Talking Integrability: Spins, Fields, and Strings

Correlations and large deviations in a very simple integrable system

The BBS Models

with Atsuo Kuniba et Grégoire Misguich

Motivation

- Find a simple model where to test ideas of GGE, GHD, G…
- Study a model with direct relation to TBA and solitons.
- Soliton decomposition of the Box-Ball System
  Pablo A. Ferrari, Chi Nguyen, Leonardo T. Rolla, Minmin Wang

Origin of the model

Journal of The Physical Society of Japan
Vol. 59, No. 10, October, 1990, pp. 3514–3519

A Soliton Cellular Automaton

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(Received May 10, 1990)

A cellular automaton (CA) of filter automata type is proposed. Any state of the CA consists only of solitary wave solutions. It is shown that the solitary waves interact with one another preserving their identities during a time evolution. It is also shown that the CA has infinitely many conserved quantities. Hence, this CA may be considered to be one of the simplest soliton systems.

Quantum Group

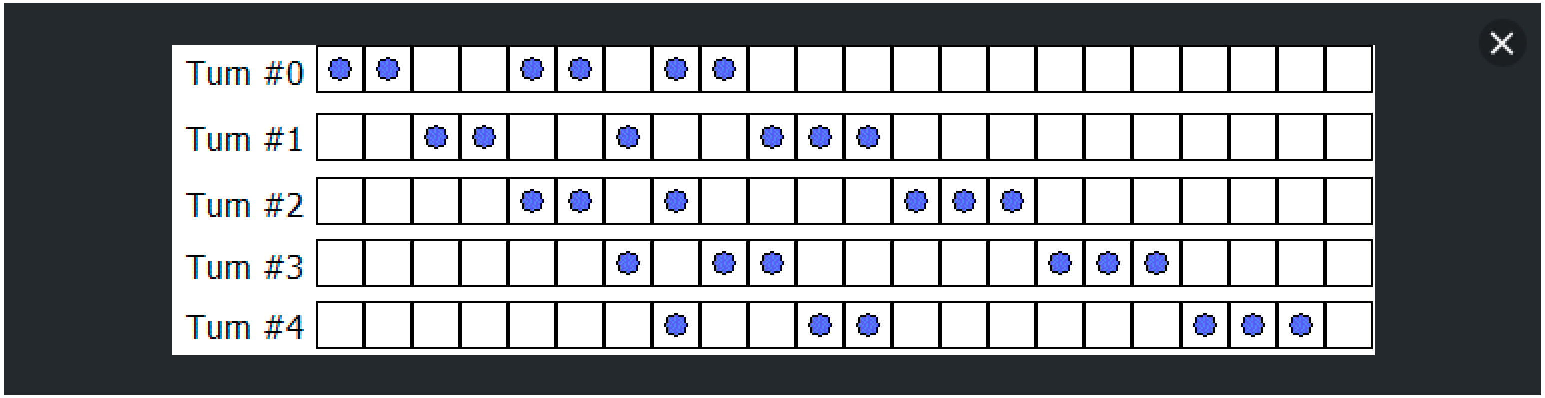
- Nuclear Physics 2000

Soliton Cellular Automata Associated With Crystal Bases

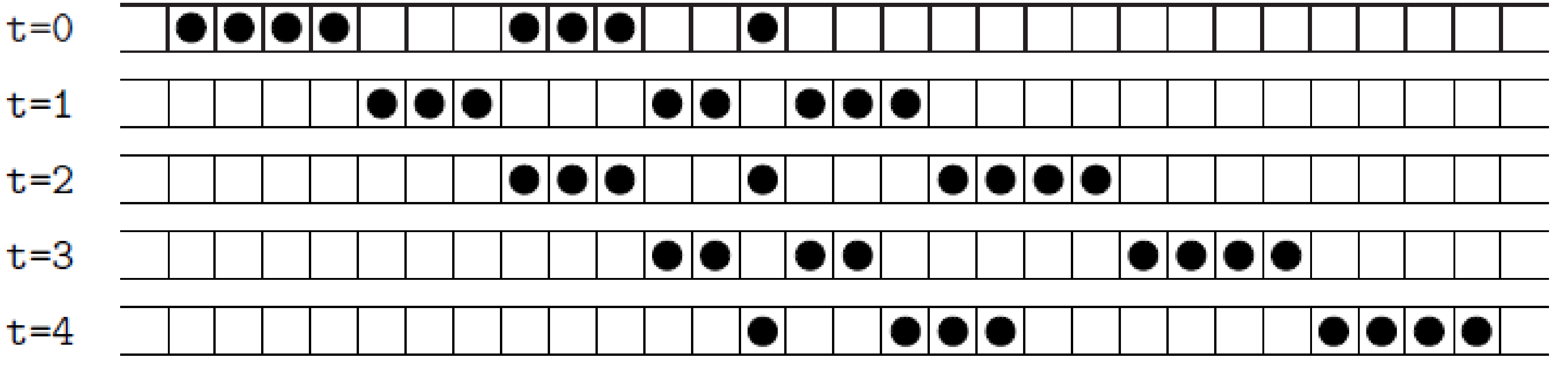
Goro Hatayama,* Atsuo Kuniba,* and Taichiro Takagi†

Hydrodynamics of BBS

Wat is BBS ?



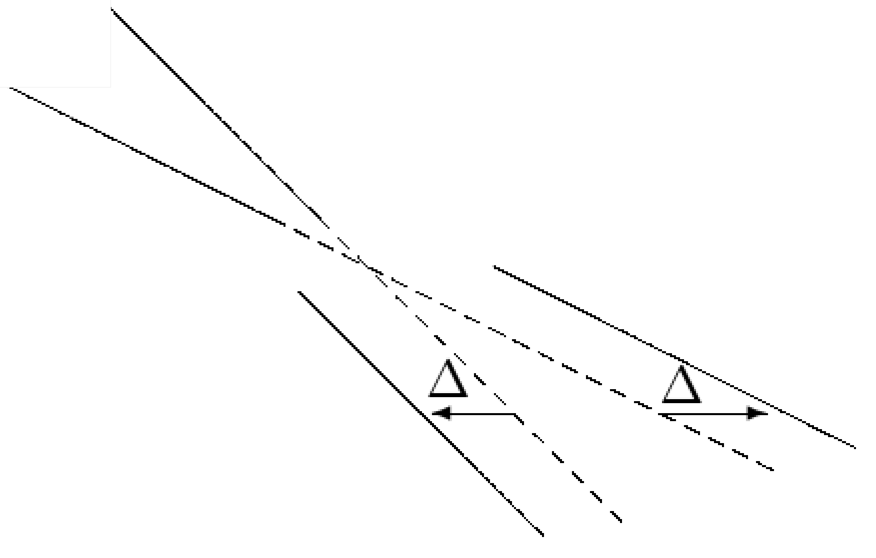
Large Solitons overpass small solitons:



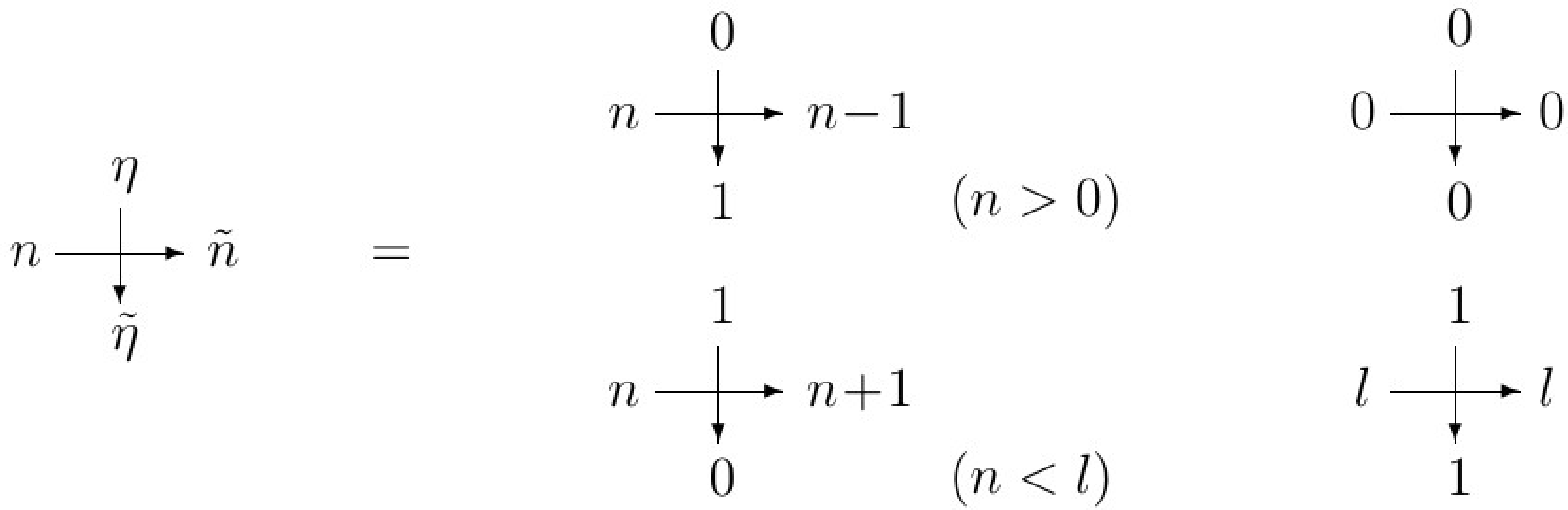
Bare velocity = size

Phase shift

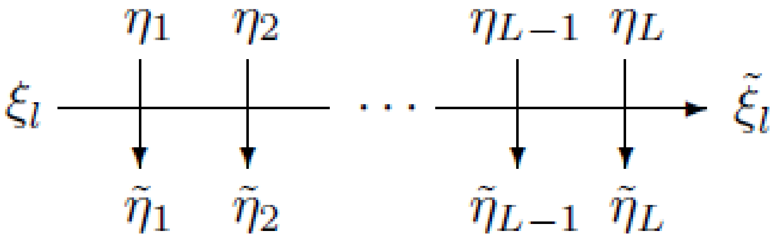
$$\Delta(k,l) = 2\min(k,l)$$



Cristal Vertex



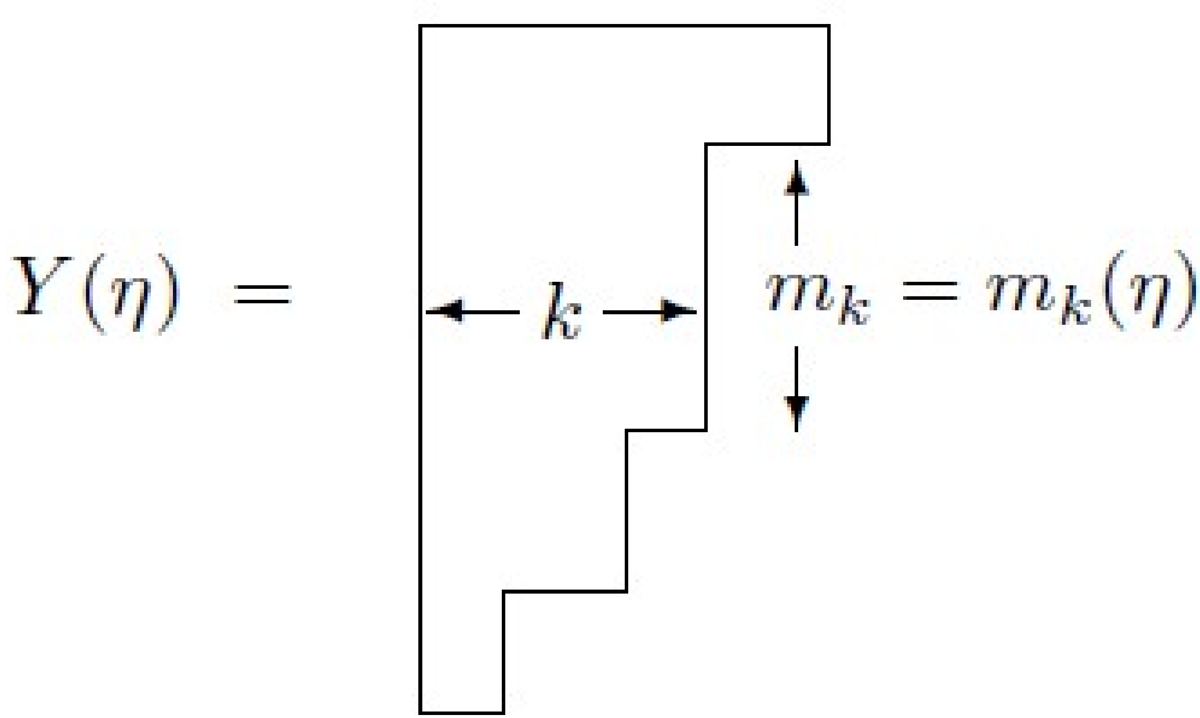
Transfer Matrix



Can be made periodic

Commute for l different from l'

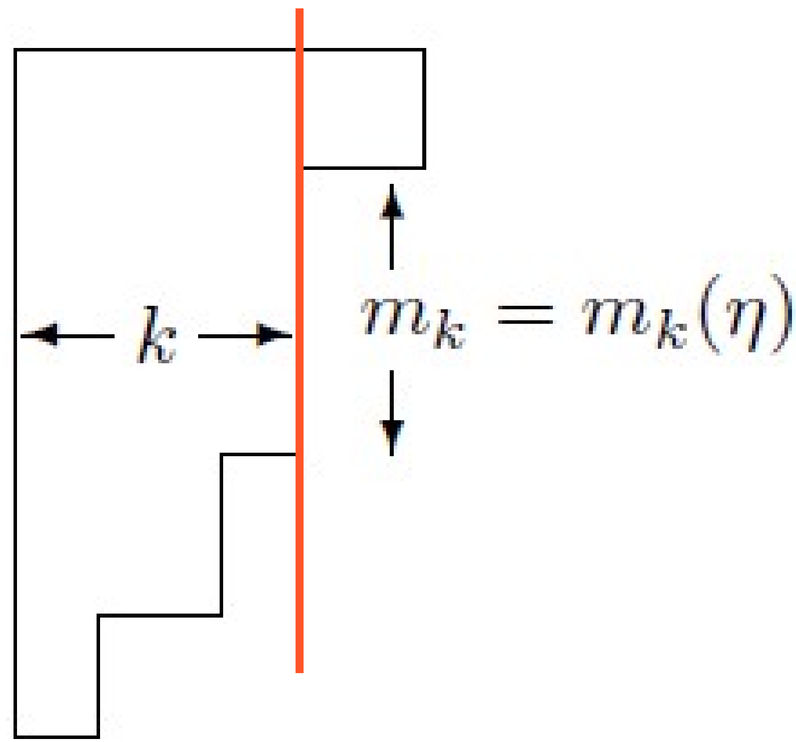
Young Diagram representation of soliton configurations



m_k =number of solitons k

Conserved quantities

$$Y(\eta) =$$



$$e_k = \sum_l \min(k, l) m_l$$

Entropy Factor

$$e^S = \prod_k \binom{m_k + p_k}{p_k}$$

$$p_k = L - 2\sum_l \min_{kl} m_l$$

Fermionic formula

Kirillov, Kerov, Reshethikin

Thermodynamics

Bethe equations convert entropy factor into hole density relation

$$\sigma_j = 1 - M_{ij}\rho_k$$

$$M_{ij} = 2\min(j,k)$$

Thermodynamics

$$\sigma_j = 1 - M_{ij}\rho_j = 1 - 2\epsilon_j$$

σ=hole density=p/L

ρ=density=m/L

$$Z = \sum_Y \prod_k e^{-\beta_k \epsilon_k} \binom{\rho_k + \sigma_k}{\rho_k}$$

Thermodynamics

- Minimize free energy :

$$\mathcal{F} = \beta_1\varepsilon_1 + \cdots + \beta_s\varepsilon_s - \sum_{i=1}^{s}\big((\sigma_i + \rho_i)\log(\sigma_i + \rho_i) - \sigma_i \log\sigma_i - \rho_i \log\rho_i\big),$$

$$\sum_{j=1}^{s}\min(i,j)\beta_j = \log(1+Y_i) - 2\sum_{j=1}^{s}\min(i,j)\log(1+Y_j^{-1}), \quad Y_i = \frac{\sigma_i}{\rho_i}.$$

$$\begin{aligned} Y_1^2 &= \mathrm{e}^{\beta_1}(1+Y_2), \\ Y_i^2 &= \mathrm{e}^{\beta_i}(1+Y_{i-1})(1+Y_{i+1}) \quad (1<i<s), \\ Y_s^2 &= \mathrm{e}^{\beta_s}(1+Y_{s-1})(1+Y_s). \end{aligned}$$

GGE

$$\mathcal{F} = -\sum_{i=1}^{s}\log(1+Y_i^{-1}).$$

Thermodynamics

Distribute the balls independantly on each site of the lattice with probability z/(1+z) only one fugacity nonzero.

- Partition function sum on configurations z raised to the number of balls :

$$Z = (1+z)^N = \sum_Y z^{\sum_k k m_k} \prod_k \binom{m_k + p_k}{m_k}$$

TBA

- Another way to write tba :

$$\epsilon = \beta.h - T * \ln(1 + e^{-\epsilon})$$

$$T = (1 - M)^{-1}$$

Taking derivative with respect to β gives an expression of dressing $\partial_\beta \epsilon = h^{\mathrm{dr}}$
which we do NOT use

Q system

- Change variable $$Q_i = \sqrt{1 + Y_i}$$

- Obtain Q-system (Schur functions):

$$Q_{i-1}Q_{i+1} + 1 = Q_i^2$$

$$Q_0 = 1 \qquad \frac{Q_{k+1}}{Q_k} \to \frac{1}{z}$$

Thermodynamics

$$\sigma_k = \frac{(1-z)(1+z^{k+1})}{(1+z)(1-z^{k+1})}$$

$$\rho_k = \frac{z^k(1-z)^3(1+z^{k+1})}{(1+z)(1-z^k)(1-z^{k+1})(1-z^{k+2})}$$

Speed of solitons in the medium

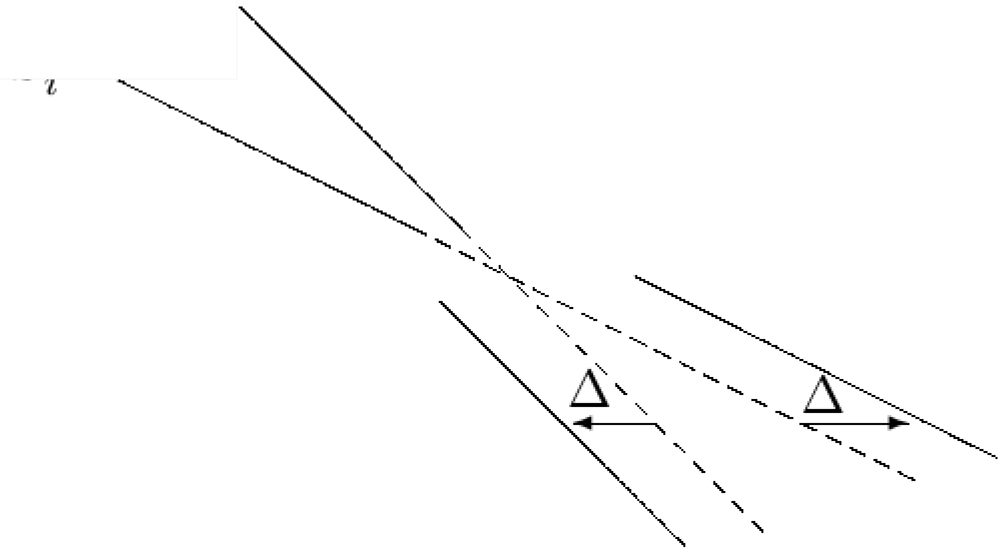
Speed of Soliton=

speed in vacuum

+

Nb of solitons of type p crossed per time unit

x relative shift



Local determination of speed

$$V^{l}{}_{k} \;=\; min(k,l) + 2\sum_{p} min(k,p)(V^{l}{}_{k} - V^{l}{}_{p})\rho_{p}$$

Linear equation for the speeds.
Related to inverse of tropical matrix.

GHD formulation

$$A^{dr} = (1 + My)^{-1} A$$

$$y_k = \sigma_k / \rho_k$$

$$V^{l} = \frac{(\kappa^{l})^{\mathrm{dr}}}{1^{\mathrm{dr}}}$$

V and densities are functions of filling fraction y

Normal modes, current

$$(j_\sigma)_k = \sigma_k v_k$$

$$\partial_t \sigma + \partial_x j_\sigma = 0.$$

Use dressing to get :

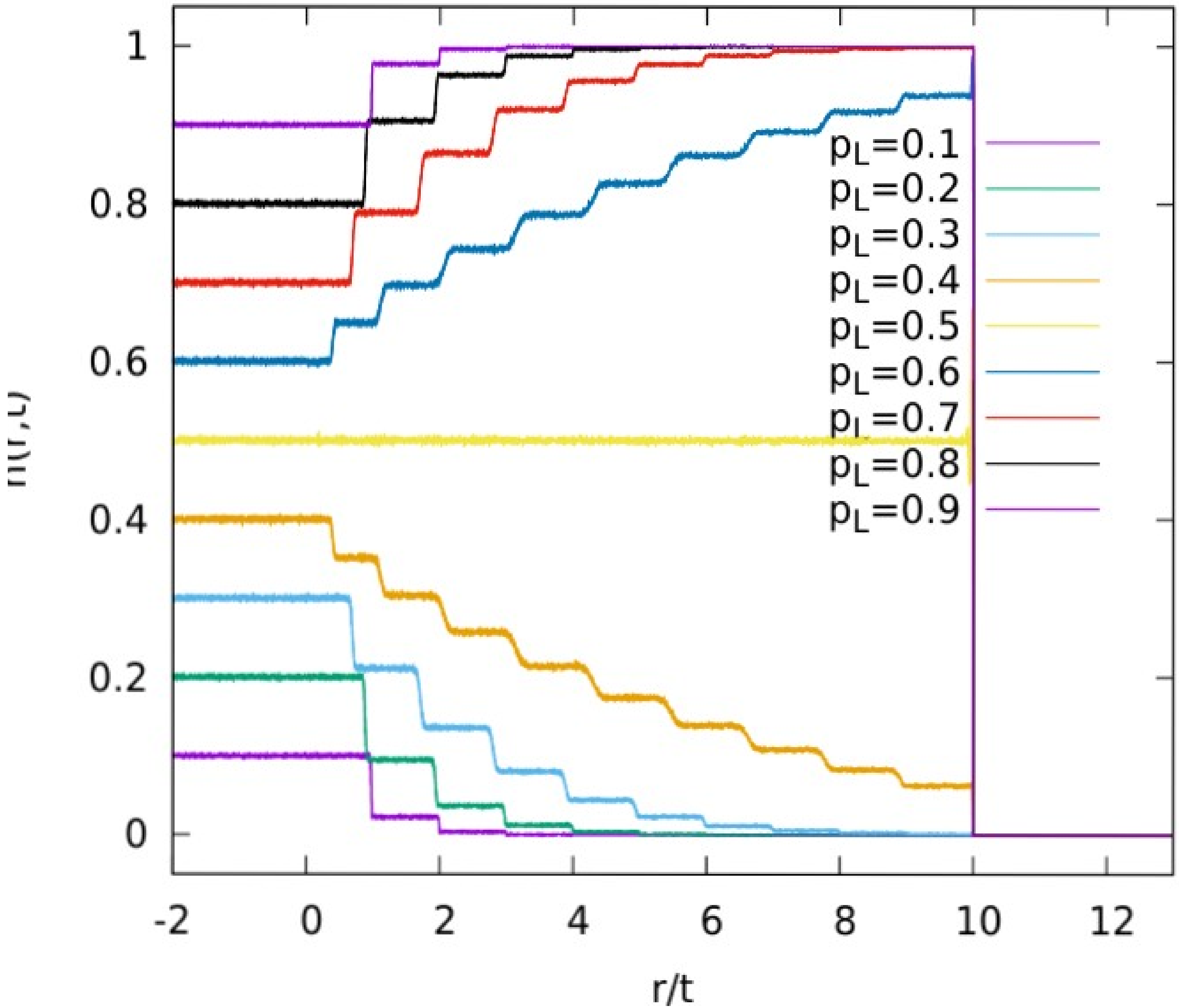
$$\partial_t y + \mathbf{v} \partial_x y = 0,$$

GHD

$$y_k = \sigma_k / \rho_k = e^{-\epsilon_k}$$

$$\langle \delta\epsilon_k \delta\epsilon_p \rangle = \delta_{kp} \frac{1 + \mathrm{e}^{\epsilon_k}}{\sigma_k}.$$

Yang-Yang



n(r,t)
r/t
-2
0
2
4
6
8
10
12
0
0.2
0.4
0.6
0.8
1
p_L=0.1
p_L=0.2
p_L=0.3
p_L=0.4
p_L=0.5
p_L=0.6
p_L=0.7
p_L=0.8
p_L=0.9

Disappearance of solitons:

$$x/t < V_1{}^{l}: \quad y_1, y_2, \cdots$$

homogeneous

$$V_1{}^{l} < x/t < V_2{}^{l}: \quad 0, y_2, \cdots$$

....

$$V_l{}^{l} < x/t: \quad 0, 0, \cdots$$

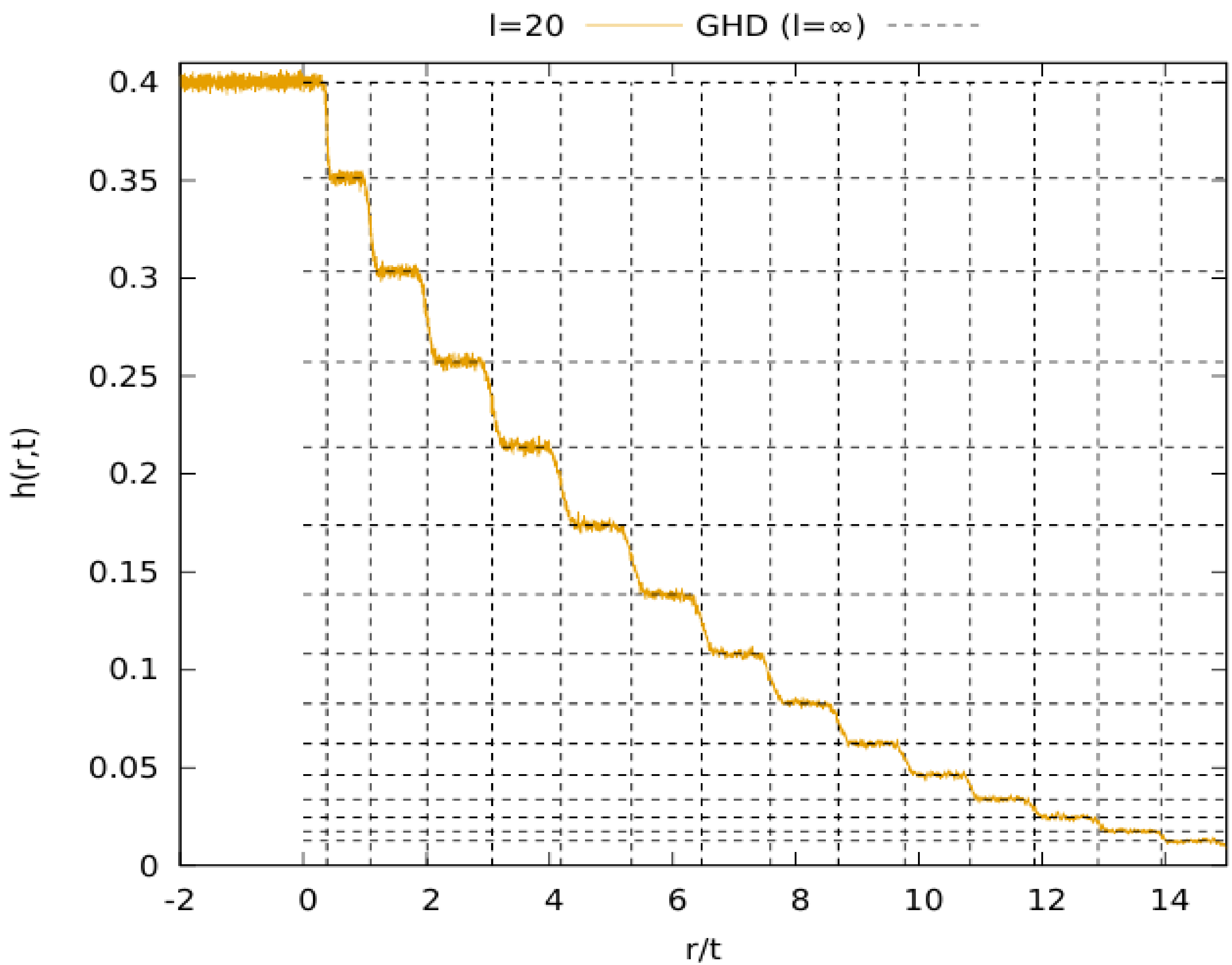
empty

Many exact predictions

- Positions of speeds and heights of density plateaus empty on the right :

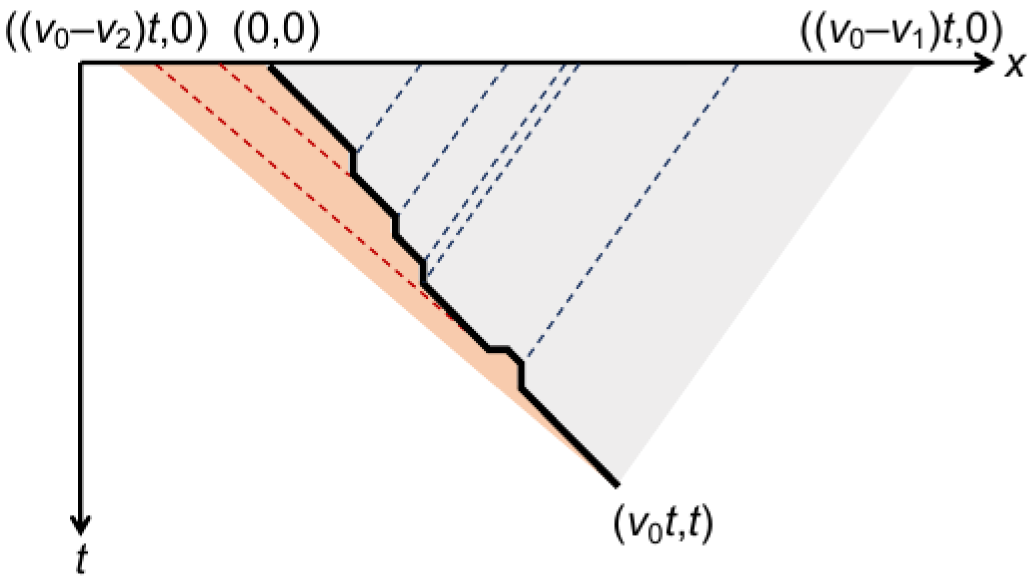
$$v_k = k\frac{1-z^{k+1}}{1+z^{k+1}}$$

$$h(k) = \sum_{j=1}^{\infty} j\rho_j(k) = \sum_{j=k+1}^{\infty} jn_j(k)r_j(k) = \frac{z^{k+1}([k+2]+k[1])}{[2k+3]+(2k+1)[1]z^{k+1}}.$$



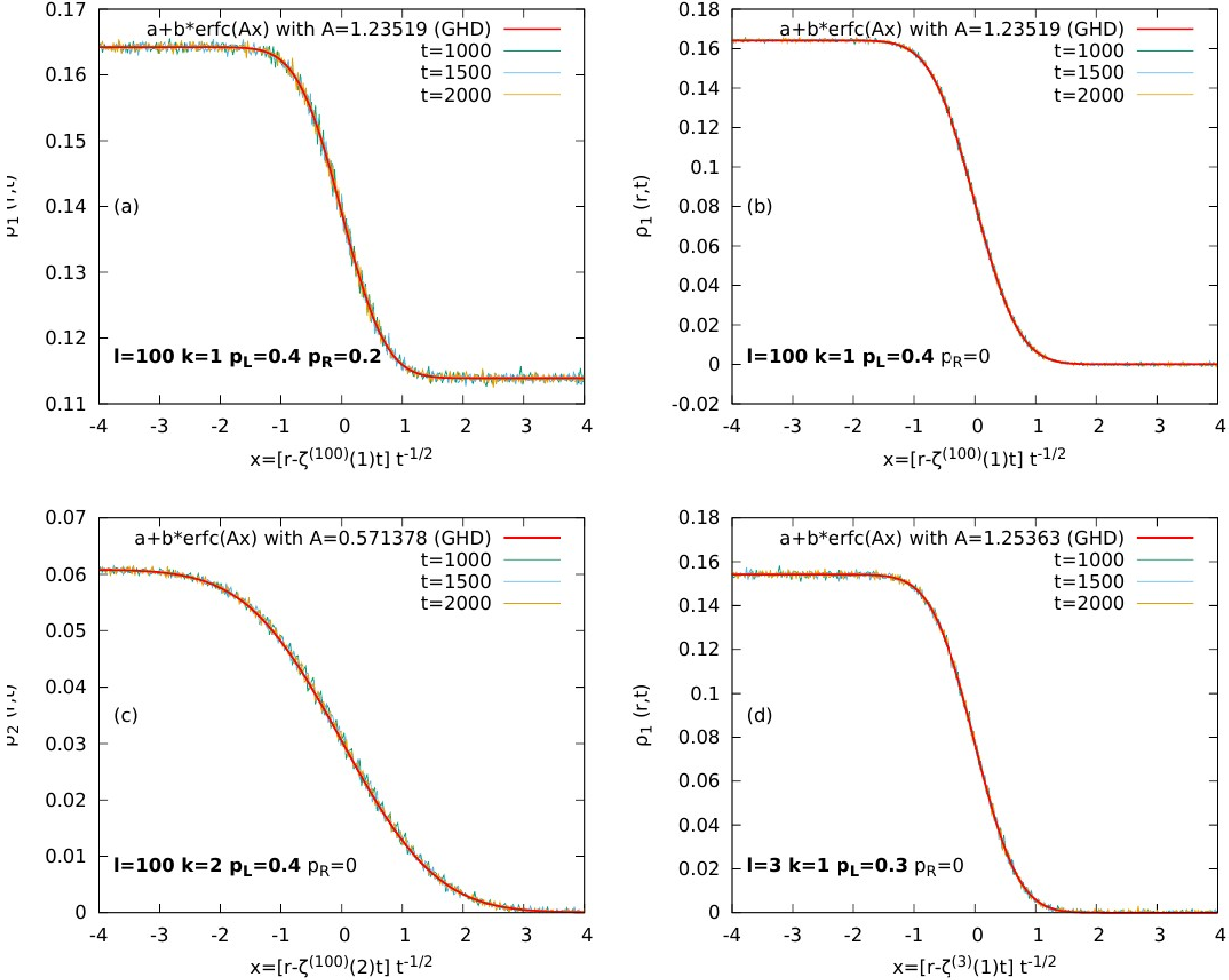
l=20
GHD (l=∞)
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
h(r,t)
-2
0
2
4
6
8
10
12
14
r/t

Diffusive corrections



J. De Nardis, D. Bernard and B. Doyon, Diffusion in generalized hydrodynamics and quasiparticle scattering, SciPost Phys. **6**, 049 (2019).

S. Gopalakrishnan, D. A. Huse, V. Khemani and R. Vasseur, Hydrodynamics of operator spreading and quasiparticle diffusion in interacting integrable systems, Phys. Rev. B **98**, 220303(R) (2018).



a+b*erfc(Ax) with A=1.23519 (GHD)
t=1000
t=1500
t=2000
(a)
l=100 k=1 pL=0.4 pR=0.2
x=[r-ζ(100)(1)t] t-1/2
a+b*erfc(Ax) with A=1.23519 (GHD)
t=1000
t=1500
t=2000
ρ1 (r,t)
(b)
l=100 k=1 pL=0.4 pR=0
x=[r-ζ(100)(1)t] t-1/2
a+b*erfc(Ax) with A=0.571378 (GHD)
t=1000
t=1500
t=2000
(c)
l=100 k=2 pL=0.4 pR=0
x=[r-ζ(100)(2)t] t-1/2
a+b*erfc(Ax) with A=1.25363 (GHD)
t=1000
t=1500
t=2000
ρ1 (r,t)
(d)
l=3 k=1 pL=0.3 pR=0
x=[r-ζ(3)(1)t] t-1/2

Transport proprties

$$\langle QQ \rangle = p(1-p) = \sum_i \rho_i \sigma_i (\rho_i + \sigma_i)(v_i^{\infty})^2$$

Second cumulant

$$c_2 = \langle j(0)Q \rangle = p(1-p) = \sum_i \rho_i \sigma_i (\rho_i + \sigma_i)(v_i^{l} v_i^{\infty})^2$$

Drude

$$D = \langle J(0)J(t) \rangle = p(1-p) = \sum_i \rho_i \sigma_i (\rho_i + \sigma_i)((v_i^{l})^2 v_i^{\infty})^2$$

Completely analogous to :

Doyon B and Spohn H 2017 Drude Weight for the Lieb-Liniger Bose Gas *SciPost Physics* **3** 039

Transport proprties

$$\langle QQ \rangle = p(1-p) = \sum_i \rho_i \sigma_i (\rho_i + \sigma_i)(v_i^\infty)^2$$

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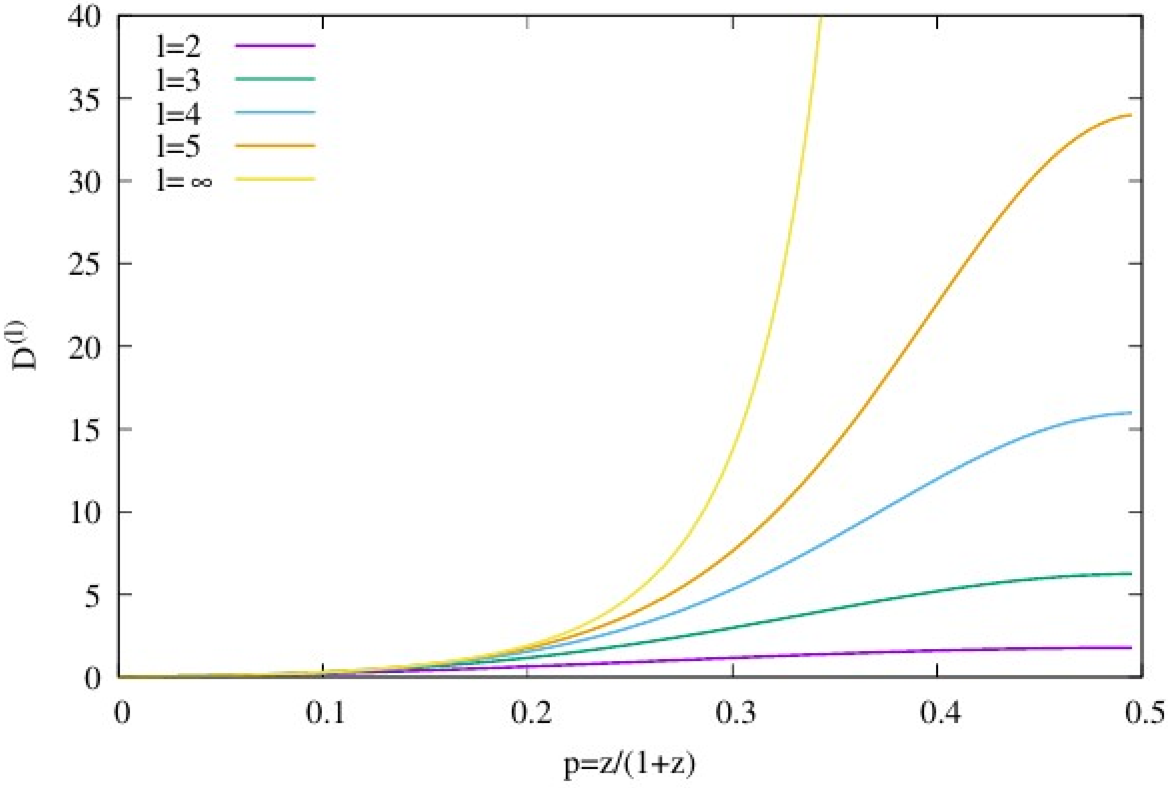
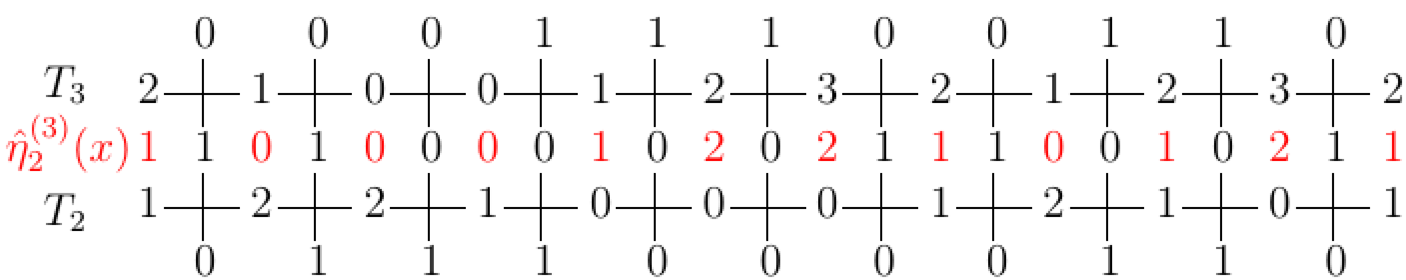
Drude weight



Generalized identity

$$
\begin{aligned}
C_{i,j}^{l,m} &= \sum_{k=1}^{\infty} \frac{\partial \eta_i^{(l)}}{\partial \epsilon_k} \frac{\partial \eta_j^{(m)}}{\partial \bar{\epsilon}_k} \frac{1+e^{\bar{\epsilon}_k}}{\sigma_k} \\
&= \sum_{k\geq 1} \rho_k \sigma_k (\rho_k + \sigma_k) v_k^{(i)} v_k^{(j)} v_k^{(l)} v_k^{(m)}.
\end{aligned}
$$

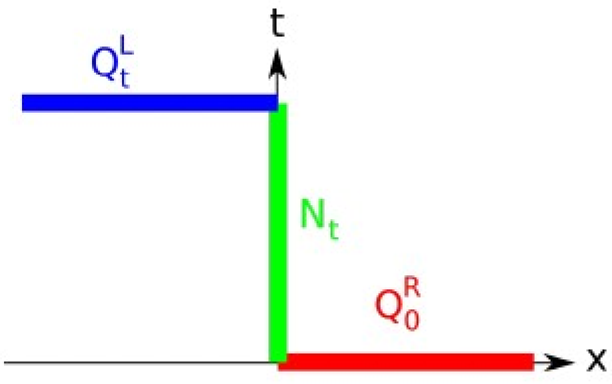


Current for energy j under time evolution l

$$
\eta_j^{(l)} = \sum_k \min(j,k) \rho_k v_k^{(l)}
$$

Large deviation function

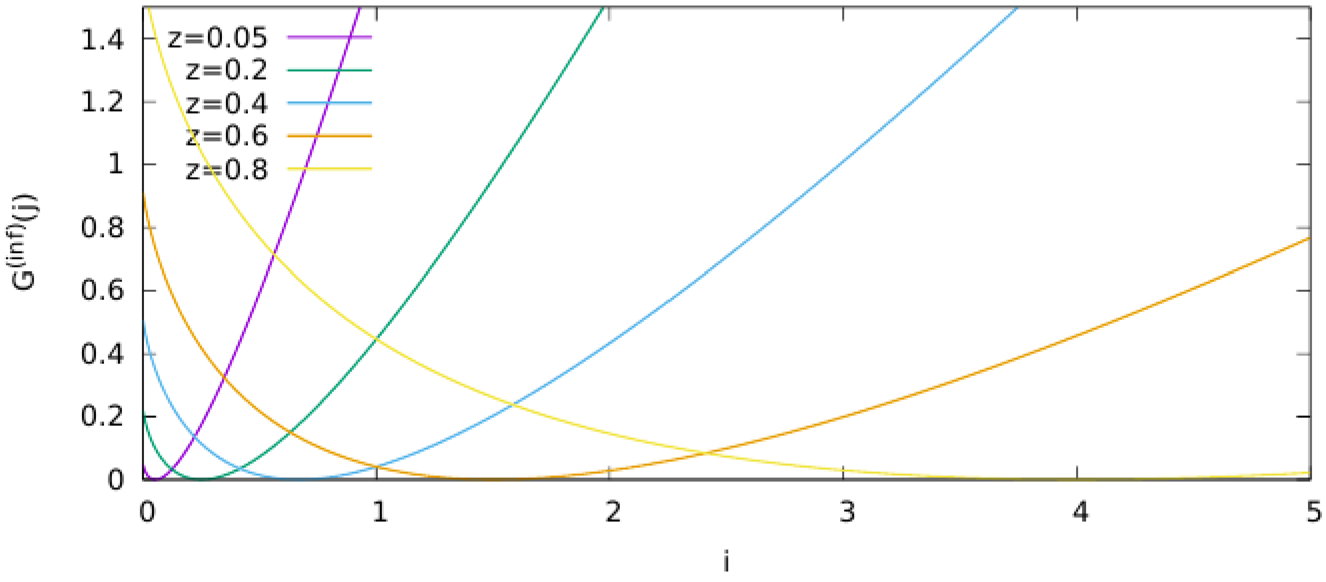
$$e^{tF(\lambda)} = e^{\int_0^t j(0,s)ds}$$



$$\frac{dF(\lambda)}{d\lambda} = \langle j(0) \rangle_{\beta-\lambda}$$

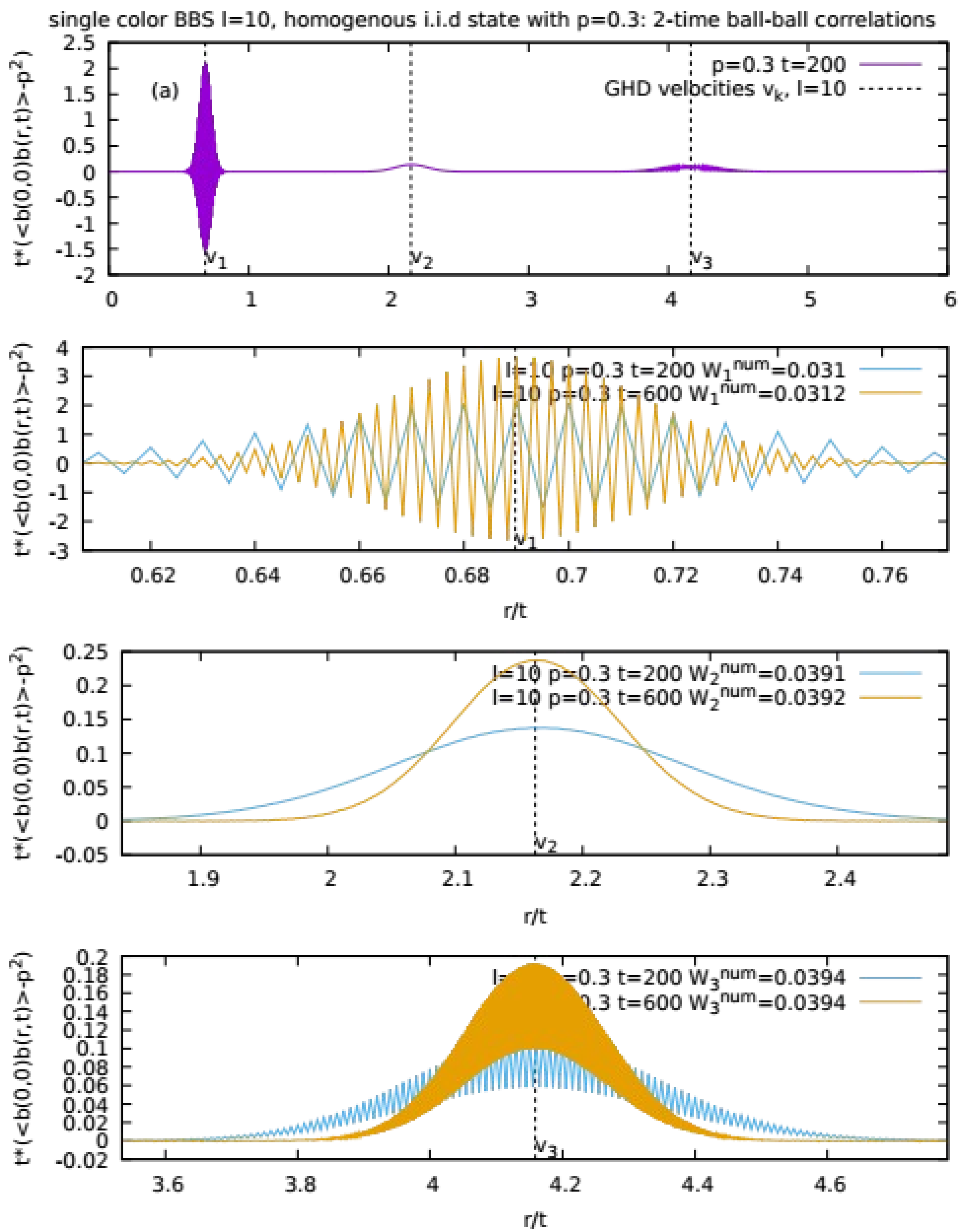
Large deviation function

$$F^{(l)}(\lambda) = \ln\left(\frac{1-(ze^{\lambda})^{l+1}}{1-ze^{\lambda}}\right) - \ln\left(\frac{1-z^{l+1}}{1-z}\right).$$



Rate function

Density correlator



Theory :

Ballistic propagation of normal modes : $$\delta\epsilon_t + v\delta\epsilon_x = 0$$

Integrate the t=0 correlator (YY) : $$\langle \delta\epsilon_i(x,t)\delta\epsilon_j(0,0)\rangle = \delta_{ij}\frac{1+e^{\epsilon_i}}{\sigma_i}\delta(x - v_i t)$$

Energy correlator : $$C_{i,j}(x,t) = \partial_k \eta_i^{(1)} \partial_k \eta_j^{(1)} \langle \delta\epsilon_i(x,t)\delta\epsilon_j(0,0)\rangle$$

$$C_{i,j}(x,t) = \sum_{k\geq 1} \rho_k \sigma_k (\rho_k + \sigma_k) v_k^{(i)} v_k^{(j)} \delta(x - v_k t)$$

Outlook

- Simple integrable deterministic model gives amazing agreement with hydrodynamical predictions
- Related to interesting combinatorics such as Schensted's row bumping algorithm.

Some remarks on $A_1^{(1)}$ soliton cellular automata

Susumu Ariki

SOLVABLE MODELS IN THE KPZ CLASS: APPROACH THROUGH PERIODIC AND FREE BOUNDARY SCHUR MEASURES

TAKASHI IMAMURA, MATTEO MUCCICONI, AND TOMOHIRO SASAMOTO