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# D-branes, III

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1. Coherent sheaves
2. Quivers
3. Matrix factorizations

$\mathbb{C}^3/\mathbb{Z}_3$ and blowup $\mathcal{O}_{\mathbb{P}^2}(-3)$

$$D(X) = \frac{D(\text{gr-}S)}{T}$$

$T$ is subset gen. by modules ann. by $(x,y,z)$

$= D(\text{mod-}$  $)$

Compact CY - Elliptic curve $X$

$x^3+y^3+z^3$ in $\mathbb{P}^2$

Enforce $f = x^3+y^3+z^3 = 0$

A <u>matrix factorization of $f$</u> consists of a pair of $M\times M$ matrices $\alpha, \beta$ s.t.

$$\alpha \cdot \beta = \beta \cdot \alpha = f \cdot \mathrm{id}_{M\times M}$$

Let $A = \dfrac{k[x,y,z]}{(x^3+y^3+z^3)}$, homog. coord. ring of $X$

Sheaves on $X$ are given by $A$-modules

$$D(X) = \frac{D(\text{gr-}A)}{T}$$

E.g. $M = \dfrac{A}{(x,y,z)}$

$$\cdots \to A(-2)^{\oplus 3} \oplus A(-3) \xrightarrow{\begin{pmatrix} 0 & -z & -y & x^2 \\ -z & 0 & x & y^2 \\ y & x & 0 & z^2 \end{pmatrix}} A(-1)^{\oplus 3} \xrightarrow{(x\ y\ z)} A \longrightarrow M \to 0$$

Eisenbud:
$$\begin{pmatrix} x^2 & -y^2 & z^2 & 0 \\ -y & -x & 0 & z^2 \\ z & 0 & -x & y^2 \\ 0 & z & y & x^2 \end{pmatrix} \text{ and } \begin{pmatrix} x & -y^2 & z^2 & 0 \\ -y & -x^2 & 0 & z^2 \\ z & 0 & -x^2 & y^2 \\ 0 & z & y & x \end{pmatrix}$$

$$= (x^3+y^3+z^3)\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$M$ is trivial in $D(X)$, but gives an interesting matrix fact.

$A$ has a finite free resolution given trivial matrix fact.

Avramov-Grayson: algorithm for Ext in Macaulay 2 - computer alg. package

Given an $A$-module, where $A$ is assoc. to hypersurface or complete intersection, define a new ring $S$ by going to ambient space and adding a new variable $p$, and an extra grading.

View $M$ as an ~~$S$-module (ignoring $p$ action for now) and write a free $S$-resolution.~~

$S = k[p, x, y, z]$ , $k[x, y, z]$

Write a free $k[x,y,z]$ resolution

$$d_c : C^\bullet \to C^\bullet$$

$$F_2 \to F_1 \to F_0 \to$$

$$(F_2 \oplus F_1 \oplus F_0) \to (F_2 \oplus F_1 \oplus F_0)$$

$C^\bullet$ is a sum of free $k[x,y,z]$-modules.

Then define $\quad d_0 = d_C$

$$\{d_0, d_n\} = \begin{cases} -f \cdot 1_C & \text{if } n = 1 \\ -\sum\limits_{\substack{\alpha+\beta=n \\ \alpha,\beta\neq 0}} d_\alpha d_\beta & \text{otherwise} \end{cases}$$

$$\| $$

$$d_0 d_n + d_n d_0$$

Define $d = \sum p^n d_n$

as a map $d : C^{\cdot} \to C^{\cdot}$

One can easily prove that $d^2 = -pf\ \mathrm{id}$

e.g. for the case $M = \frac{A}{(x,y,z)}$

$$d = \begin{pmatrix} 0 & -px^2 & -py^2 & -pz^2 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & py^2 & pz^2 & 0 & 0 \\ y & 0 & 0 & 0 & -px^2 & 0 & pz^2 & 0 \\ & & & \vdots & & & & \end{pmatrix}$$

Can always split $C^{\cdot}$ into even and odd parts

$$d = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$$

$$\Rightarrow \quad \alpha\beta = \beta\alpha = -\mathrm{id}\, f\, p$$

This yields an equivalence between $D(gr\text{-}A)$ and a category

$$D\,Gr\,S(W)$$

where $W = pf$

Objects in $DGrS(W)$ are graded matrix factorization of $W = -pf$

Morphisms are maps modulo homotopy

$$\begin{tikzcd} C_0 \arrow[r, bend left, "c_0"] \arrow[d, "f_0"'] \arrow[dr, dashed, "h_0" near end] & C_1 \arrow[l, bend left, "c_1"] \arrow[d, "f_1"] \arrow[dl, dashed, "h_1"' near start] \\ D_0 \arrow[r, bend left, "d_0"] & D_1 \arrow[l, bend left, "d_1"] \end{tikzcd}$$

null homotopic map has
$$f_0 = d_1 h_0 + h_1 c_0$$
$$f_1 = d_0 h_1 + h_0 c_1$$

$$D(X) = \frac{D(\mathrm{gr}\text{-}A)}{T} = \frac{DGrS(W)}{T}$$

One can also show that...

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In the resolved phase we have $\mathcal{O}_{\mathbb{P}^2}(-3)$

$T$ is ann by powers of $(x,y,z)$.

Other phase is $\mathbb{C}^3/\mathbb{Z}_3$

$T$ is ann by powers of $p$.

$$A = k[x,y,z]/(f) \qquad - \otimes k[p]$$

Dividing by T is the same thing as setting $p=1$ in orbifold phase.

In this phase, $D(X) = D\,Gr\,B(f)$

$B = k[x,y,z]$

$f = x^3+y^3+z^3$

L.G. orbifold phase

Kontsevich
Kapustin + Li
H. H. P.
- Orlov

Given an element of $D(X)$, what is the associated matrix fact?

e.g. a point, e.g. $x+y=0$, $z=0$ on $x^3+y^3+z^3=0$

$$S \xrightarrow{\begin{smallmatrix} -z \\ x+y \end{smallmatrix}} S(1) \oplus S(1) \xrightarrow{x+y,\, z} S(2) \longrightarrow \mathcal{O}_p(2) \to 0$$

$$C^\bullet = S \oplus S(1) \oplus S(1) \oplus S(2)$$

This corresponds to
$$\begin{pmatrix} -(x+y) & z^2 \\ z & x^2-xy+y^2 \end{pmatrix} \begin{pmatrix} -x^2+xy-y^2 & z^2 \\ z & x+y \end{pmatrix}$$

What is $\mathcal{O}_X(3)$?

$$0 \to S \xrightarrow{\times f} \underbrace{S(3)}_{\text{not in my basic set}} \longrightarrow \mathcal{O}_X(3) \to 0$$

need to write in terms of $S, S(1), S(2)$

$$0 \to S \to S(1)^{\oplus 3} \to S(2)^{\oplus 3} \to S(3) \to M(3) \to 0$$

$$M = \frac{S}{(x,y,z)}$$

$S \to S(3) \to M(3)$ is written in terms of $S, S(1), S(2)$

Now go to LG phase

$$S \to S(3) \to M(3)$$

In LG phase, $S \to S(3)$ corresponds to matrix fact $f = 1 \cdot f$, which is trivial.

So we are left with $M(3)$. This yields $4 \times 4$ matrix factorization we wrote down earlier.