Wild Hitchin Spaces & Kac-Moody Root Systems

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Aim Understand Hitchin moduli spaces in cases where the Higgs fields / connections have irregular singularities

[Basically poles of order $\geqslant 2$]

Basic questions

1. Algebraically integrable systems?

2. Complete hyperkähler metrics?

3. Correspondence mero. Higgs $\longleftrightarrow$ mero. conn's

4. Torelli type theorems?

5. When are the moduli spaces non-empty?
   [irregular Deligne - Simpson]

⋮

○ Geometric Langlands?

Original motivation ~ 1995

Dubrovin's approach to 2d TQFT

- axiomatize deformations of Frobenius algebras

$\Rightarrow$ notion of Frobenius Manifold

Thm (Dubrovin)

Local moduli space of 2d semisimple Frobenius manifolds $\cong$ a moduli space of mero. connections on $\mathbb{P}^1$ with 1 simple pole & 1 pole of order 2

+ (& interesting symplectic braid group action)

Hitchin spaces (usual picture with punctures)

Choose
- complex reductive group $G = K_{\mathbb{C}}$
- smooth projective curve Σ
- distinct points $a_1, \dots, a_m \in \Sigma$
- conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_m \subset G$

(+ parabolic str.)

$$\Downarrow$$

Hyperkähler manifold $\mathcal{M}$

$\mathcal{M}_{Dol}$ $\qquad$ $\mathcal{M}_{DR}$ $\qquad$ $\mathcal{M}_{Betti}$

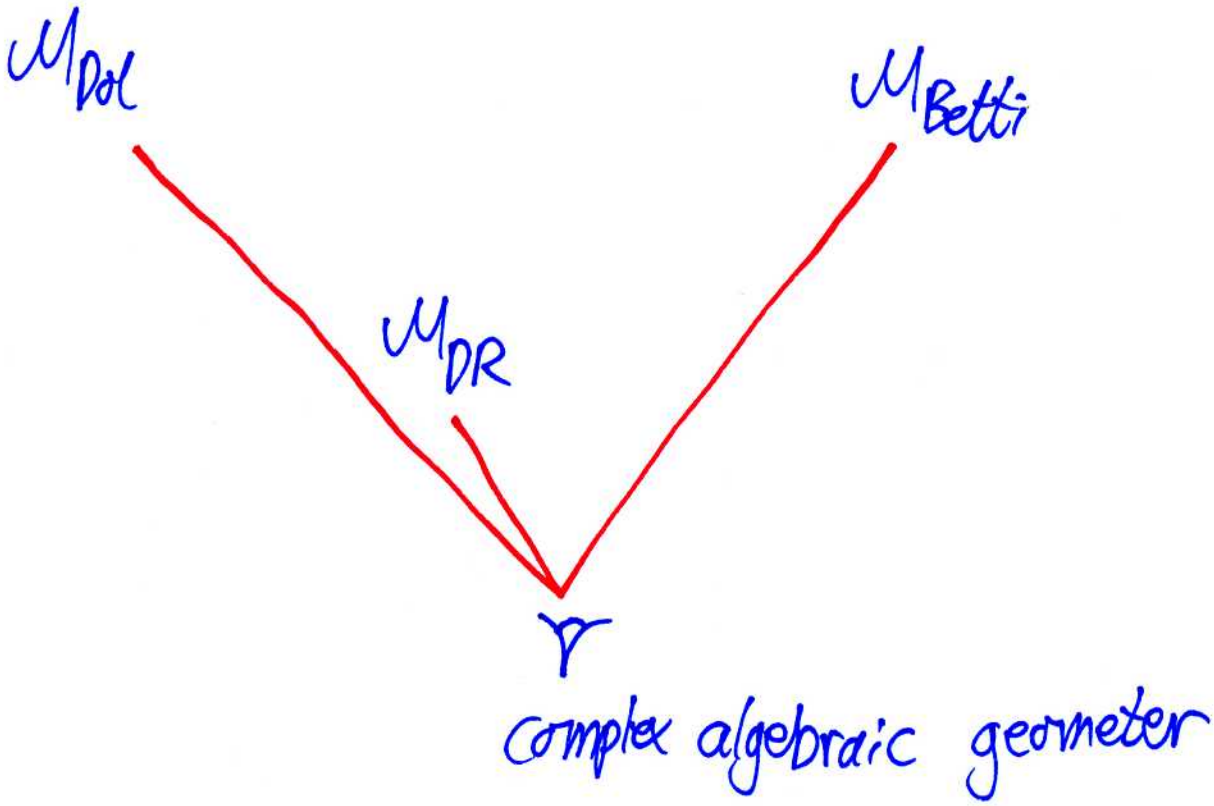
Hitchin, Donaldson, Corlette, Simpson, Nakajima, ...

$\mathcal{M}_{Dol}$

$\mathcal{M}_{DR} \underset{\text{RH isom.}}{\cong} \mathcal{M}_{Betti}$

complex analytic geometer

two spaces!



three spaces, two of which are very close (deformation)

Eg ("E_8 ALG space") GL_6, $\mathbb{P}^1$, $m=3$, special orbits

Blow up 9 points on the smooth locus of a cuspidal cubic in $\mathbb{P}^2$ & remove strict transform of cubic

1. Get $\mathcal{M}_{Dol}$ if 9 points sum to zero
   (elliptically fibred — Hitchin fibration)

2. Else get $\mathcal{M}_{DR}$ — (deformation)

3. $\mathcal{M}_{Betti}$ got by blowing up $\mathbb{P}^2$ in 8 points & removing a nodal $\mathbb{P}^1$ (Etingof-Oblomkov-Rains

Wild Hitchin Spaces

Basically fixing conjugacy class ~~of~~ of monodromy around puncture $\Leftrightarrow$ mero. connection with simple pole & residue in fixed adjoint orbit

$$\frac{A}{z}\,dz \qquad A \in \mathcal{O} \subset \mathfrak{g}$$
$$\exp(2\pi i \mathcal{O}) = \mathcal{C} \subset G$$

$\Leftrightarrow$ fixing $G[[z]]$ isom. class of connection

Generalization — allow higher order poles in fixed formal isom. class

$$\left(\frac{A_k}{z^k} + \frac{A_{k-1}}{z^{k-1}} + \dots + \frac{A_2}{z^2} + \frac{A_1}{z}\right)dz + \dots$$

Here:— assume $A_i \in \mathfrak{h}$ (Cartan sub alg. $\mathfrak{g}$)

— generic condition (e.g follows if $A_k \in \mathfrak{h}_{reg}$)

Similarly fix $G[[z]]$ orbit of principal part of Higgs fields (at each pole)

1

Thm (Bottacin, Markman) ~ 1993 & Beauville, Adams-Harnad-Hurtubise, Reiman-Semenov Tian Shansky, Adler-van Moerbeke if $g=0$

$\mathcal{M}_{Dol}$ is an algebraically completely integrable system

[Hitchin map - take remaining invariants of the Higgs field (those not already fixed]

$\mathcal{M}_{Betti}$ — can be described as space of certain representations of the "wild fundamental gp" of Martinet–Ramis (Tannakian viewpoint)

— or more directly via Stokes multipliers

(quasi Hamiltonian spaces generalizing the complex conjugacy classes $\mathcal{C} \subset G$)

[PB, Duke '07]

$\mathcal{M}_{dR}$ — complex symplectic structure obtained by extending the Atiyah–Bott approach

— locally indept of
- complex str. of Σ
- positions of $a_1, \dots, a_m$
- Irregular parts of formal types $A_k, \dots, A_2$

"Symplectic nature of the wild fundamental group"

(Cf. PB Adv. Math. '01)

"Isomonodromy is a symplectic connection"

(2) & (3) (GLn)

Thm (O. Biquard - PB)
2004

- Correspondence $\mathcal{M}_{Dol} \cong \mathcal{M}_{DR}$
  - map ← earlier by Sabbah
  - same "rotation" of eigenvalues / parb. weights as found by Simpson in simple pole case

- Complete hyperkahler metrics, if moduli spaces smooth
  - generic formal types ⇒ smoothness

Basic examples

Approximations $\mathcal{M}^*$	$\mathcal{M}$
① $\mathcal{O} /\!\!/_{\Lambda} H$ $\mathcal{O} \subset \mathfrak{g}^*$	$\mathcal{L} /\!\!/_{\Lambda} H$ $\mathcal{L} \subset G^*$ dual Poisson Lie gp
② $H \,{}_{\Lambda_2}\!\backslash\!\!\backslash T^*G /\!\!/_{\Lambda_1} H$	$H \,{}_{\Lambda_2}\!\backslash\!\!\backslash \mathcal{D} /\!\!/_{\Lambda_1} H$ $\mathcal{D} \subset (G \times G^*)^2$ Lu-Weinstein sympl. double groupoid
③ special ALE spaces e.g. A_{1-3}, D_4, E_{6-8} $\sim \widetilde{\mathbb{C}^2/\Gamma}$	Okamoto Painlevé spaces "2d Hitchin systems"

(4) Torelli type theorem?

Is the map: G, curve + points + formal types...

$$\downarrow$$

$$\mathcal{M}$$

injective?

(Yes) $g>1$, $m=0$, SL_n (Biswas-Gomez '01)

(No) In general:

Can do Fourier-Laplace / Nahm transform of mero. connections on $\mathbb{P}^1$

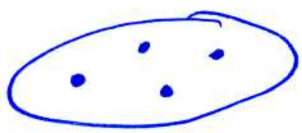
– changes pole orders & rk of G

(Hyperkahler isometry by S. Szabo '05)

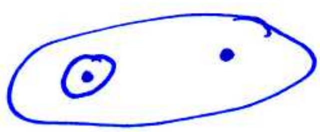
[Lots of examples when $\dim \mathcal{M} = 0$ — rigid differential equations]

Simple example (cf. Harnad's dual Lax pair for Painlevé V)

① $G = GL_2$, $m = 4$, $g = 0$, all simple poles



② $G = GL_3$, $m = 2$, $g = 0$, 1 simple pole
1 second order pole



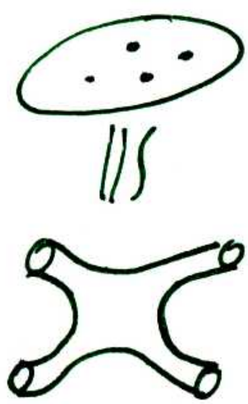
Viewpoint: Two "representations" / "realizations" of the same (abstract) hyperkahler moduli space $\mathcal{M}$

Approach

Attach a graph ("Dynkin diagram") to certain wild Hitchin systems & explain how to read off various realizations from the graph

Definition

A graph is "Hitchin" if it arises in this way from a (meromorphic) Hitchin moduli space

e.g.  4 simple poles, GL_2

$\longmapsto$ 

affine D_4 Dynkin diagram

Nakajima quiver var.

$$\mathcal{M} \hookleftarrow \mathcal{M}^*$$

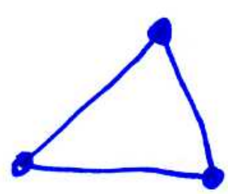
open subset where underlying bundle on $\mathbb{P}^1$ holomorphically trivial

"Known" cases of Hitchin graphs

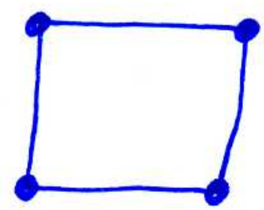
① Okamoto - Painlevé moduli spaces

$$\dim_{\mathbb{C}}(\mathcal{M}) = 2$$

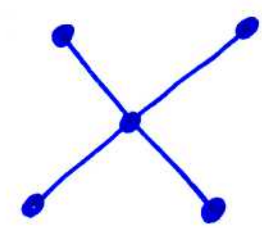
e.g



$\hat{A}_2$

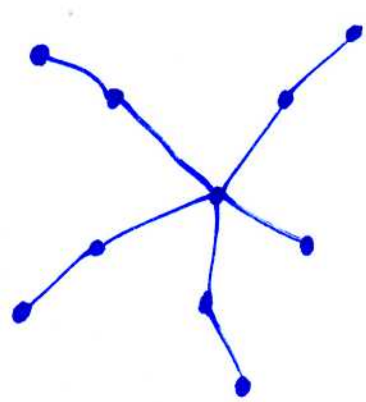


$\hat{A}_3$



$\hat{D}_4$

② Star-shaped graphs



$\mathcal{M}$ realized as connections with simple poles on $\mathbb{P}^1$

Pattern?

Let $I_1, \dots, I_k$ be finite sets $n_i = \# I_i$

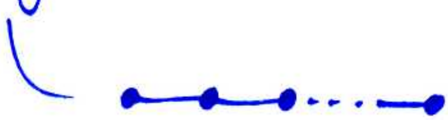
Let $\Gamma_c = \Gamma(n_1, \dots, n_k)$ be the

complete k-partite graph on the sets I_i

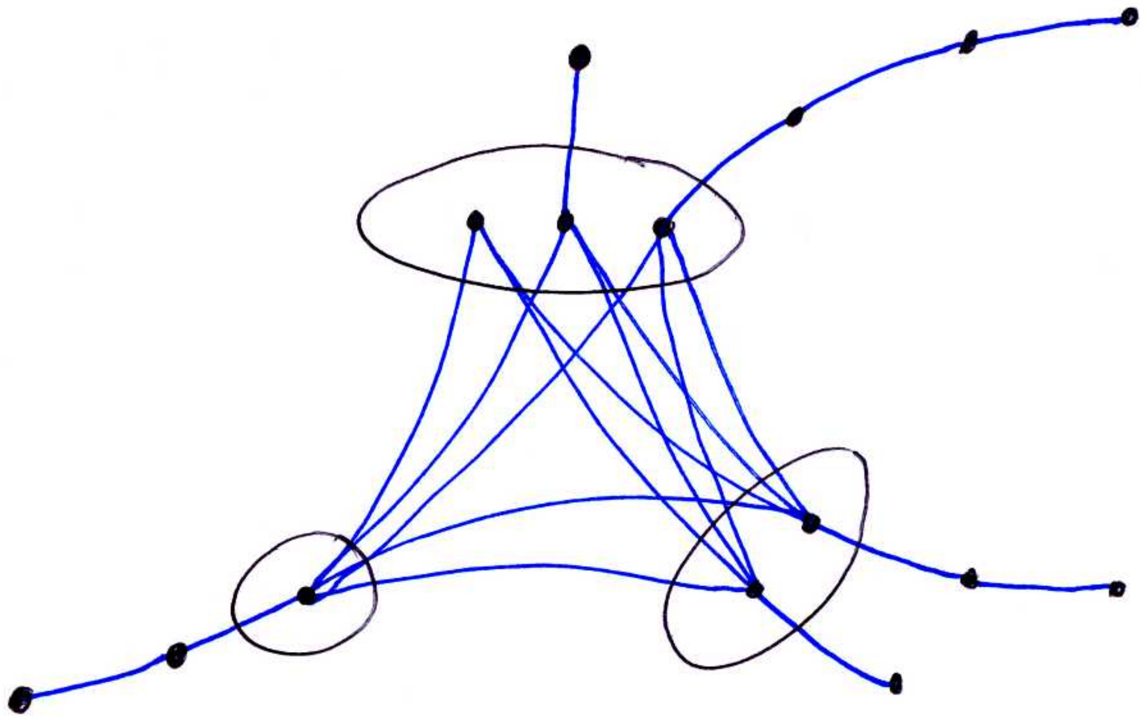
[with nodes $I = \sqcup I_i$ and an edge joining $e \in I_i$ to $f \in I_j$ iff $i \neq j$]

Theorem (PB 0806-1050)

- Γ_c is a Hitchin graph
- The same is true if we first glue an arbitrary "leg" onto each node of Γ_c



— i.e. we get a rich class of "Dynkin diagrams"



Example "complete tripartite graph with legs"

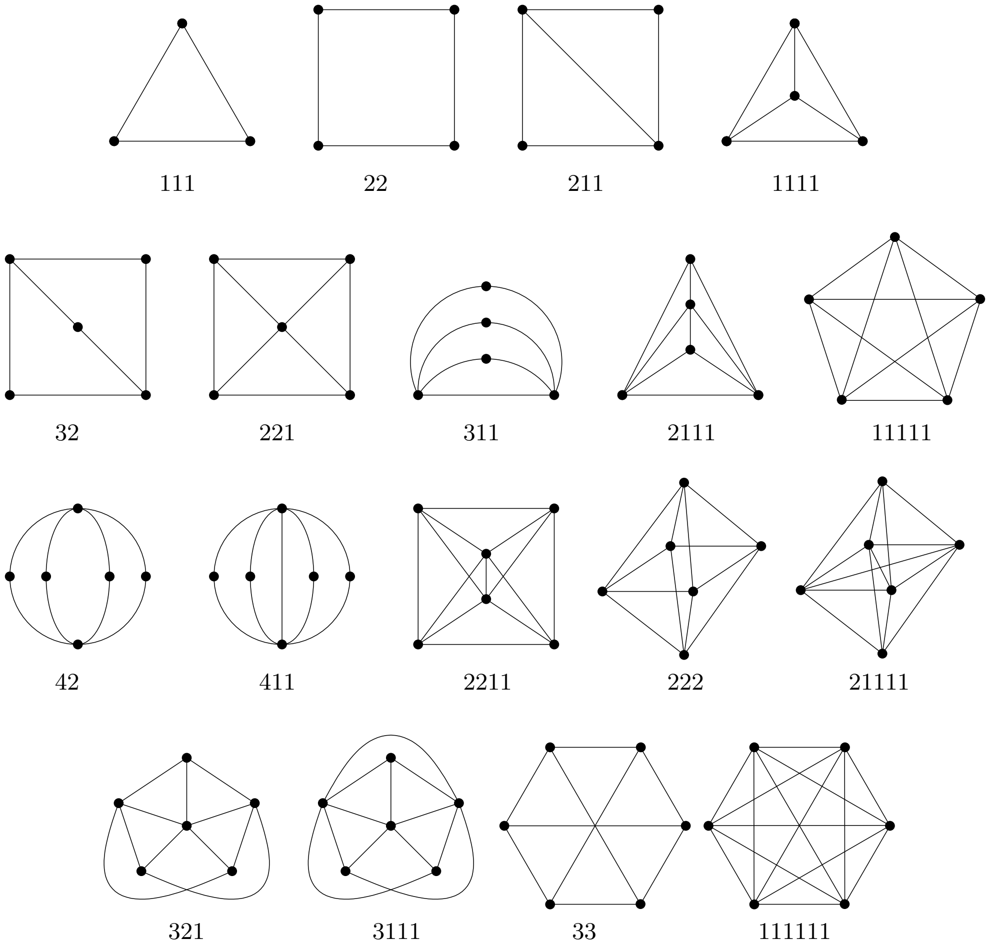


FIGURE 1. Graphs from partitions of $N \leq 6$
(omitting the stars $\Gamma(n, 1)$ and the totally disconnected graphs $\Gamma(n)$)

$k+1$ ways to 'read' complete k-partite graph w. legs

$$\Gamma_C = \Gamma(n_1, \dots, n_k) = \Gamma(I_1, \dots, I_k)$$

Auxiliary data

- dimension $d_i \in \mathbb{N}$ $\forall$ nodes i
- real no.s $\alpha_i, \beta_i, \gamma_i$ $\forall$ nodes i
- distinct complex no.s $a_1, \dots, a_k$
- complex no.s b_i $\forall$ nodes of Γ_C $(i \in I_j)$
  ($b_i \neq b_{i'}$ if i, i' in same part)

Principal reading

- $G = GL_N \qquad N = \sum_{\text{nodes of } \Gamma_c} d_i$
- 1 pole of order 3 & no others on $\mathbb{P}^1$

Set $N_j = \sum_{i \in I_j} d_i \qquad j = 1, .., k \quad$ so $\quad N = \sum N_j$

- Formal type $\left(\frac{A_3}{z^3} + \frac{A_2}{z^2} + \frac{A_1}{z}\right) dz$

  A_3 evals a_j (multiplicities N_j)

  A_2 evals $b_i \quad i \in I_j$ in a_j eigenspace A_3
  (multiplicities d_i)

Let $H = \mathrm{Stab}(A_2, A_3) \cong \prod GL_{d_i}(\mathbb{C})$

A_1 is in fixed adjoint orbit of H, determined by the legs

[~ Reduces to simple pole case in each simultaneous eigenspace of A_2, A_3]

Other k readings

Choose one of the k 'parts' – say I_1

1. Delete this part
   & obtain (as above) formal type at pole of order 3

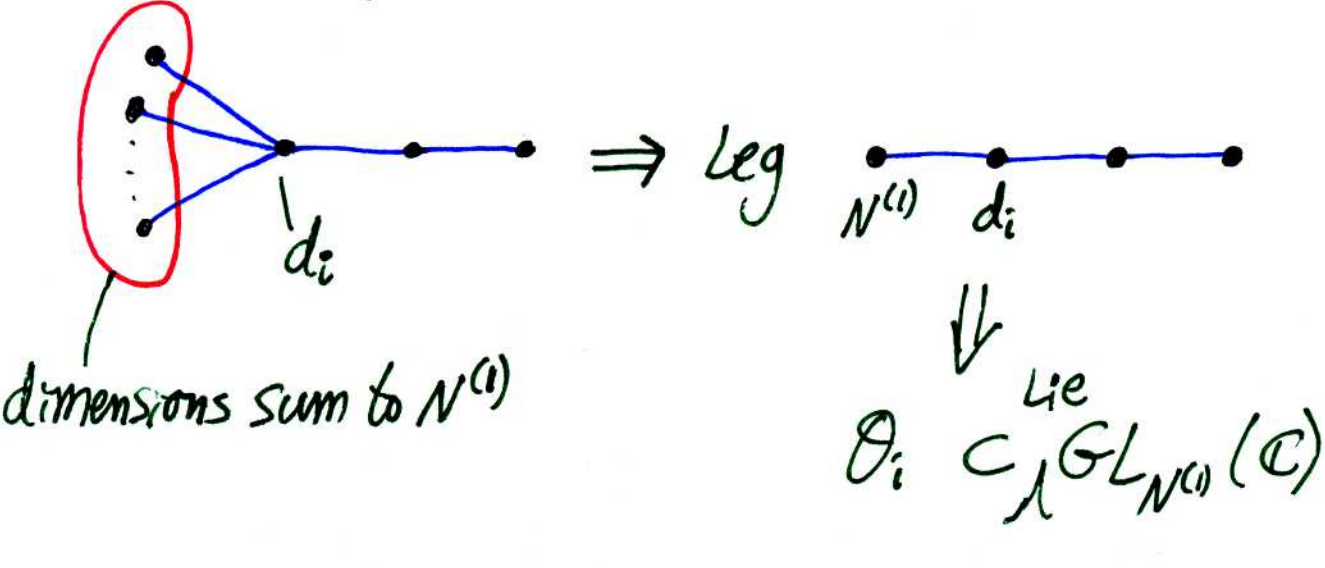
$$G = GL_{N^{(1)}}(\mathbb{C}) \quad , \quad N^{(1)} = N - N_1$$

2. Put a simple pole at $z = 1/b_i \quad \forall\, i \in I_1$

   $n_1 = \# I_1$ simple poles

   Need to fix adjoint orbits of residues at the simple poles.

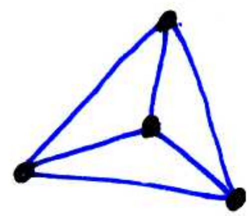
   near $i \in I_1$ graph looks like



$$
\begin{tikzcd}
\text{(Certain) input data } G, \text{ formal type, divisor}, \ldots \arrow[d] \\
\mathcal{M} \arrow[d] \\
\text{Graph } \Gamma + \text{auxiliary data} \arrow[u, bend right=60] \arrow[u, bend right=70] \arrow[u, bend right=80, "\text{Different readings of } \Gamma"'] \arrow[uu, phantom]
\end{tikzcd}
$$

In particular we are saying the open part $\mathcal{M}^*$ of $\mathcal{M}_{dR}$ (or $\mathcal{M}_{Dol}$) where the underlying holom. bdle is trivial, is complex symplectically isomorphic to a Nakajima quiver variety attached to Γ

Tetrahedron



$\Gamma(1,1,1,1)$

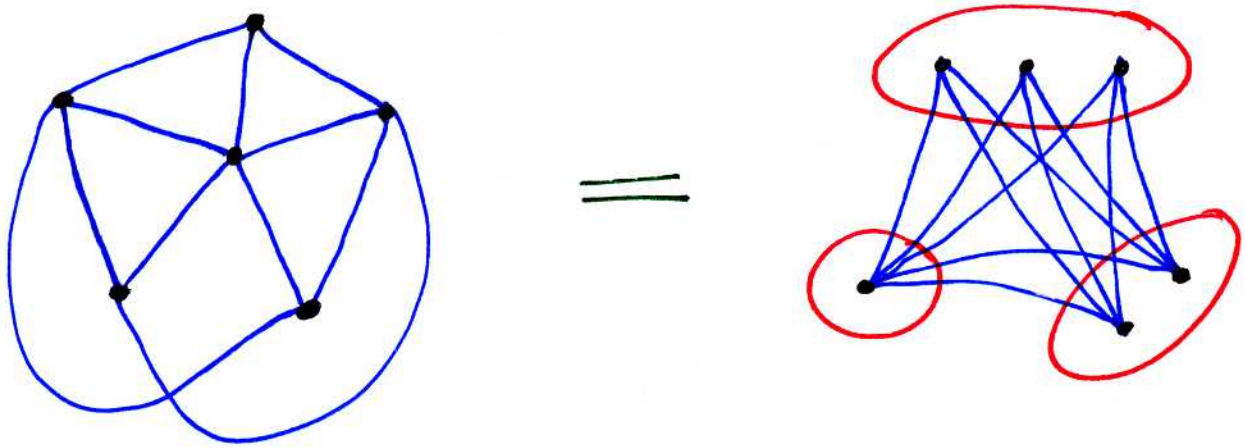
dimensions 1, 2, 3, 4 say

$\dim_{\mathbb{C}} \mathcal{M} = 12$, four-partite graph

$\Rightarrow$ 5 readings:

rank of vector bundles	pole orders
10	3
9	3+1
8	3+1
7	3+1
6	3+1

$\Gamma(3,2,1)$



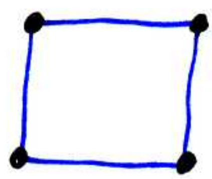
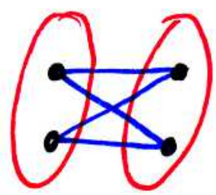
$d_i = 1 \quad \forall\, i$ say

Rank	poles
6	3
5	3+1
4	3+1+1
3	3+1+1+1

Bipartite cases — 3 readings

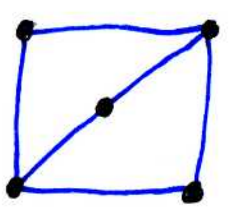
Compare: Jimbo – Miwa – Mori – Sato (Bose gas, Painlevé 5)
with & Harnad's duality

In 2 readings only one eigenvalue of A_3
~ can shift it to zero ⇒ order 2 pole

e.g. 1  =  $\hat{A}_3$

$d_i = 1 \ \forall i$

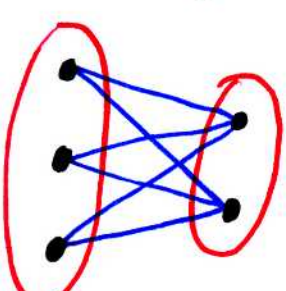
Rank	poles
4	3
2	2+1+1
2	2+1+1

2 

$d_i = 1 \ \forall i$

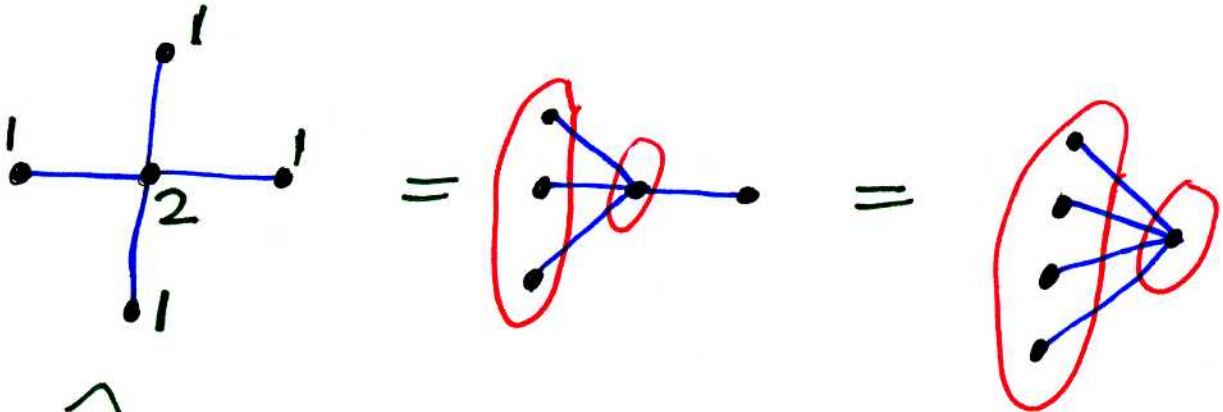
Rank	poles
5	3
3	2+1+1
2	2+1+1

$\Gamma(3,2)$



Stars: $\Gamma(n,1)$ (with legs)

e.g



$\widehat{D}_4$

Rank	poles
6	3
4	2+1
3	2+1
2	1+1+1+1

[Additive / $\mathcal{M}^*$ version of isom. $2 \cong 4$ for $\Gamma(n,1)$ is complexification of "Gelfand MacPherson duality" ~ dilogarithm]

Extensions

1. Higher order pole ✓ (need multiple edges)

2. $\geqslant 2$ irregular singularities

   $\mathcal{M}^*$ not a quiver variety

   $\rightsquigarrow$ more general picture (~bows)

(e.g. Rk 2, 2+2 $\mathcal{M}^* \cong D_2$ ALF space)

Ulterior motive for attaching a graph to $\mathcal{M}$:

$\rightsquigarrow$ Get a Kac-Moody root system

1. Get precise criteria for existence of stable connections (phrased in terms of roots)

- extending work of Crawley-Boevey on the Deligne-Simpson problem

[$\exists$? stable connections in simple pole case / $\mathbb{P}^1$]

(star shaped graphs here)

2. Get "reflection functors" - action of KM Weyl group on auxiliary data

Claim These induce more isomorphisms between $\mathcal{M}$'s

[typical orbits are infinite]

Given graph Γ, nodes I, $n = \# I$

- Cartan matrix $C = 2 - A$ $(n \times n)$

  $A_{ij} = \#$ edges node $i \leftrightarrow$ node j

- Root lattice $\mathbb{Z}^I = \bigoplus_{i \in I} \mathbb{Z} \varepsilon_i$ has bil. form $(,)$

  $$(\varepsilon_i, \varepsilon_j) = C_{ij}$$

- Weyl group $W \circlearrowleft \mathbb{Z}^I$ generated by $\{s_i\}_{i \in I}$

  $$s_i(x) = x - (x, \varepsilon_i)\varepsilon_i$$

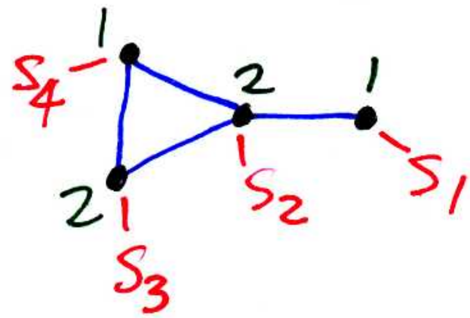
  (& dual reflections $r_i \circlearrowleft \mathbb{C}^I$ s.t $r_i(\lambda) \cdot s_i(x) = \lambda \cdot x$)

- Root system $\subset \mathbb{Z}^I$ (real & imaginary roots)

View dimension vector $\underset{\sim}{d} \in \mathbb{Z}^I$

Example of W action

A_2^{++}



Read e.g as
Rk 3, poles 3+1

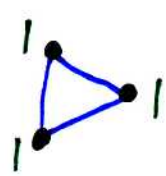
$\underset{\sim}{d} = (1,2,2,1)$

Here $W \underset{\text{index } 2}{\supset} W^+ \cong PSL_2(E)$ $\qquad E = \mathbb{Z}[\omega]$ (Eisenstein integers)

(cf. Feingold–Kleinschmidt–Nicolai '08)

1. Let $w = s_1 s_4 s_1 s_2 s_4 s_1 s_3 s_1$

   Compute $w^n(1,2,2,1) \rightsquigarrow$ Read as connections on bdles of rank $n^2+(n-1)+(n-2)^2$

2. $s_1 s_2 s_3 (1221) = (0111) \Rightarrow$ 

   So $\mathcal{M}^* \cong A_2$ ALE space $\quad (\dim_{\mathbb{C}} = 2)$