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Quasi-classical limit of the geometric Langlands correspondence

1. Quasiclassical limit
2. Hecke functors
3. Quantized Hitchin
4. Quasiclassical Hecke eigensheaves

} $PGL(n) = G$, ${}^LG = SL(n)$

~~5.~~ Other groups

① C = Riemann surface

$\hbar$-Langlands correspondence $\hbar \in \mathbb{C}$

Given a ${}^LG$-local system → ${}^LG$-bundle with $\hbar$-connection; stack of G-bundles on C

$\mathcal{E}$ on C ⇝ $Aut_{\mathcal{E}}$ on Bun_C

$\mathcal{E} = (E, \nabla)$

$D_\hbar$-module with Hecke eigenproperty

Definition: $\hbar$-connection $\nabla : E \to E \otimes \Omega$

$$\nabla(fs) = f\nabla s + \hbar s \otimes df$$

$$\text{"}\nabla = \hbar \tfrac{d}{dz} + A(z)\text{"}$$

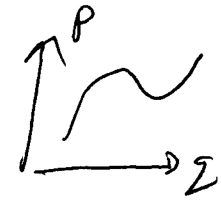
$D_\hbar$-operators:

Example $D_\hbar$-modules on $\mathbb{A}^1 \ni q$ ⟷ module $\mathbb{C}\langle p, q\rangle$, $p = \frac{d}{dq}$, $[p,q] = \hbar$

<u>Now</u> $\hbar \in \mathbb{C}[[\hbar]]$

<u>Example</u>: M $\mathbb{C}\langle p,q\rangle$, $[p,q]=\hbar$, $\hbar \in \mathbb{C}[[\hbar]]$

$\hbar=0$: $M/\hbar M$ $\mathbb{C}[p,q]$ - module



$= T^*\mathbb{A}^1$

$$\underbrace{\overset{\shortparallel}{M_0}}_{\text{a vector bundle on } \{(q, f(q))\}}$$

$f(q) = p$

On M_0, we have operator $\partial = \dfrac{p-f(q)}{\hbar}$

$$[\partial, q] = 1$$

(2) Hecke functors

$$\begin{array}{ccc} & \mathcal{H}ecke \ (= \mathcal{H}ecke^1) & \\ & {\scriptstyle p} \swarrow \quad \searrow {\scriptstyle q} & \\ Bun_G & & Bun_G \times C \end{array}$$

$\{(L, L', \varphi : L \to L'$: det φ has a simple zero at $c \in C\}$

$$\begin{array}{ccc} & \swarrow \quad \searrow & \\ L & & (L', c) \end{array}$$

$$q_* p^* At_{\mathcal{E}} = At_{\mathcal{E}} \boxtimes \mathcal{E}$$

~~Hecke~~ Hecke_c $c \in C$

$$\begin{array}{ccc} & \mathrm{Hecke}_c & \\ {}^{p}\swarrow & & \searrow^{q} \\ \mathrm{Bun} & & \mathrm{Bun} \end{array}$$

$$q^* p^* At_{\mathcal{E}} \simeq At_{\mathcal{E}} \otimes \mathbb{C}^n.$$

Makes sense for any $\hbar$

<u>For $\hbar = 0$</u>

$$D_0 \mathrm{Bun}_G = \mathcal{O}_{T^* \mathrm{Bun}_G} \; ; \; T^* \mathrm{Bun}_G = \mathrm{Higgs}_G$$

$$\mathrm{Higgs}_G = \{ (L, A : L \to L \otimes \Omega) \}$$

$$\mathrm{Higgs}_G \ni (L, A : L \to L \otimes \Omega)$$

$$\downarrow$$

$(\tilde{C} \subset T^*C, \ell \text{ on } \tilde{C})$: if $\tilde{C}$ is smooth, ℓ is a line bundle

Hitchin fibration

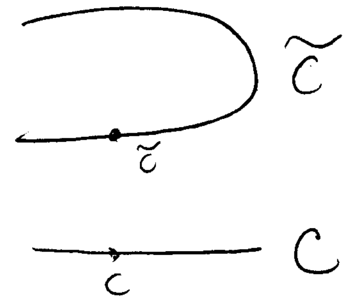
$$\begin{array}{ccccccc} & & \mathrm{Higgs}_G & & \mathrm{Higgs}_G^{sm} & & \\ & & \downarrow & & \downarrow & & \\ \mathbb{C}^k & \simeq & \mathrm{SCurves} & \supset & \mathrm{SCurves} & \ni & \tilde{C} \\ & & \Psi & & & & \\ & & \tilde{C} \subset T^*X & & \tilde{C} \text{ is smooth} & & \end{array}$$

$$\mathrm{Pic}\, \tilde{C} / \mathrm{Pic}\, C$$

Applying Hecke$_C$ to M on Higgs$_G$ gives

$$\mathbb{H}_C M$$

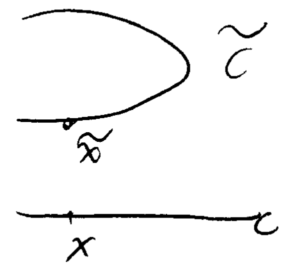
$$(\mathbb{H}_C M)_{(\tilde{C},\ell)} = \bigoplus_{\substack{\tilde{c}\in\tilde{C} \\ \tilde{c} \text{ goes to } c}} M_{(\tilde{C},\,\ell(-\tilde{c}))}$$



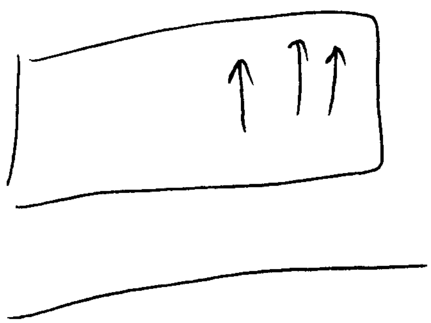
notation change

$\mathbb{H}_x M \qquad x\in C$

$$(\mathbb{H}_x M)_{(\tilde{C},\ell)} = \bigoplus_{\substack{\tilde{x}\in\tilde{C} \\ \tilde{x} \text{ goes to } x}} M_{(\tilde{C},\,\ell(-\tilde{x}))}$$



Hitch



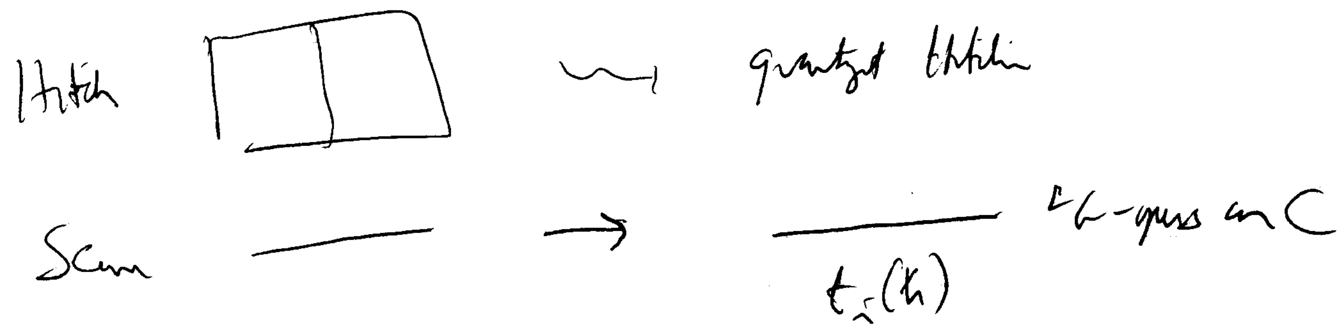
Scomer

③ Quantized Hitchin (Beilinson–Drinfeld)

Thm $\exists$ $t_1,\dots,t_k \in \mathcal{O}_{Higgs} = D_0 Bun_G$

$\rightsquigarrow$ $t_1(\hbar),\dots,t_k(\hbar) \in D_\hbar Bun_G \qquad [t_i(\hbar),t_j(\hbar)]=0$

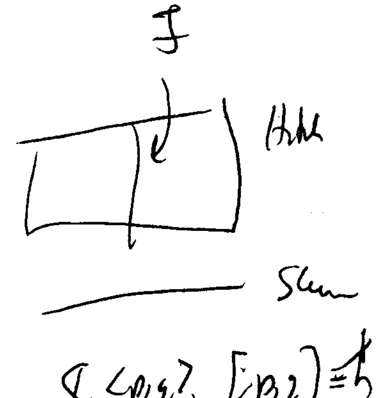
$\hbar\in\mathbb{C}$



Th (BD) $\mathcal{F} = D_{Bun_G} / (t_1^{(1)} - \lambda_1, \dots, t_k^{(k)} - \lambda_k)$ for any $\lambda_i \in \mathbb{C}$

is a Hecke eigen D-module

Prop Th stays true for $D_\hbar$-modules



④ Consider $\mathcal{F}$ for $\hbar \in \mathbb{C}[[\hbar]] \ni \lambda_1, \dots, \lambda_k$

Take any rank one local system ℓ on supp$(\mathcal{F})$

Claim $\mathcal{F} \otimes_{\mathbb{C}} \ell$ is a Hecke eigensheaf

Thm ① Suppose M is a $D_\hbar$-Bun module (top. free over $\mathbb{C}[[\hbar]]$)

s.t. $M/\hbar M$ is a line bundle on a smooth Hitchin fiber.

(ignore connected components)

Then M has a Hecke eigen property.

② Moduli space of M as in ① exists.

③ M ⟶ Hecke eigenvalue is an iso

$$\left\{ \begin{matrix} \text{moduli} \\ \text{from ②} \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{moduli space of bundles with } \hbar\text{-connection} \\ \text{whose spectral curve (at } \hbar = 0\text{) is smooth} \end{matrix} \right\}$$

$$LocSys_{\hbar}^{sm}(C, {}^L G), \ \hbar \in \mathbb{C}[[\hbar]]$$

④ Equivalence of derived categories