground state asymptotics

$$\lim_{\mathrm{Re}\theta\to\pm\infty}\ln\left[Q_\pm\left(\theta+i\frac{\tau}{2},\vec{c}\right)\right]\sim -w_0(\vec{c})e^{\theta}-\bar{w}_0(\vec{c})e^{-\theta}$$

$$q(\theta,\vec{c})=C_\pm e^{\pm\frac{i\pi}{4}\pm\left(\theta+\frac{i\pi}{2}\right)l}e^{-w_0(\vec{c})e^{\theta}-\bar{w}_0(\vec{c})e^{-\theta}}$$

$w_0(\vec{c})=M_{soliton}L=r$: RG time

- A better proportional variable

$$C_{\pm}X_{\pm}(\theta,\vec{c}) = e^{\mp\frac{i\pi}{4}} e^{\mp\left(\theta+\frac{i\pi}{2}\right)l} e^{w_0(\vec{c})e^{\theta}+\bar{w}_0(\vec{c})e^{-\theta}} Q_{\pm}\left(\theta + i\frac{\tau}{2},\vec{c}\right)$$

- satisfies a universal integral equation

- $$X_{\pm}(\theta,\vec{c}) = 1 \pm \int_{-\infty}^{+\infty} \frac{d\theta'}{4\pi} \tanh\frac{\theta-\theta'}{2} T\left(\theta' + i\frac{\tau}{2},\vec{c}\right) E(\theta',\vec{c}) X_{\pm}(\theta',\vec{c})$$

- with peculiar kernel (**solitons** around the corner)

$$E(\theta,\vec{c}) = e^{-2w_0(\vec{c})e^{\theta}-2\bar{w}_0(\vec{c})e^{-\theta}}$$

- direct consequence of the **asymptotics of Q**.

- Upon Fourier transforming $\lambda = e^{\theta}$:

$$K_{\pm}(w_0', \xi; \bar{w}_0') = \int_{-\infty-i\epsilon}^{+\infty-i\epsilon} d\lambda e^{i(\xi - w_0')\lambda}[X_{\pm}(w_0', \bar{w}_0'|\lambda) - 1]$$

- Volterra equation for $K_{\pm}(w_0', \xi; \bar{w}_0')$

$$K_{\pm}(w_0', \xi; \bar{w}_0') \pm F(w_0' + \xi; \bar{w}_0') \pm \int_{w_0'}^{+\infty} \frac{d\xi'}{2\pi} K_{\pm}(w_0', \xi'; \bar{w}_0') F(\xi' + \xi; \bar{w}_0') = 0$$

- almost Marchenko eq. but the scattering data

$$F(x; \bar{w}_0') = i \int_0^{+\infty} d\lambda' e^{-ix\lambda' + 2i\bar{w}_0'/\lambda'} T(\lambda' e^{i\frac{\tau}{2}})$$

- which depends (in a intricate way) on $w_0' = -iw_0 = -ir$ because of T

- **NEW IDEA**: promote $w_0(\overrightarrow{c}) = iw'_0(\overrightarrow{c})$ to new dynamical variables $w = iw'$ everywhere except in T:
- $$K_\pm(w',\xi;\bar{w}') \pm F(w'+\xi;\bar{w}') \pm \int_{w'}^{+\infty} \frac{d\xi'}{2\pi} K_\pm(w',\xi';\bar{w}')F(\xi'+\xi;\bar{w}') = 0\,, \quad \xi > w'$$
- **Marchenko-like eq.** (!) with ‘good’ scattering data
$$F(x;\bar{w}') = i\int_0^{+\infty} d\lambda' e^{-ix\lambda'+2i\bar{w}'/\lambda'} T(\lambda' e^{i\frac{\tau}{2}}, \vec{c})$$
-

- Finally we can derive the Schroedinger eq. for

$$K_\pm(w',\xi;\bar{w}') = \int_{-\infty-i\epsilon}^{+\infty-i\epsilon} d\lambda e^{i(\xi-w')\lambda}[X_\pm(w',\bar{w}'|\lambda)-1] \qquad X_\pm(w',\bar{w}'|\lambda)-1 = \int_{-\infty}^{+\infty}\frac{d\xi}{2\pi}e^{-i(\xi-w')\lambda}K_\pm(w',\xi;\bar{w}') = \int_{w'}^{+\infty}\frac{d\xi}{2\pi}e^{-i(\xi-w')\lambda}K_\pm(w',\xi;\bar{w}')$$

Summary: TQ eq. $\Longrightarrow$ Marchenko eq. (Fourier)

- the wave-function (plane wave multiplication)

$$\psi_\pm(w',\bar{w}'|\lambda) = X_\pm(w',\bar{w}'|\lambda)e^{-iw'\lambda+i\bar{w}'\lambda^{-1}}$$

- extension of the Q-function to this new ODE/IM space

- Promotion $w'(0) = -ir \to w'(z)$ means that it is a 'holographic' **RG space**?

More Perspectives beyond gauge and BH

- **The machine is ready: extension to more complicated higher rank systems: let us find the ODEs!**
- Non-linear integral or functional equations are powerful and describe monodromies of a ODE or PDE. There is any deep reason why these (TBA) are reproduced by an integrable Form Factor series of a 'weird' scattering theory?
- NS limit $\epsilon_1 = \hbar, \ \epsilon_2 = 0 \rightarrow$ ODE/IM description: $\epsilon_2 \neq 0$ quantum ODE/IM? q-TBA? Similarly about classical string.
- On the contrary: meaning of $b \neq 1$ for our Liouville field theory (not AGT)?

Thanks