Off-diagonal long-range order in low dimensional quantum systems

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Talking Integrability: Spins, Fields, and Strings

KITP. Santa Barbara - 31 August 2022

Outline

- **Reminder on off-diagonal long-range order (ODLRO)**
- **ODLRO order for 1D quantum systems**
  (including effects of external potentials and 1D anyons)
- **ODLRO in 2D: quasi-ODLRO at finite temperature (including coupled bilayer 2D systems)**
- **ODLRO & quasi-ODLRO with long-range interactions → BKT transition with long-range couplings**

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Off-diagonal long-range order (I)

One-body density matrix of a generic system having N particles and its eigenvalues:

$$\rho_1(\vec{r}, \overrightarrow{r'}) \equiv \langle \hat{\psi}^+(\vec{r})\, \hat{\psi}(\overrightarrow{r'}) \rangle$$

$$\int \rho_1(\vec{r}, \overrightarrow{r'})\, \vartheta_i(\overrightarrow{r'}) d\overrightarrow{r'} = \lambda_i \vartheta_i(\vec{r})$$

ODLRO: largest eigenvalue λ_0 scale with N, while the others are O(1) [Penrose-Onsager, PR (1956) – Stringari-Pitaevskii, Bose-Einstein Condensation and Superfluidity (2016)]

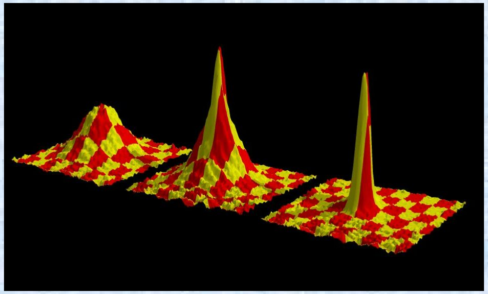
Off-diagonal long-range order (II)

largest eigenvalue $\lambda_0 \sim N^{\mathcal{C}_0(T)}$

in general $\lambda_k \sim N^{\mathcal{C}_k(T)}$

$\mathcal{C}_0(T)$

- 0 → Fermi-like
- between 0 and 1 → quasi-ODLRO
- 1 → ODLRO & Bose-Einstein Condensation



Off-diagonal long-range order (III)

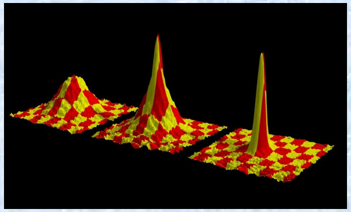
$$\mathcal{C}_0(T) \begin{cases} 0 \rightarrow \text{Fermi-like} \\ \text{between 0 and 1} \rightarrow \textit{quasi}\text{-ODLRO} \\ 1 \rightarrow \text{ODLRO \& Bose-Einstein Condensation} \end{cases}$$

If two or more $\mathcal{C}$ are equal to 1→ fragmentation

If two or more $\mathcal{C}$ are larger than 0 and at least one is smaller than 1→ *quasi*-fragmentation

Off-diagonal long-range order (IV)

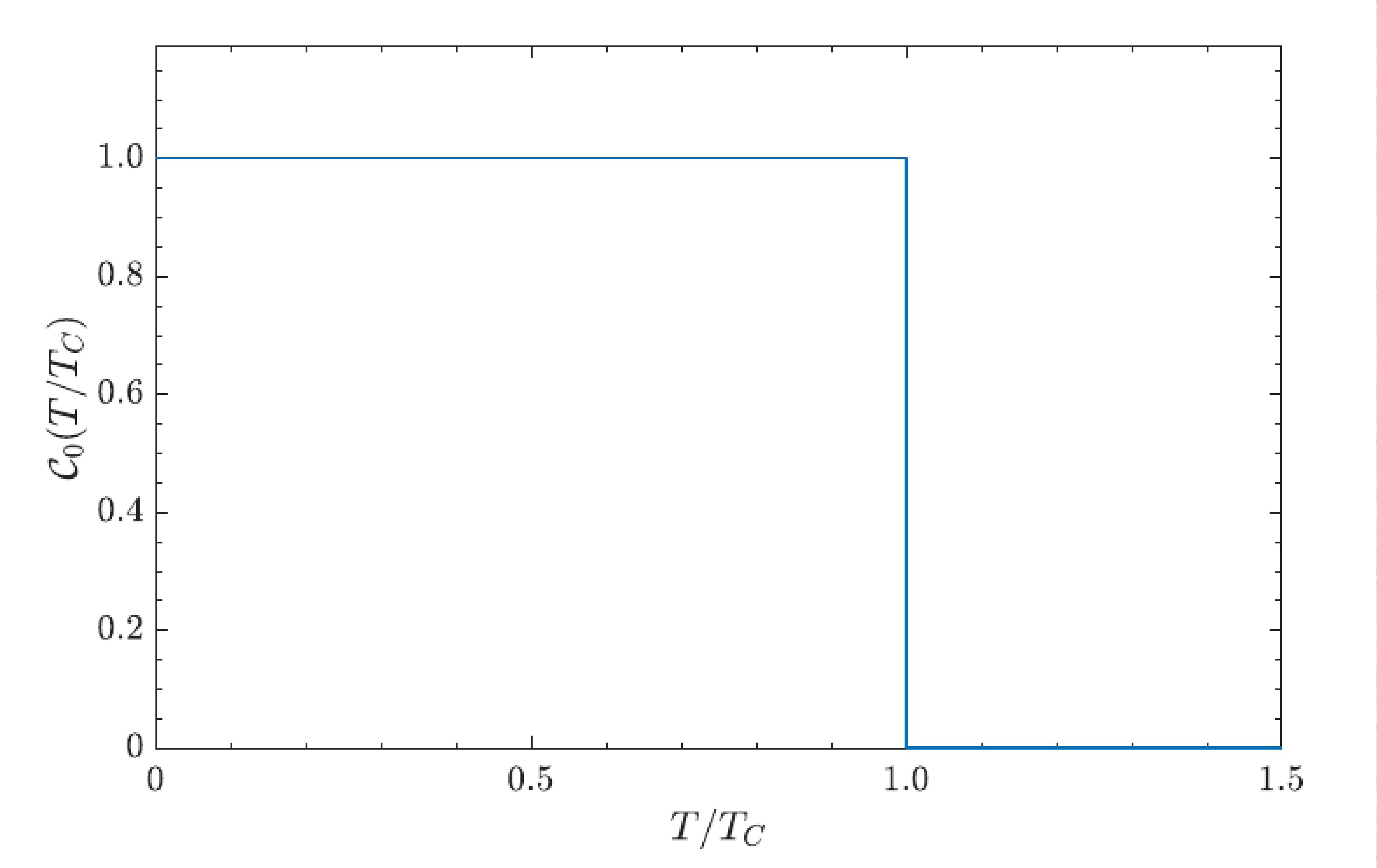
$\mathcal{C}_0(T)$ $\begin{cases} 0 \rightarrow \text{Fermi-like} \\ \text{between 0 and 1} \rightarrow \textit{quasi}\text{-ODLRO} \\ 1 \rightarrow \text{ODLRO \& Bose-Einstein Condensation} \end{cases}$



ODLRO → correlation functions tend to a constant (in the translational invariant case)

quasi-ODLRO → correlation functions decay as power-law (in the translational invariant case)

ODLRO in 3D *[if Tc is different from zero]*



For a 1D Bose gas...

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Lieb-Liniger Hamiltonian (I)

N interacting bosons in 1D:

$$H = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2\lambda \sum_{i<j} \delta(x_i - x_j)$$

One non-trivial coupling constant:

$$\gamma = \frac{2m\lambda}{\hbar^2 n}$$

n ← density

Temperature typically in units of the degeneracy temperature:

$$k_B T_D = \frac{\hbar^2 n^2}{2m}$$

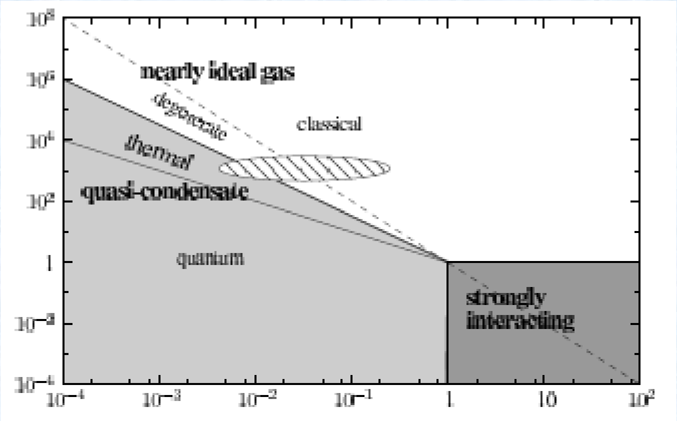
Coupling controllable from the 3D setup:

$$\lambda = \frac{\hbar^2 a_{3D}}{m a_\perp^2} \frac{1}{1 - C a_{3D}/a_\perp}$$

Lieb-Liniger Hamiltonian (II)

$$H=-\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}+2\lambda\sum_{i<j}\delta\left(x_i-x_j\right)$$

τ



[from I. Bouchole et al., 2009]

γ

$$\gamma=\frac{2m\lambda}{\hbar^2 n}$$

$$\tau=\frac{T}{T_D}$$

Large γ → Tonks-Girardeau limit

Lieb-Liniger Hamiltonian (III)

$$|\psi_N\rangle = \frac{1}{\sqrt{N!}}\int_0^L d^N z\, \chi_N(z_1, \dots, z_N)\psi_B^+(z_1)\dots\psi_B^+(z_N)|0\rangle$$

$$\lambda_j = \frac{2\pi}{L}\left(j - \frac{N+1}{2}\right) + \frac{2}{L}\sum_{k=1}^{N} \arctan\left(\frac{\lambda_j - \lambda_k}{c}\right), \qquad j = 1, \dots, N$$

$$\chi_N(z_1, \dots, z_N) = \mathcal{N} \det\left(e^{i\lambda_j z_m}\right)\prod_{n<l}[\lambda_l - \lambda_n - ic\, sign(z_l - z_n)]$$

Bethe ansatz solution

Deviations from ODLRO

$$\frac{n(p=0)}{L} \propto N^{C}$$

$$n(p) = \frac{L}{2\pi}\int_0^L \rho(z)\, e^{ipz}\, dz$$

Large distance $\rho(z \gg 1)$ ⟹ Small momenta $n(p \approx 0)$

Luttinger Liquid prediction: $\rho(z \to \infty) \propto z^{-1/2K}$

$$K(\gamma) = \frac{v_F}{s(\gamma)}$$

$$p_{min} \approx \frac{2\pi}{L} \propto N^{-1} \qquad n_{1D} = \frac{N}{L}$$

$$\frac{n(p\to 0)}{L} \propto \frac{1}{p^{1-1/2K}} \propto N^{1-1/2K}$$

$$\mathcal{C}(\gamma) = 1 - \frac{1}{2K}$$

To go to large N:

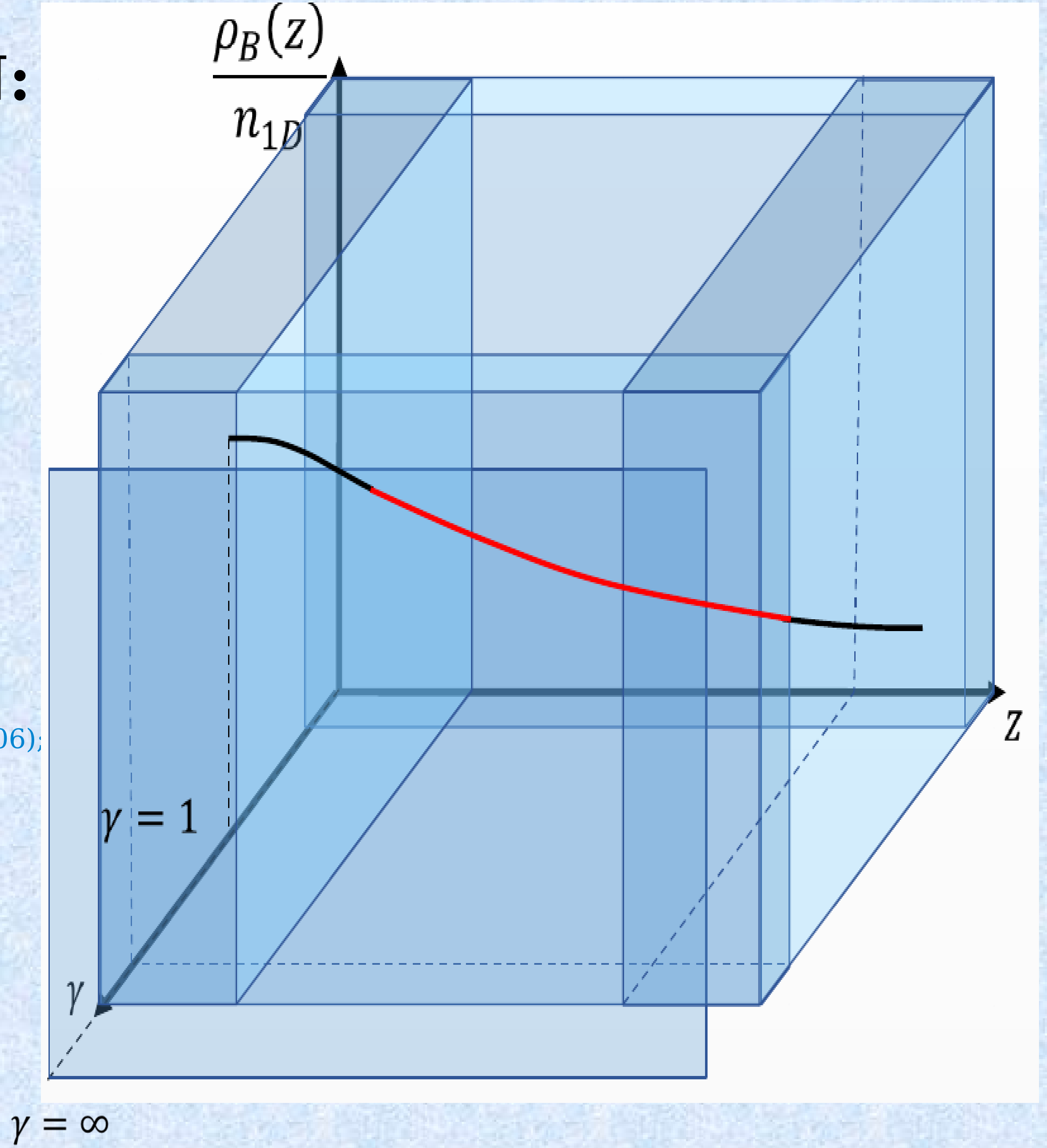
Small γ
[C. Mora, Y. Castin, PRA (2003);...]

Large γ
[M. Jimbo, T. Miwa, PRD (1981);
P.J. Forrester, N.E. Frankel,
M.I. Makin, PRA (2006);...]

TG limit – and beyond
[A. Lenard, J. Math. Phys. (1964);
G.E. Astrakharchik, J. Boronat,
J. Casulleras, S. Giorgini, PRL (2005);
D. Gangardt, G.V. Shlyapnikov, NJP (2006);
P. Vignolo, A. Minguzzi, PRL (2013);...]

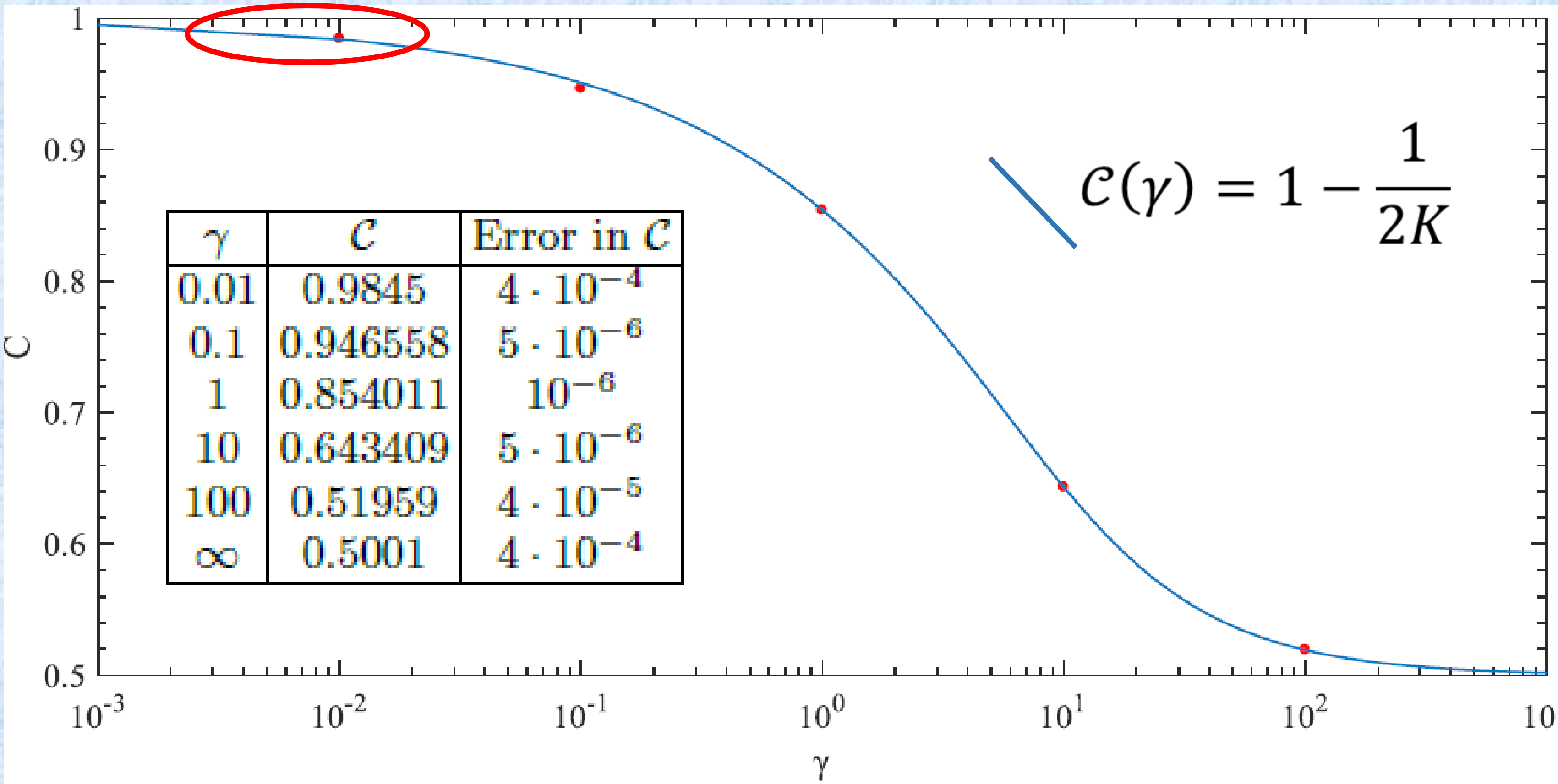
Small z
[M. Olshanii, V. Dunjko,
PRL (2003);...]

Large z
[A. Shashi, M. Panfil, J.-S. Caux,
A.I. Alexander, PRB (2012);...]



From γ=0 to infinite γ

For a 1D Lieb-Liniger Bose gas: if the largest eigenvalue of the one-body density matrix scale with N to the power C

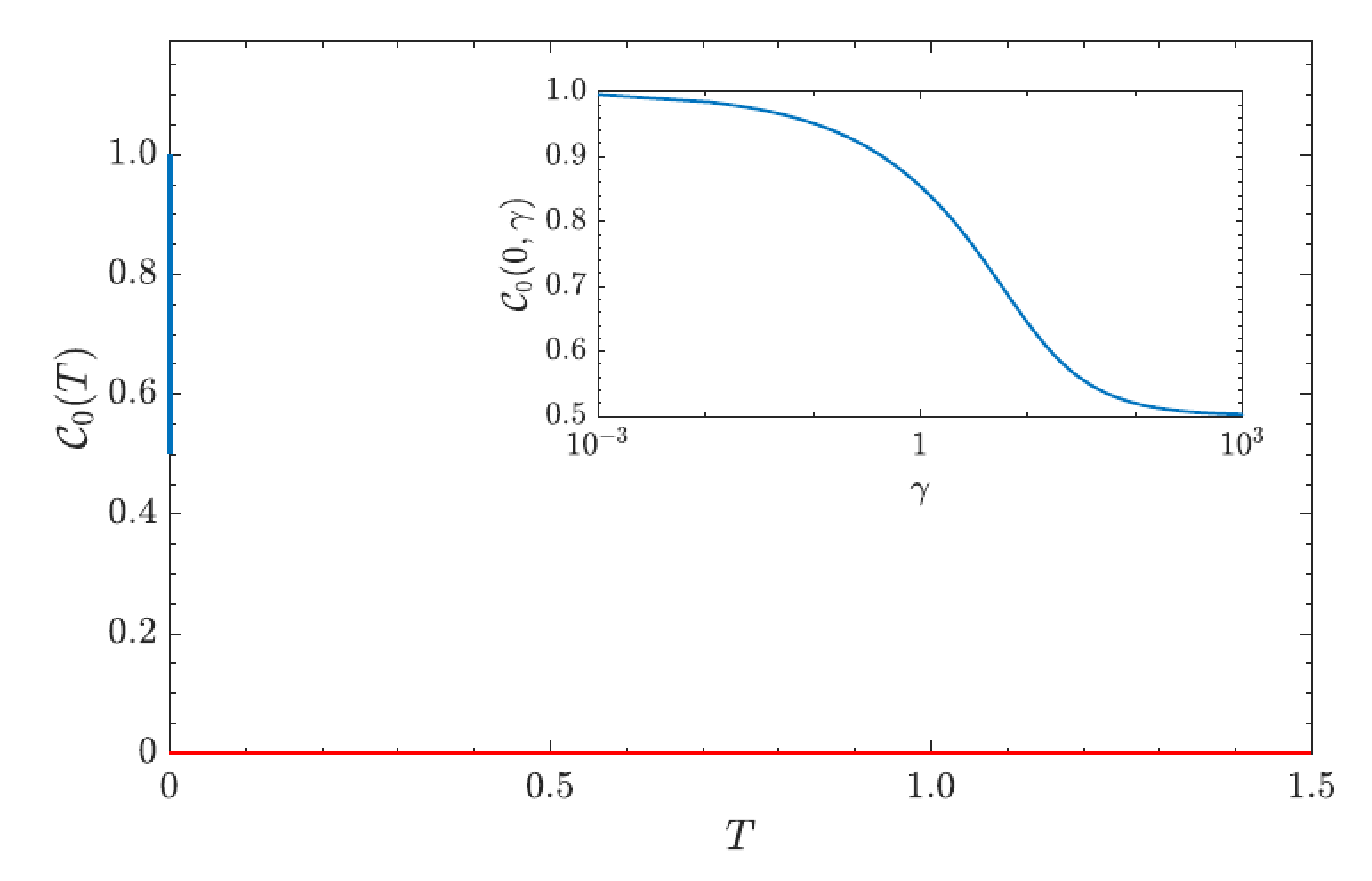


γ	$\mathcal{C}$	Error in $\mathcal{C}$
0.01	0.9845	$4 \cdot 10^{-4}$
0.1	0.946558	$5 \cdot 10^{-6}$
1	0.854011	10^{-6}
10	0.643409	$5 \cdot 10^{-6}$
100	0.51959	$4 \cdot 10^{-5}$
∞	0.5001	$4 \cdot 10^{-4}$

$$\mathcal{C}(\gamma) = 1 - \frac{1}{2K}$$

[A. Colcelli, G. Mussardo and A. Trombettoni, Europhys. Lett. (2018)]

ODLRO in 1D



Notice: no quasi-fragmentation

[A. Colcelli, G. Mussardo, and A. Trombettoni, Europhys. Lett. (2018); A. Colcelli, N. Defenu, G. Mussardo, and A. Trombettoni, PRB (2020)]

Hard-Core Anyons

$$\chi_N^\kappa(z_1,\dots,z_N) = \left[\prod_{1\le i<j\le N} A(z_j - z_i)\right]\chi_N^1(z_1,\dots,z_N) \qquad \kappa = \frac{m}{n} \in \mathbb{Q}$$

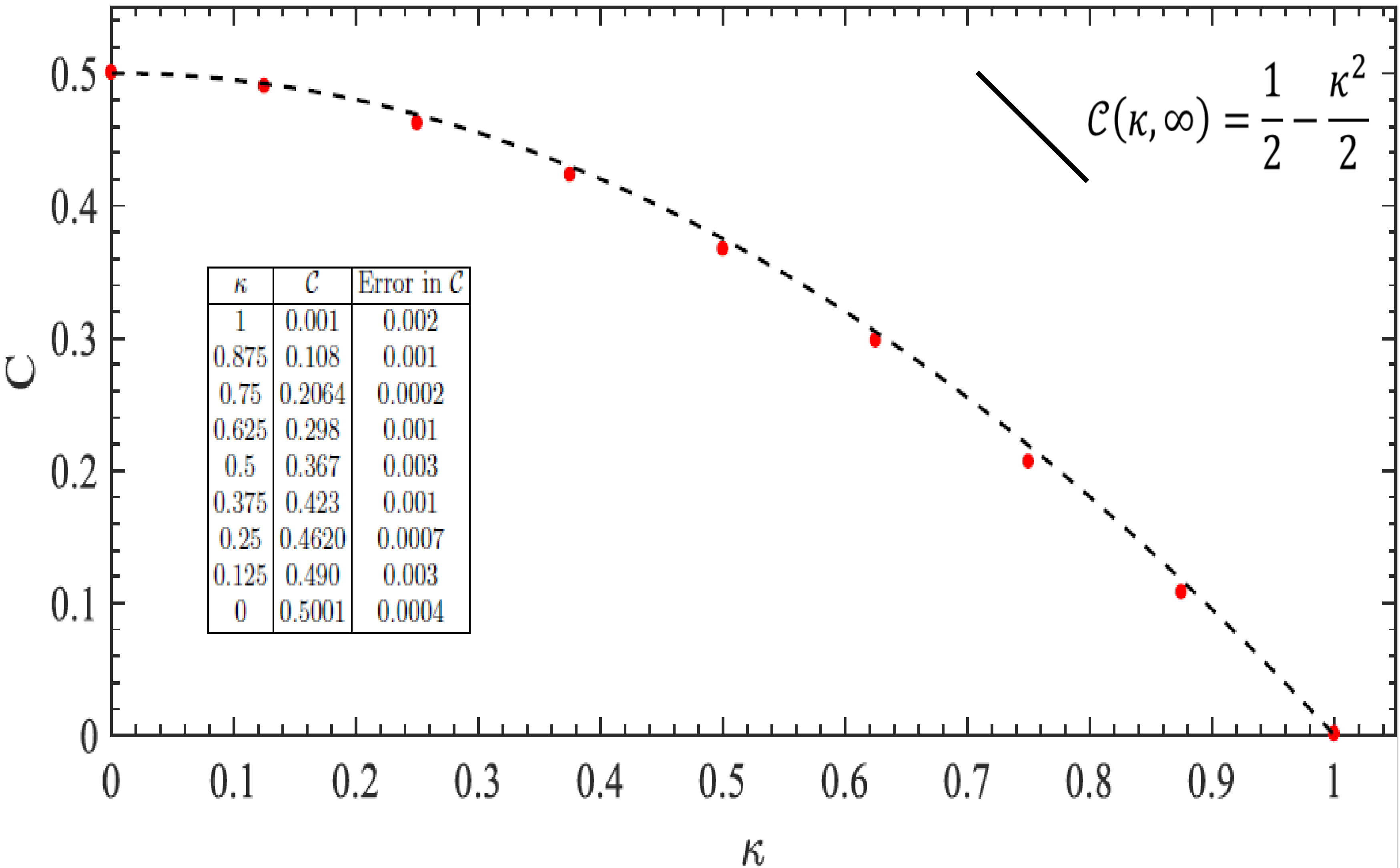
Anyon – Fermi mapping: $A(z_j - z_i) = [\theta(z_j - z_i) + \theta(z_i - z_j)e^{i\pi(1-\kappa)}]$ $\qquad \theta(0) = 0$

$\boxed{\kappa = 0}$ Boson – Fermi mapping: $A(z_j - z_i) = sign(z_j - z_i)$

$$\rho_A(t) = \det\left[\frac{2}{\pi}\int_0^{2\pi} d\tau\, e^{i(j-l)\tau} A(\tau - t)\sin\left(\frac{\tau - t}{2}\right)\sin\left(\frac{\tau}{2}\right)\right]_{j,l=1,\dots,N-1} \qquad t \equiv \frac{2\pi x}{L}$$

[R. Santachiara and P. Calabrese, JSTAT (2008)]

Hard-Core Anyons



κ	$\mathcal{C}$	Error in $\mathcal{C}$
1	0.001	0.002
0.875	0.108	0.001
0.75	0.2064	0.0002
0.625	0.298	0.001
0.5	0.367	0.003
0.375	0.423	0.001
0.25	0.4620	0.0007
0.125	0.490	0.003
0	0.5001	0.0004

Anyonic Lieb-Liniger Model (I)

$$H = \int_0^L \{[\partial_z \psi_A^+(z)]\,[\partial_z \psi_A(z)] + c\,\psi_A^+(z)\psi_A^+(z)\psi_A(z)\psi_A(z)\}\,dz$$

$$\psi_A^+(z_1)\psi_A^+(z_2) = e^{i\pi\kappa\,sign(z_1 - z_2)}\psi_A^+(z_2)\psi_A^+(z_1)$$

$$\psi_A(z_1)\psi_A(z_2) = e^{i\pi\kappa\,sign(z_1 - z_2)}\psi_A(z_2)\psi_A(z_1)$$

$$\psi_A(z_1)\psi_A^+(z_2) = e^{-i\pi\kappa\,sign(z_1 - z_2)}\psi_A^+(z_2)\psi_A(z_1) + \delta(z_1 - z_2)$$

$$\kappa = \begin{cases} 0 & \text{Bosons} \\ 1 & \text{Fermions,} \end{cases}$$

$$\chi_N(z_1, \dots, z_i, z_{i+1}, \dots, z_N) = e^{-i\pi\kappa\,sign(z_i - z_{i+1})}\chi_N(z_1, \dots, z_{i+1}, z_i, \dots, z_N)$$

Anyonic Lieb-Liniger Model (II)

Twisted BC: $\chi_N^\kappa(0, x_2, \dots) = e^{i\pi\kappa(N-1)}\chi_N^\kappa(L, x_2, \dots) \longleftrightarrow \chi_N^0(0, x_2, \dots) = \chi_N^0(L, x_2, \dots)$

Periodic BC

$$\lambda_j = \frac{2\pi}{L}\left(j - \frac{N+1}{2}\right) + \frac{2}{L}\sum_{k=1}^{N} \arctan\left(\frac{\lambda_j - \lambda_k}{c'}\right), \qquad j = 1, \dots, N \qquad c' = \frac{c}{\cos(\pi\kappa/2)} > 0$$

$$\chi_N^\kappa(z_1, \dots, z_N) = \mathcal{N} \exp\left(i\frac{\pi\kappa}{2}\sum_{j<k} sign(z_j - z_k)\right)\det\left(e^{i\lambda_j z_m}\right) \cdot$$

$$\cdot \prod_{n<l} [\lambda_l - \lambda_n - ic' sign(z_l - z_n)]$$

[O.I. Pâţu, V.E. Korepin, D.V. Averin, JPA (2007);
M.T. Batchelor, X.-W. Guan, N. Oelkers, PRL (2006)]

Predictions for the ODLRO scaling

$$n_A(p=0) \propto N^{\mathcal{C}(\kappa)}$$

$$n_A(p) = \frac{L}{2\pi}\int_0^L \rho_A(z)\, e^{ipz}\, dz$$

Large distance $\rho_A(z \gg 1)$ ⟹ "Small" momenta $n_A(p \approx -p_F\kappa)$

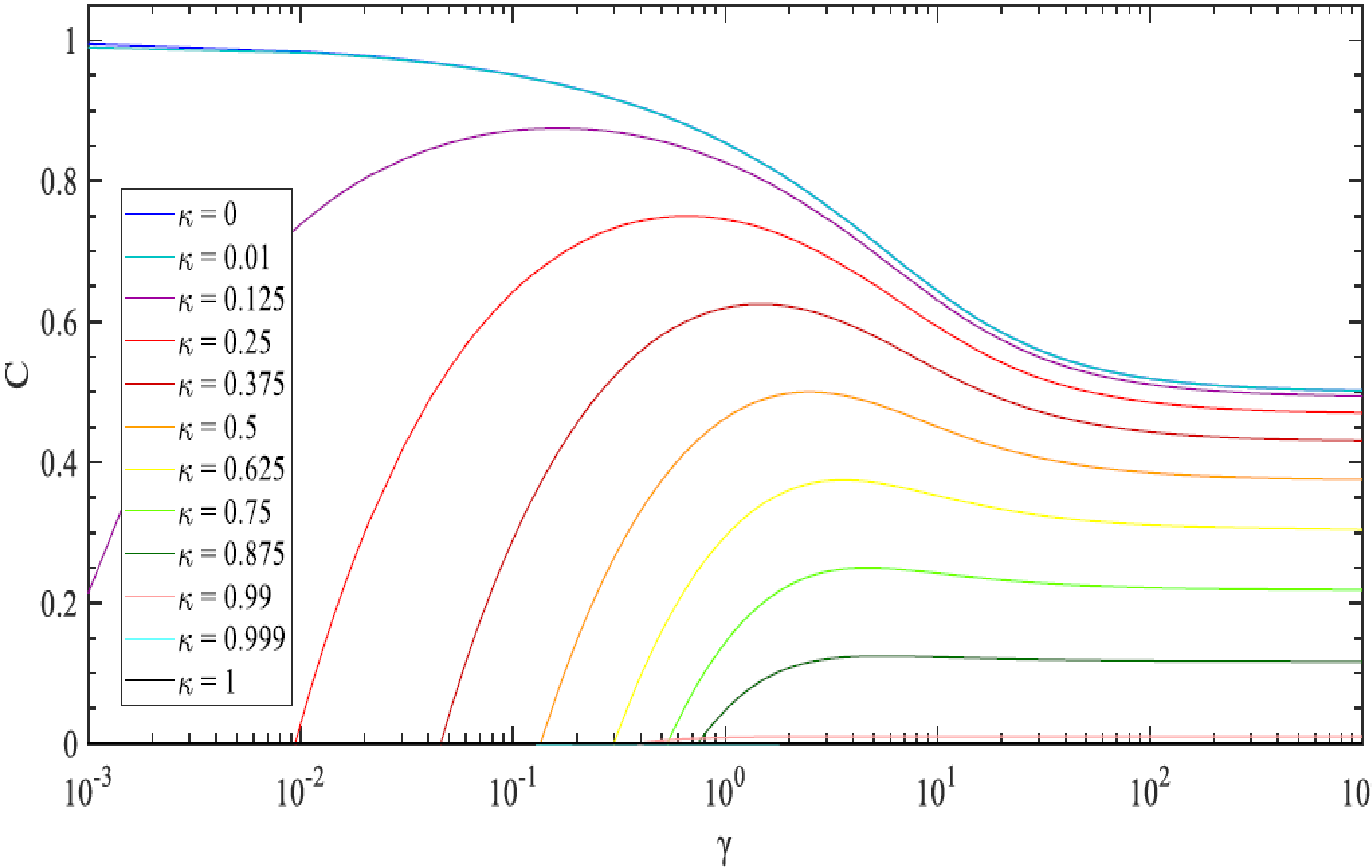
$\boxed{\kappa = 0}$ $\mathcal{C}(\kappa=0,\gamma) = 1 - \frac{1}{2K}$ $K(\gamma) = \frac{v_F}{s(\gamma)}$

$$\frac{n_A(p)}{L} \propto \frac{1}{(p+p_F\kappa)^{1-\frac{1}{2K}-\frac{K\kappa^2}{2}}} \Longrightarrow \boxed{\mathcal{C}(\kappa,\gamma) = 1 - \frac{1}{2K} - \frac{K\kappa^2}{2}}$$

Results for 1D Lieb-Liniger anyons →

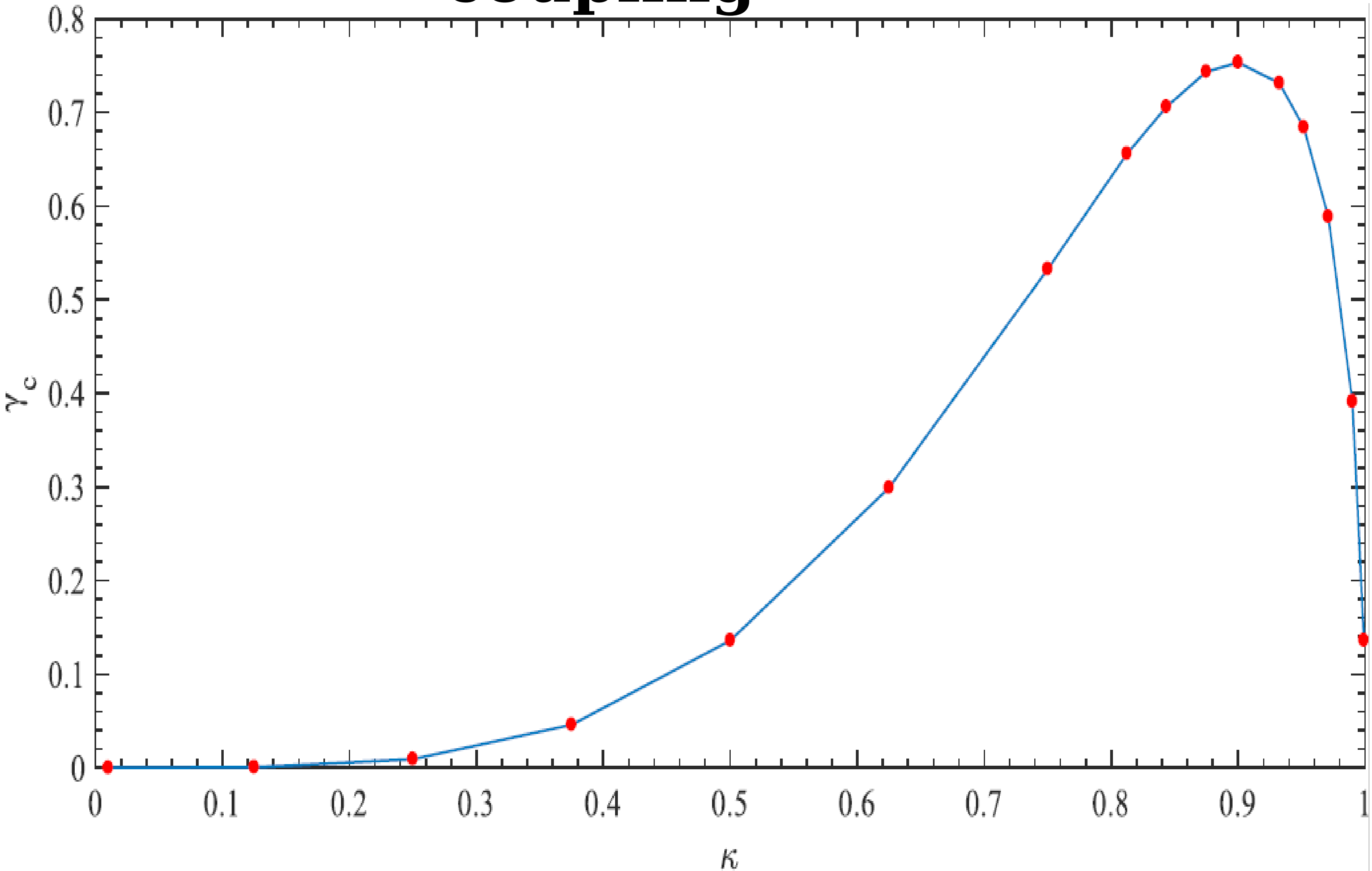
$$\mathcal{C}(\kappa,\gamma) = 1 - \frac{1}{2K} - \frac{K\kappa^2}{2}$$

Statistical parameter
1 → fermions
0 → bosons



“Critical” coupling

$$\mathcal{C}(\kappa, \gamma_c) = 0$$



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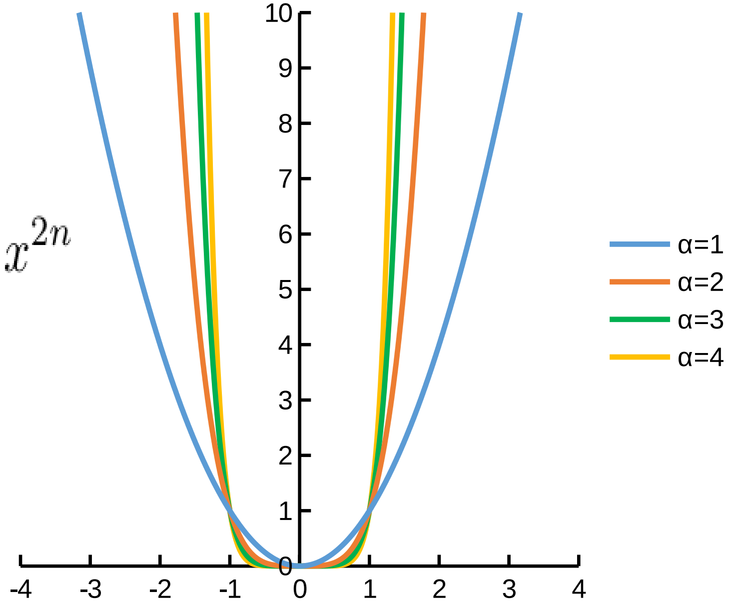
Tonks-Girardeau gas: effect of parabola

$$H = \sum_{i=1}^{N} \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_i^2} + V(x_i) \right]$$

$$V(x) = \Lambda x^{2n}$$



$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \phi_k(x) = \varepsilon_k \phi_k(x)$$

$$\psi|_{x_i = x_j} = 0\,, \quad \forall i, j = 1, \ldots, N \; with \; i \neq j$$

Bose – Fermi mapping: $\psi(x_1, \ldots, x_N) = \frac{1}{\sqrt{N!}} \det\left[\phi_k(x_l)\right]_{k,l=1,\ldots,N} \prod_{1 \leq i < j \leq N} sgn(x_i - x_j)$

$$\rho(x,y) = \sum_{i,j=1}^{N} (-1)^{i+j} \phi_i(y)\, \phi_j^*(x) \det \left[\delta_{k,l} - 2 \int_y^x dt\, \phi_l(t)\, \phi_k^*(t) \right]_{\substack{k,l=1,\ldots,N \\ k \neq j,\, l \neq i}}$$

Universality of the scaling of the ODLRO

$$\lambda_0 \sim \mathcal{B} N^{\mathcal{C}}$$

Semiclassical analysis:

$\hbar \to 0, \qquad \text{with } m,\ V(x),\ \mu \text{ fixed} \qquad\qquad N\hbar = \text{const}$

$$\rho(x) = \frac{1}{\pi\hbar}\sqrt{2m\left[\mu - V(x)\right]}$$

$$\rho_{\text{cft}}(\tilde{x},\tilde{y}) = \sqrt{\frac{m}{2\hbar\widetilde{L}}}\,\frac{|C|^2\left|\sin\left(\frac{\pi\widetilde{x}}{\widetilde{L}}\right)\right|^{\frac{1}{4}}\left|\sin\left(\frac{\pi\widetilde{y}}{\widetilde{L}}\right)\right|^{\frac{1}{4}}}{\left|\sin\left(\frac{\pi}{\widetilde{L}}\frac{\widetilde{x}-\widetilde{y}}{2}\right)\right|^{\frac{1}{2}}\left|\sin\left(\frac{\pi}{\widetilde{L}}\frac{\widetilde{x}+\widetilde{y}}{2}\right)\right|^{\frac{1}{2}}}$$

$$\widetilde{x}(x) = \int_{x_1}^{x}\frac{du}{v(u)} \qquad v(x) = \sqrt{\frac{2}{m}\left[\mu - V(x)\right]}$$

[Y. Brun and J. Dubail, SciPost Physics (2017)]

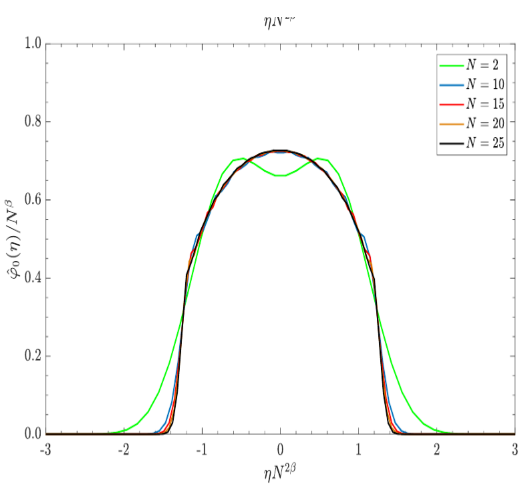
Universality of the scaling of the ODLRO

$$\lambda_0 = O(\hbar^{-1/2}) \rightarrow \lambda_0 \sim \mathcal{B} N^{1/2}$$ for *every* potential!

Regarding the coefficient $\mathcal{B}$.

$$\eta \equiv \frac{x}{\xi} \qquad \hat{\varphi}_0(\eta) \equiv \varphi_0(x)\sqrt{\xi} \qquad \int |\hat{\varphi}_0(\eta)|^2 \, d\eta = 1$$

$$\hat{\varphi}_0(\eta) \sim N^{\beta}$$



Universality of the scaling of the ODLRO

$$\tilde{\rho}(k) = \frac{1}{2\pi}\int dx \int dy\, \rho(x,y) e^{-i\,k\,(x-y)}$$

$$\frac{n_{\mathrm{peak}}}{\xi} \sim N^{\gamma}$$

$$\boxed{\gamma + 2\beta = C}$$

Role of boundary conditions

$$\mathcal{B}_{\mathrm{fit}} = 1.32(1)$$

Hard-wall with open BC: different from the analytical results for periodic BC! [1.54269..., see P.J. Forrester et al., PRA (2003)]

In other words: the thermodynamic limit "remembers" the boundary conditions (for $\mathcal{B}$, but not for $\mathcal{C}$)

Comparison of semiclassical limit with numerical results

$$\lambda_0 = \mathcal{A} + \mathcal{B}\, N^{\mathcal{C}} + \frac{\mathcal{D}}{N^{\mathcal{E}}}$$

n	$\mathcal{B}_{\rm fit}$	$\mathcal{B}$
1	1.4304(2)	1.430(4)
2	1.400(4)	1.392(4)
3	1.380(4)	1.378(3)
4	1.372(5)	1.368(2)
∞	1.31(1)	1.308(3)

n	$\mathcal{A}_{\rm fit}$	$\mathcal{B}_{\rm fit}$	$\mathcal{D}_{\rm fit}$	$\mathcal{E}_{\rm fit}$
1	−0.554(2)	1.4304(2)	0.122(1)	0.60(1)
2	−0.55(4)	1.400(4)	0.141(8)	0.79(6)
3	−0.53(3)	1.380(4)	0.16(2)	1.1(5)
4	−0.56(3)	1.372(5)	0.20(1)	0.9(3)
∞	−0.6(1)	1.31(1)	0.31(9)	0.3(1)

Fixing V and increasing the number (agreement with the CFT results for $\mathcal{B}$.)

Comparison with numerical results

n	$\mathcal{A}_{\mathrm{fit}}$	$\mathcal{B}_{\mathrm{fit}}$	$\mathcal{D}_{\mathrm{fit}}$	$\mathcal{E}_{\mathrm{fit}}$
1	$-0.56(3)$	$1.432(4)$	$0.13(3)$	$0.57(2)$
2	$-0.55(2)$	$1.407(4)$	$0.15(3)$	$1.0(2)$
3	$-0.56(2)$	$1.391(1)$	$0.18(2)$	$1.0(1)$
4	$-0.56(1)$	$1.38(2)$	$0.18(4)$	$0.8(1)$

Fixing instead the density at the center: almost the same results!

Comparison with numerical results

$$\int d\eta \propto \frac{1}{\xi} \propto \hbar^{-\frac{1}{n+1}} \qquad\qquad \beta = -\frac{1}{2n+2}$$

$$N^{-2\beta} \propto N^{\frac{1}{n+1}} \quad \rightarrow \quad \gamma = \frac{n+3}{2(n+1)}$$

n	$\mathcal{C}_{\text{fit}}$	$\mathcal{C}^{wkb}$	β_{fit}	γ_{fit}	β	γ
1	0.500(2)	0.496(8)	−0.25(1)	1.02(4)	$-\frac{1}{4}$	1
2	0.501(1)	0.54(3)	−0.16(1)	0.85(2)	$-\frac{1}{6}$	$\frac{5}{6}$
3	0.501(2)	0.54(7)	−0.12(2)	0.76(1)	$-\frac{1}{8}$	$\frac{3}{4}$
4	0.500(3)	0.54(9)	−0.10(1)	0.70(1)	$-\frac{1}{10}$	$\frac{7}{10}$
∞	0.500(1)		0.00(1)	0.502(2)	0	$\frac{1}{2}$

[A. Colcelli, J. Viti, G. Mussardo, and A. Trombettoni, PRA (2018)]

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ODLRO in 2D

Mermin-Wagner theorem → $\mathcal{C}_0(T) < 1$ for T>0

Moreover $\mathcal{C}_0(T=0) = 1$

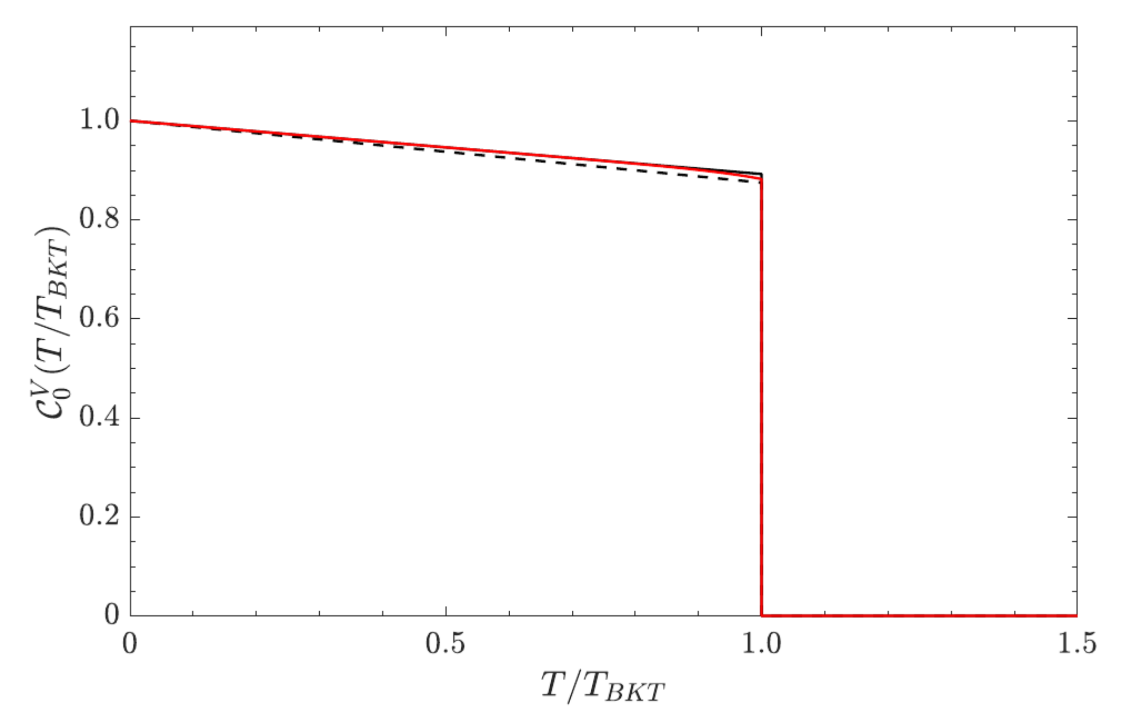
Connection with anomalous dimension

$$\rho(r) \sim \frac{1}{r^{D-2+\eta}} \quad \rightarrow \quad \mathcal{C}_0 = 1 - \frac{\eta}{2}$$

Therefore $\mathcal{C}_0(T = T_{\mathrm{BKT}}) = \frac{7}{8}$

Notice that for temperature between 0 and T the anomalous dimension is not universal, and so also $\mathcal{C}_0(T)$ is <u>not</u> universal

ODLRO in 2D: Villain model



$$\frac{k_B T_{\rm BKT}}{A} = \frac{1}{0.74} \simeq 1.351$$

[from: W. Janke and K. Nather, PRB (1993)]

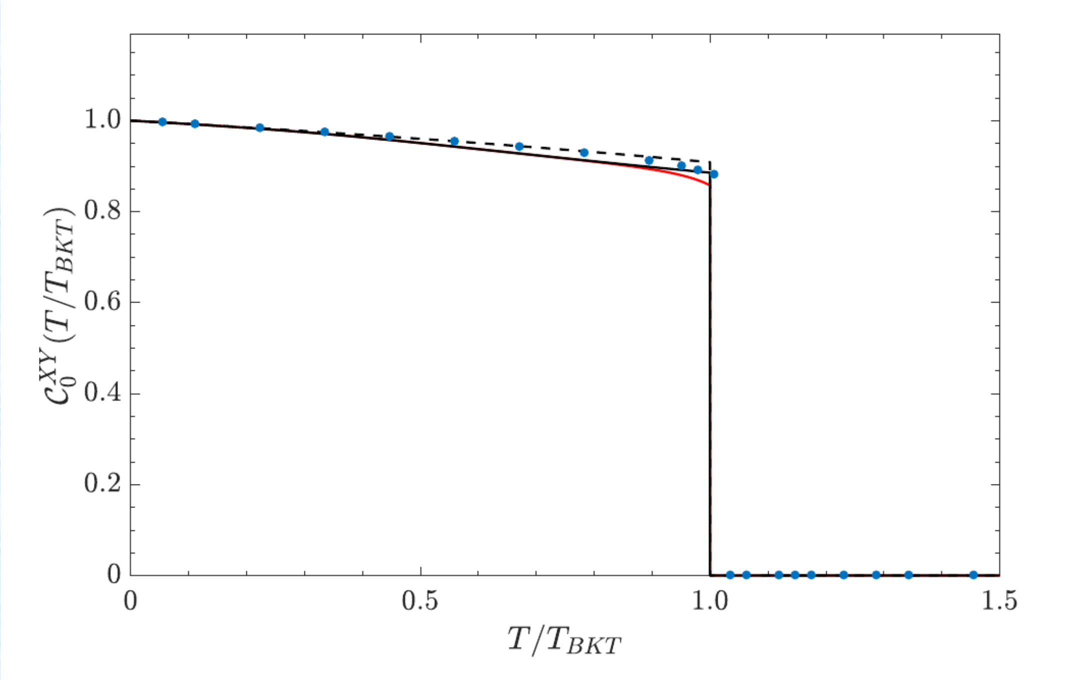
$$\eta_V(T/T_{\rm BKT}) = \mathcal{A}\,\frac{T}{T_{\rm BKT}} + \frac{\pi^2}{2}\,\frac{e^{-\mathcal{B}\,T_{\rm BKT}/T}}{\left(-1+\mathcal{D}\frac{T_{\rm BKT}}{T}\right)}$$

$\mathcal{A} \approx 0.215$, $\mathcal{B} \approx 7.304$, and $\mathcal{D} \approx 1.162$

At low temperatures

$$\mathcal{C}_0^V(T/T_{\rm BKT}) \simeq 1 - \frac{1}{2}\left(\frac{T}{2\,\pi\,T_{\rm BKT}}\,\frac{1}{0.74}\right)$$

ODLRO in 2D: XY model



$$H = -J \sum_{i,j} \cos(\phi_i - \phi_j)$$

$$\eta_{XY} = \frac{k_B T}{2\pi J_s(T)}$$

[data for stiffness from
I. Maccari, L. Benfatto, and
C. Castellani, PRB (2017)]

$$\eta_{XY} = -\frac{1}{\pi}\ln\left[\frac{I_1\left(\frac{J}{k_BT}\right)}{I_0\left(\frac{J}{k_BT}\right)}\right] + \frac{\pi^2}{2} e^{(\pi^2/2)\{\ln[I_1(\frac{J}{k_BT})/I_0(\frac{J}{k_BT})]\}^{-1}} \times \left\{-1 + \frac{\pi}{\pi + 4\ln\left[I_1\left(\frac{J}{k_BT}\right)\right] - 4\ln\left[I_0\left(\frac{J}{k_BT}\right)\right]}\right\}.$$

[A. Colcelli, N. Defenu, G. Mussardo, and A. Trombettoni, PRB (2020)]

ODLRO in 2D: Bose gas (I)

$$\rho(r) \sim \frac{1}{r^{m^2 k_B T/2\pi\hbar^2 \rho_s}}$$

$$\rho_s = \frac{2m^2 k_B T}{\hbar^2 \pi} f(X)$$

$$X = \frac{\hbar^2(\mu - \mu_c)}{m k_B T U}$$

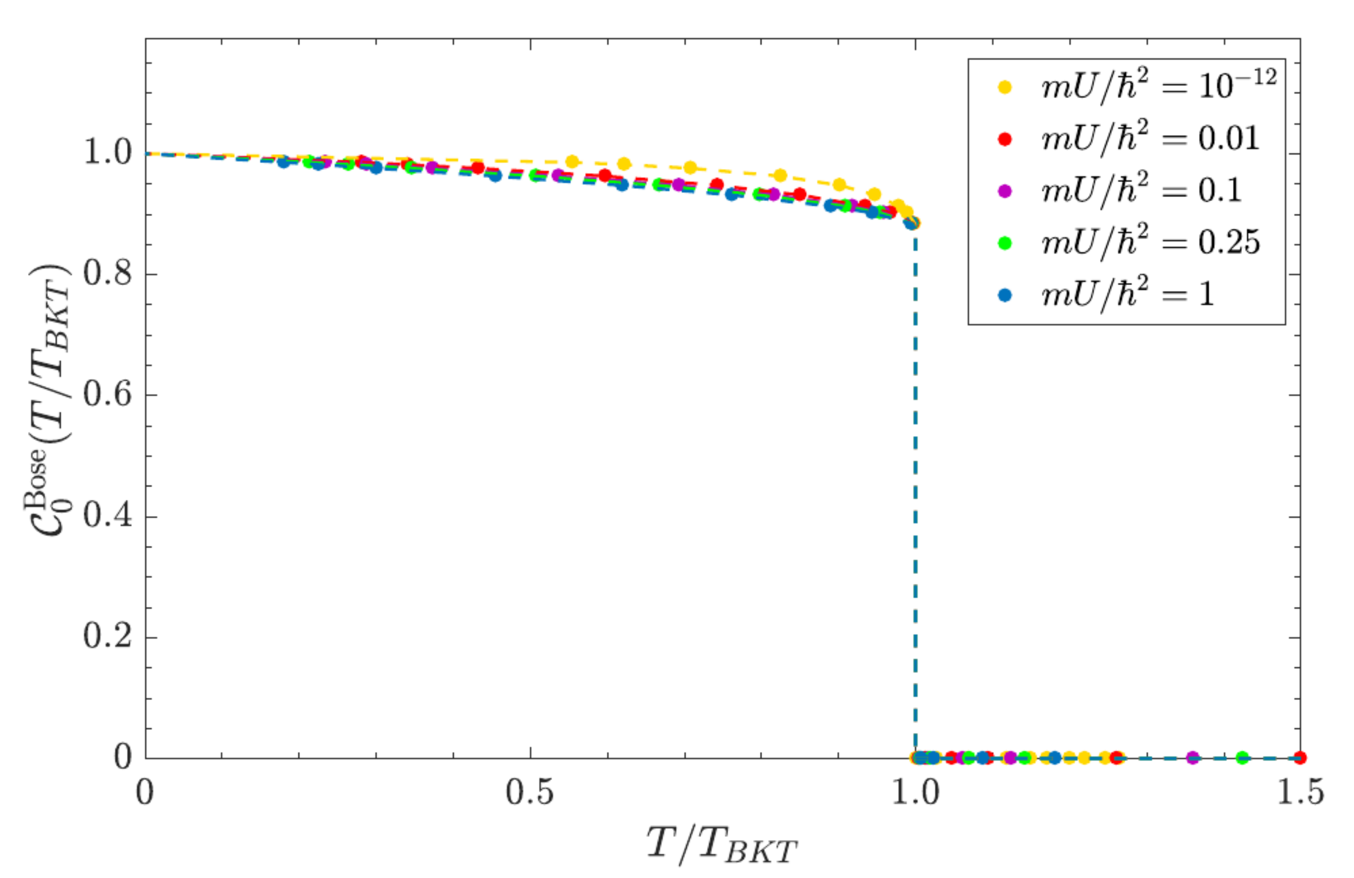
$$\frac{T}{T_{\mathrm{BKT}}}(X) = \frac{1}{1 + 2\pi\lambda(X)/\ln(\hbar^2\xi/mU)}$$

$$\xi = 380 \pm 3$$

f(X) and λ(X) dimensionless universal functions

[N. Prokof’ev and B. Svistunov, PRA (2002)]

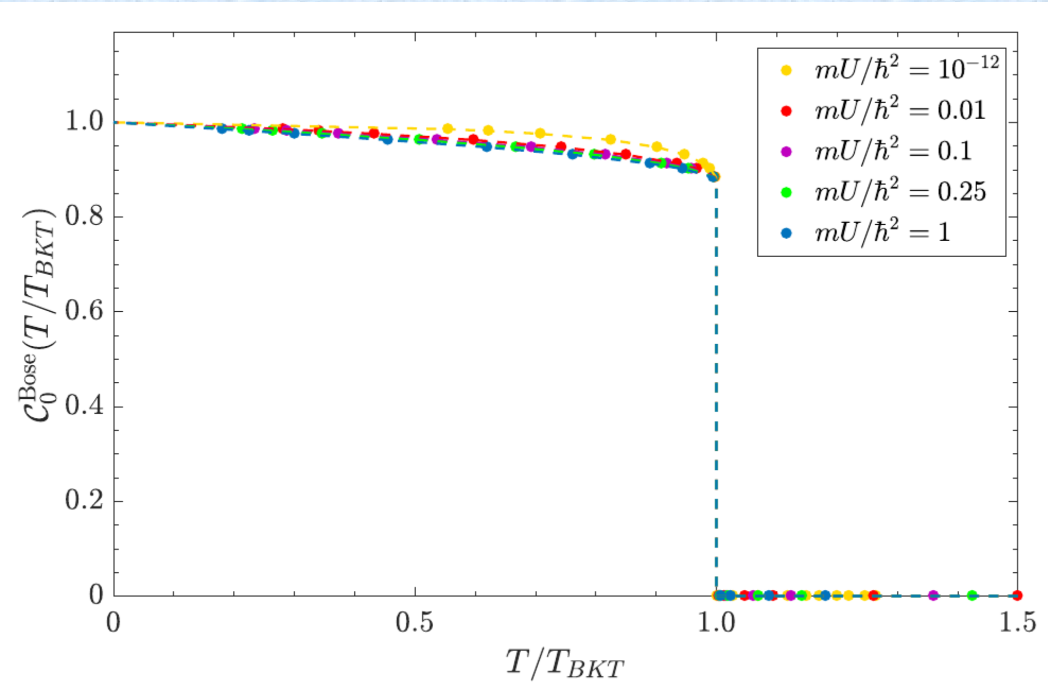
ODLRO in 2D: Bose gas (II)



$$\mathcal{C}_0^{\rm Bose}(X) = 1 - \frac{1}{8f(X)}$$

"double jump" for extremely small interactions: increasing T towards T_{BKT}, before → 0.912, then → 7/8 and then → 0

ODLRO in 2D: Bose gas (III)



At low T:

$$\mathcal{C}_0^{\rm Bose(0)}(T/T_{\rm BKT}) \simeq 1 + \frac{1}{2}\Bigg[1 - \ln(2\xi) - \ln\left(\frac{\hbar^2\xi}{mU}\right)\left(\frac{T_{\rm BKT}}{T} - 1\right)\Bigg]^{-}$$

Equating the low T expressions for the 2D XY gas and for the 2D Bose gas:

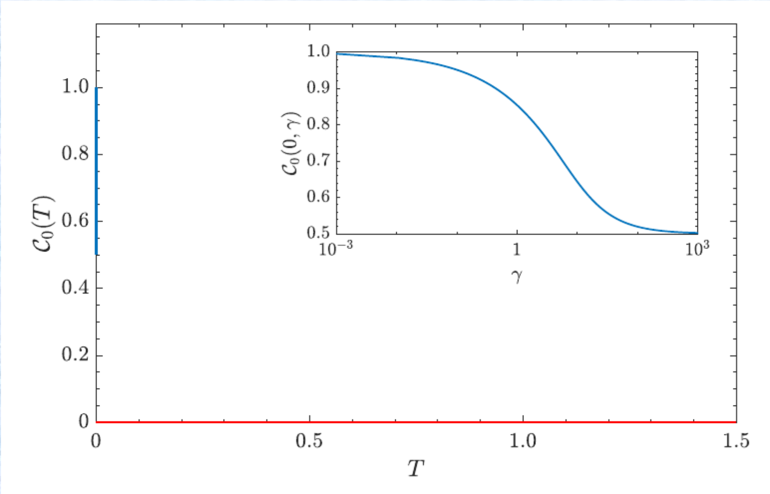
$$\xi = \frac{1}{2}e^{1+(\pi/2)(\mathcal{T}-1)} \qquad \mathcal{T} \equiv 4J/k_B T_{\rm BKT}^{(XY)}$$

Since for the XY model it is $k_B T_{\rm BKT}^{(XY)}/J = 0.893 \pm 0.001$

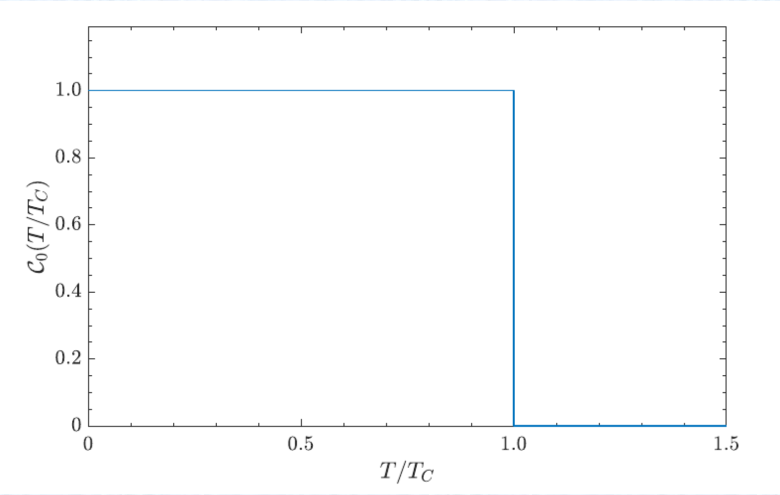
we get the approximate but reasonable value $\xi = 321 \pm 3$

Recapitulating:

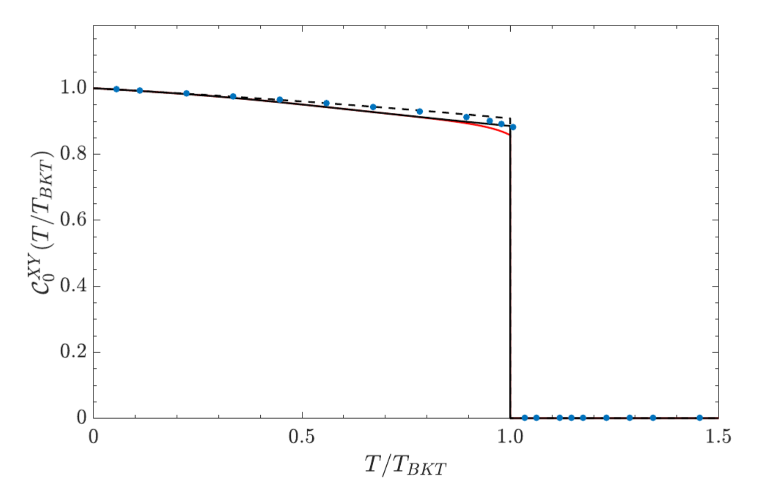
[A. Colcelli, N. Defenu, G. Mussardo, and A. Trombettoni, PRB (2020)]



1D



3D



No quasi-fragmentation

2D

Outline

- Reminder on off-diagonal long-range order (ODLRO)

- ODLRO order for 1D quantum systems

  (including effects of external potentials and 1D anyons)

- **ODLRO in 2D: quasi-ODLRO at finite temperature** (including coupled bilayer 2D systems)

- ODLRO & quasi-ODLRO with long-range interactions → BKT transition with long-range couplings

Higher-order quasi-ODLRO: bilayer XY model

Two coupled 2D XY models:

$$\mathcal{H} = -J\sum_{\langle ij\rangle} \cos(\phi_i - \phi_j) - J\sum_{\langle ij\rangle} \cos(\psi_i - \psi_j) + \\ - K\sum_i \cos(\phi_i - \psi_i),$$

Let define

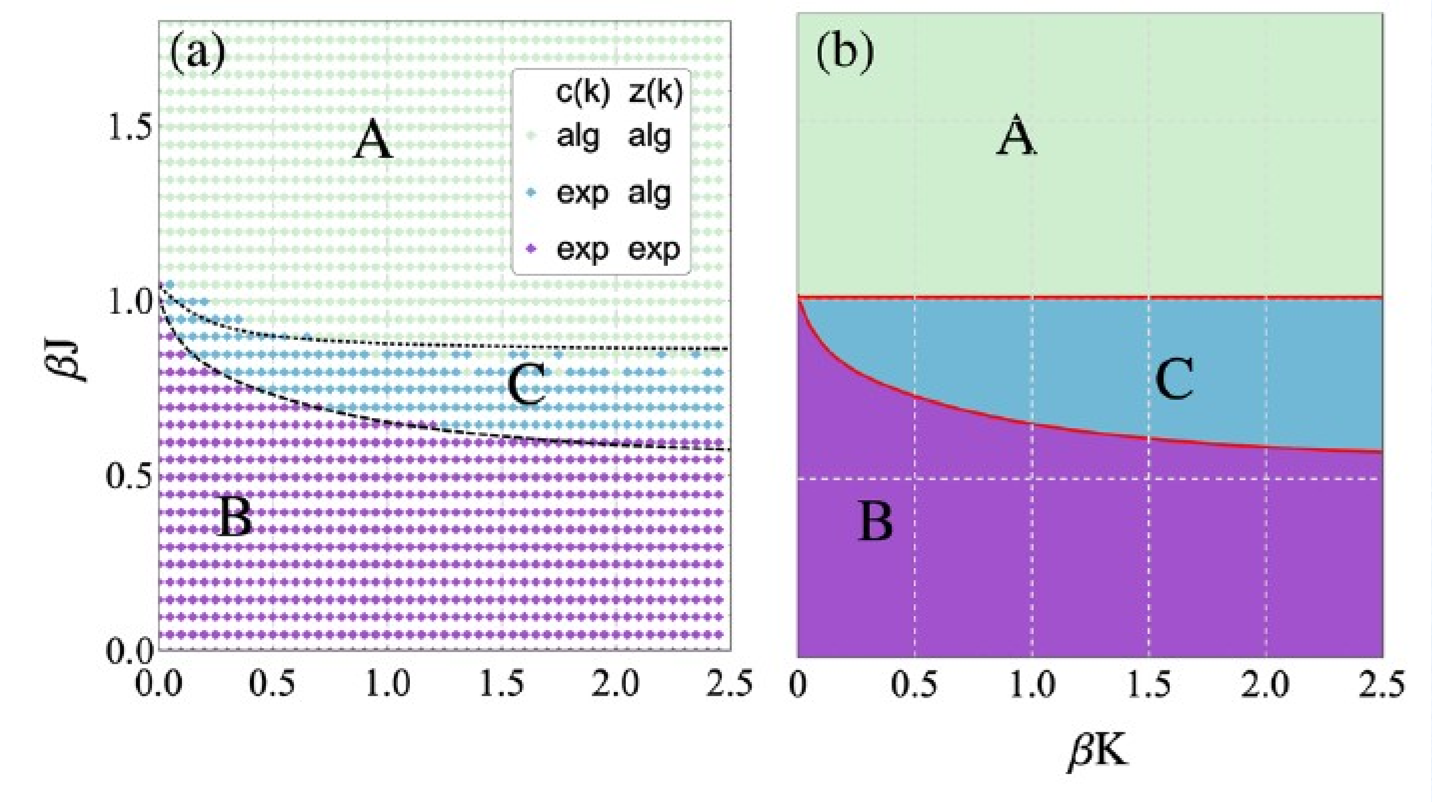
$$c_\uparrow(k) = \sum_{|i-j|=k} \exp(i\phi_i - i\phi_j),$$

$$c_\downarrow(k) = \sum_{|i-j|=k} \exp(i\psi_i - i\psi_j),$$

$$z(k) = \sum_{|i-j|=k} \exp(i\phi_i + i\psi_i - i\phi_j - i\psi_j)$$

and then determine the average <...>: one has the one-body density matrix (c) and the density matrix for the pairs (z)

Bilayer XY model



C is a "paired" BKT phase having quasi-ODLRO for pairs

Right: calculation relying on a FRG approach presented in [N. Defenu, A. Trombettoni, I. Nandori and T. Enss, PRB (2017)]

[G. Bighin, N. Defenu, L. Salasnich, and A. Trombettoni, PRL (2019)]

Outline

- ➢ **Reminder on off-diagonal long-range order (ODLRO)**

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  (including effects of external potentials and 1D anyons)

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How to “interpolate” between different dimensionality ODLRO and quasi-ODLRO?

How to “interpolate” between different dimensionality ODLRO and quasi-ODLRO?

Use long-range interactions, which effectively change the dimensionality

2D XY model with long-range couplings

$$\beta H = \frac{1}{2}\sum_{\mathbf{i},\mathbf{j}} J_{|\mathbf{i}-\mathbf{j}|}\left[1-\cos(\theta_{\mathbf{j}}-\theta_{\mathbf{i}})\right]$$

$$J_{|\mathbf{i}-\mathbf{j}|} \sim \frac{g}{|\mathbf{i}-\mathbf{j}|^{2+\sigma}}$$ for large distances

Reminder on long-range systems

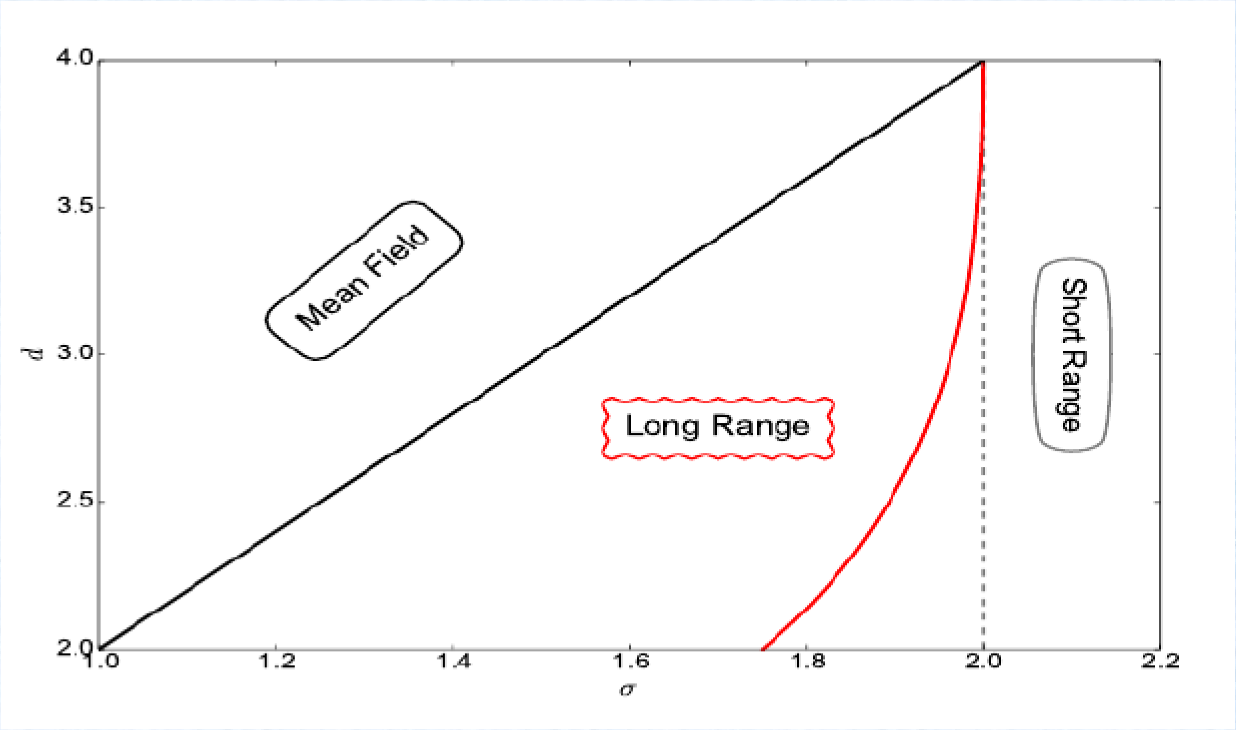
$$\frac{1}{p^{2-\eta_{\rm sr}}} \quad \text{vs} \quad \frac{1}{p^{\sigma}}$$

Sak criterion [J. Sak, PRB (1973)]:

$\sigma > 2-\eta_{\rm sr}$ → system behaves as its short-range counterpart (at criticality)

$\sigma < 2-\eta_{\rm sr}$ → genuine long-range behaviour

Reminder on long-range systems (II)



Phase diagram valid for O(n) models with couplings of the form $1/r^{d+\sigma}$

[recent reviews: N. Defenu, A. Codello, S. Ruffo, and A. Trombettoni, JPA (2020); N. Defenu et al., arXiv (2021)]

However this does not apply to d=2 and n=2...

2D long-range XY model

Effective description:

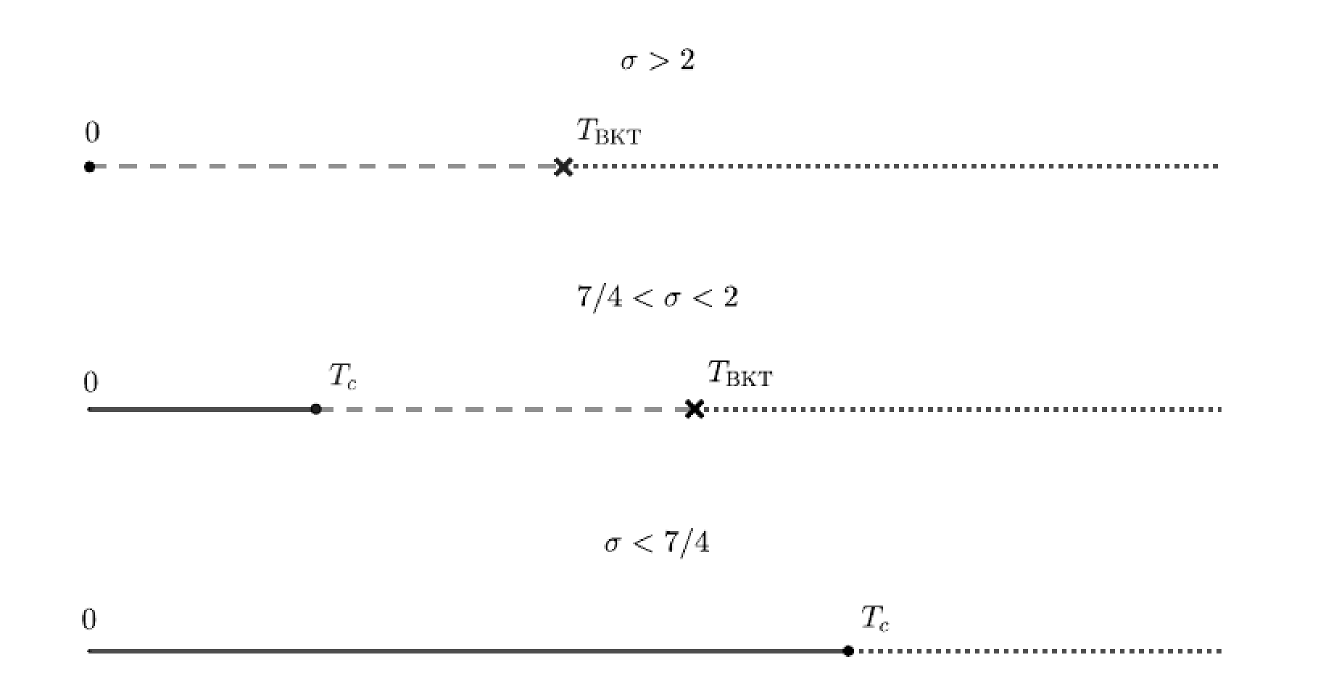
$$S[\theta] = \frac{J}{2}\int d^2x |\nabla\theta|^2 + S_{\mathrm{L}R}$$

$$S_{\mathrm{L}R} = -\frac{g}{2\gamma_{2,\sigma}}\int d^2x (\cos\theta\,\nabla^\sigma\cos\theta + \sin\theta\,\nabla^\sigma\sin\theta)$$

$$\gamma_{2,\sigma} = 2^\sigma\Gamma\left(\tfrac{1+\sigma}{2}\right)\pi^{-1}\left|\Gamma\left(-\tfrac{\sigma}{2}\right)\right|^{-1}$$

Studying the flow equations →

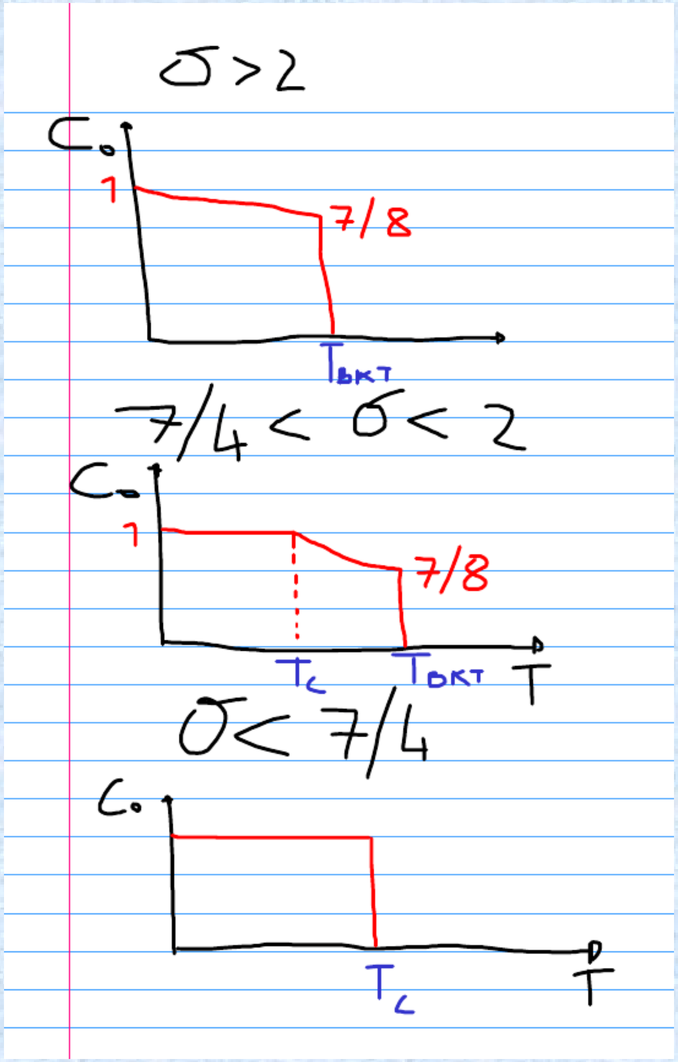
Phase diagram and ODLRO properties



Dashed: quasi-ODLRO / BKT phase
Dotted: no quasi-ODLRO / disordered phase
Solid: ODLRO / magnetized phase

[G. Giachetti, N. Defenu, S. Ruffo, and A. Trombettoni, PRL (2021); PRB (2022)]

The ODLRO diagram



Conclusions

→ Quantified deviations from ODLRO in 1D and 2D systems

→ Effect of statistics: interpolation between bosonic ODLRO and fermionic behavior

→ Effect of trapping potentials: exponent of ODLRO not changed in 1D

→ Higher-order quasi-ODLRO in bilayer models

→ Phase diagram and ODLRO/quasi-ODLRO properties of BKT phases with long-range interactions

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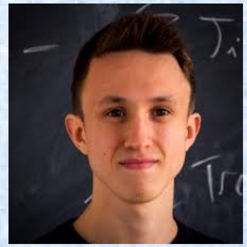

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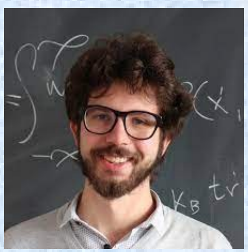

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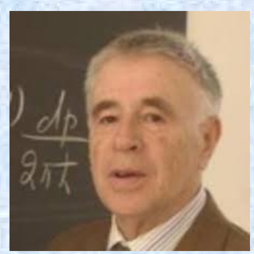

Thank you!