28 July 2008

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Conformal field theory realization of the geometric Langlands correspondence

Refs about questions asked last week

R. Schimmrigk, "The Langlands program and string modular K3 surfaces", hep-th/0603234

" A modularity test for elliptic mirror symmetry"
hep-th/0705.2427

About connections to classical Langlands.

Geometric Langlands and 2D CFT

X - smooth projective curve / $\mathbb{C}$

G - simple connected simply-connected Lie group / $\mathbb{C}$

${}^L G$ - Langlands dual gp.

Bun_G - moduli stack of G-bundles on X

Geometric Langlands Correspondence

$$\left\{ \begin{matrix} \text{flat } {}^L G\text{-bundle} \\ \text{on } X \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{Hecke eigensheaf} \\ \text{on } Bun_G \end{matrix} \right\}$$

$\mathcal{E} = (E, \nabla) \longmapsto \mathcal{F}_{\mathcal{E}}$ = $\mathcal{D}$-module on Bun_G

principal (hol.) ${}^L G$-bundle on X; ∇ - (hol) conn on E

$\mathcal{H}_{V,x}$ - Hecke functors
$V \in Rep\, {}^L G$, $x \in X$

$\mathcal{H}_{V,x}(\mathcal{F}_{\mathcal{E}}) \cong V \otimes \mathcal{F}_{\mathcal{E}}$

$$\mathcal{H}_{V,x} : \mathcal{D}_{Bun_G}\text{-mod} \longrightarrow \mathcal{D}_{Bun_G}\text{-mod}$$

$$\begin{tikzcd} & \mathcal{H}ecke_{V,x} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ Bun_G & & Bun_G \end{tikzcd}$$

$\mathcal{H}ecke_{V,x} \leftarrow \mathcal{K}_{V,x}$ – $\mathcal{D}$-module ("kernel")

$\mathcal{F} \mapsto Bun_G$

$$\mathcal{H}_{V,x}(\mathcal{F}) := p_{2*}(p_1^*(\mathcal{F}) \otimes \mathcal{K}_{V,x})$$

→ intersection coh sheaves on the affine Grassmannian of G (Nadler)

<u>Example</u> $G = GL_2$, $V = \mathbb{C}^2$

($\mathcal{K}_{V,x}$ – trivial (omit it))

$$\mathcal{H}ecke_{\mathbb{C}^2,x} = \{\mathcal{M}, \mathcal{M}' \text{ – rk 2 v.b. on } X,\ \mathcal{M} \hookrightarrow \mathcal{M}' \mid \mathcal{M}'/\mathcal{M} \cong \mathbb{C}_x\}$$

↑ ↑ coherent sheaves

↗ skyscraper sheaf

$$\begin{tikzcd} & \mathcal{H}ecke_{\mathbb{C}^2,x} \arrow[dl] \arrow[dr] & \\ \mathcal{M}' \in Bun_{GL_2,d} & & Bun_{GL_2,d+1} \ni \mathcal{M} \end{tikzcd}$$

$Bun_{GL_2,d}$ = moduli stack of rank 2 bundles on X

$\mathcal{M} \subset \mathcal{M}'$

$$\boxed{\text{fiber over } \mathcal{M} \cong \mathbb{P}^1 = \mathbb{P}(\mathcal{M}_x)}$$

$$\begin{tikzcd} & \mathcal{H}ecke_{V,x} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathrm{Bun}_G & & X \times \mathrm{Bun}_G \end{tikzcd}$$

allow to vary

When x varies, $\mathcal{H}_V(\mathcal{F}_\mathcal{E}) \simeq V_\mathcal{E} \boxtimes \mathcal{F}_\mathcal{E}$

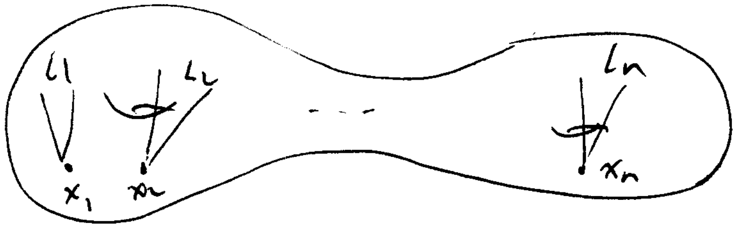
$$V_\mathcal{E} = \mathcal{E} \times_{{}^L G} V$$

$$\mathcal{H}_{V,x}(\mathcal{F}_\mathcal{E}) \cong V_{\mathcal{E}_x} \otimes \mathcal{F}_\mathcal{E}$$

CFT with $\hat{\mathfrak{g}}$-symmetry is a machine for constructing $\mathcal{D}$-modules on Bun_G. ("$\mathcal{D}$-modules of conformal blocks")

Beilinson–Drinfeld: use it (at the "critical level") to get Hecke eigensheaves

RCFT: WZW of $k \in \mathbb{Z}_+$-level



L_λ – integrable reps of $\hat{\mathfrak{g}}$ level k

$$\mathcal{C}(L_1, \dots, L_n) := \mathrm{Hom}_{\mathfrak{g}_{out}}(L_1 \otimes \dots \otimes L_n, \mathbb{C})$$

– f.d. vector space

acts

$$\mathfrak{g}_{out} := \mathfrak{g} \otimes \mathbb{C}[X \setminus \{x_1, \dots, x_n\}]$$

{cpx str on X, position of x_i} $\in \mathcal{M}_{g,n}$

dim - the same, so get a vector bundle $\mathcal{C}$ on $\mathcal{M}_{g,n}$ (extends to $\overline{\mathcal{M}_{g,n}}$).

Moreover, get a projectively flat connection on $\mathcal{C}$ (Friedan-Shenker)
Use Virasoro algebra ($T(z)$)
flat v.b. $\longrightarrow$ $\mathcal{D}$-module
projectively flat v.b. $\longrightarrow$ twisted $\mathcal{D}$-module

$\mathcal{D}_{\mathcal{L}}$-module, $\mathcal{L}$ - l.b. on $\mathcal{M}_{g,n}$

$\mathcal{L}$ - "$\det^{\otimes c}$"

Use $\hat{g}$-action to get a (twisted, twist is determined by the level k) $\mathcal{D}$-module on $\mathrm{Bun}_G \ni P$ - prin. G-bundle

$$\mathcal{C}_P(L_1, \dots, L_n) := \mathrm{Hom}_{g^P_{out}}(L_1 \otimes \dots \otimes L_n, \mathbb{C})$$

$$g^P_{out} := H^0(X \setminus \{x_1, \dots, x_n\}, g_P), \quad g_P = P \times_G g$$

Lie algebra (as was g_{out})

WZW model — get trivial bundle with trivial connection
$k \in \mathbb{Z}_+$ because L_i's are integrable (G acts on them)

But use this construction to get twisted $\mathcal{D}$-modules at other levels!

In general, $\dim \mathcal{C}_P(L_1, \dots, L_n)$ will jump along some loci in Bun_G.

Remark Back to WZW

$$C(\overset{\swarrow \text{vacuum}}{L_0}) = H^0(\mathrm{Bun}_G, \Theta^{\otimes k}), \ k \in \mathbb{Z}_+$$

$(n=1)$

Analogy: Borel-Weil-Bott

$V_\lambda = H^0(G/B, \mathcal{L}_\lambda)$ - finite-dim reps of G (integrable)

Many $\mathfrak{g}$-modules, not integrable.

BB) These more general $\mathfrak{g}$-mod. $\rightsquigarrow$ $\mathcal{D}$-modules on G/B

Fibers = space of (co)invariants

Apply this at $k = -h^\vee$ ($k = -n$ for $\mathfrak{g} = \mathfrak{sl}_n$)

Then Virasoro "disappears" so cannot get a $\mathcal{D}$-mod. on $\mathcal{M}_{g,n}$, but still get a twisted $\mathcal{D}$-module on Bun_G (twist: $K^{1/2}_{\mathrm{Bun}_G}$)

$$T(z) = \frac{1}{k+h^\vee} \cdot \frac{1}{2} \sum_a :J^a(z) J_a(z):$$

$k=-h^\vee$: $S_1(z) = \frac{1}{2} \sum_a :J^a(z) J_a(z):$ become central

$\deg S_1 = 1 + 1 = 2$

Moreover, have higher order Sugawaras

$S_i(z)$, $i = 1, \dots, l = \mathrm{rk}\,\mathfrak{g}$, $\deg S_i = d_i + 1$.

Look at the "center of the chiral algebra at the critical level",

$\mathfrak{z}(\hat{\mathfrak{g}})$ $\qquad \mathbb{C}\left[\partial_z^{n_i} S_i(z)\right]_{\substack{n_i \geq 0 \\ i = 1, \dots, l}}$

Thm (Feigin-Frenkel)

$$\mathfrak{z}(\hat{\mathfrak{g}}) \simeq \mathrm{Fun}\, \underbrace{\mathrm{Op}_{^LG}(D)}_{^LG\text{-opers on the disc}}$$

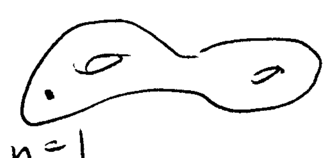
^LG-oper: $(\underbrace{E, \nabla}_{\text{flat bundle}}, E_B)$ ↖ reduction of E to $B \subset G$, Borel subgroup

$$\nabla = \partial_z + \begin{pmatrix} 1 & 1 & & * \\ 0 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

Ref: hep-th/0512172, Part III

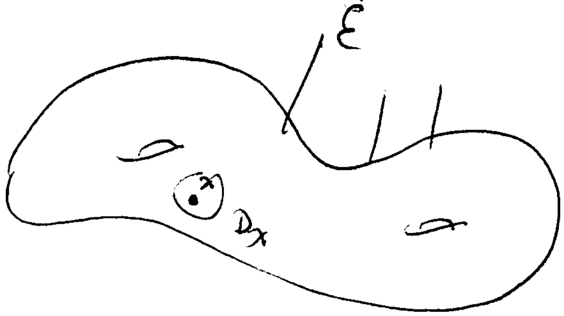
$\{\text{flat } {}^LG\text{-}\cancel{\text{bdle}}\text{ bundle on } X\} \longmapsto \mathcal{F}_E$ - $\mathcal{D}$-module on Bun_G

$\mathcal{E} \longrightarrow$ Rep of $\hat{\mathfrak{g}}$ ↗



$n = 1$

Suppose $\mathcal{E}$ has an oper structure (this means $E = E_0$, ∇-satisfying) conn. on E_0 then E_B is unique



$$\mathcal{E} \mapsto \chi$$
$$(E,\nabla) \qquad (E,\nabla,E_B)$$

$$\chi|_{D_x} = \chi_x = \{ \chi_i(z) \in \mathbb{C}[[z]] \}_{i=1,\dots,l}$$

$$\rightsquigarrow V_{\chi_x} := \mathrm{Vac}/(S_i(z) = \chi_i(z))$$

(B-D) The corresponding $\mathcal{D}$-module is the sought-after Hecke eigensheaf.