4 August 2008
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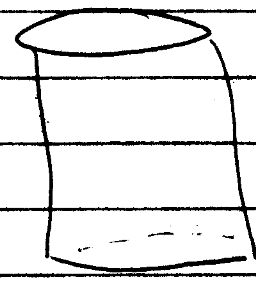
Branes and representations

w/ E. Witten

TQFT (cohomological)

Q_{BRST}



time

$M = N \times \mathbb{R}$

$$\rightsquigarrow \mathcal{H} = H^*_{Q_{BRST}}$$

Ex: GL twist of $\mathcal{N}=4$ SYM

quantum TQFT (Chern-Simons)

$\mathcal{H}$ = quantization of (M, ω)
symplectic m-fold

Q: Can a coh-TQFT give a Hilbert space $\mathcal{H}$, quantization of (M,ω)?

A: yes

$\mathbb{R} \times [0,1] \times N$

$\uparrow$ time

$\nwarrow \begin{cases} B_0 \text{ at } 0 \\ B_1 \text{ at } 1 \end{cases}$

$\rightsquigarrow \mathcal{H} = \mathrm{Hom}(B_0, B_1)$

$= \text{quantization of } (M, \omega)$

<u>Quantization</u>

$(M,\omega) \rightsquigarrow \mathcal{H}$

symplectic mfold

("phase space")

function $f(p,q) \rightsquigarrow$ operator on $\mathcal{H}$ $\mathcal{O}_f$

$$[\mathcal{O}_f, \mathcal{O}_g] = -i\hbar\,\mathcal{O}_{\{f,g\}} + \cdots$$

Ex: $M = \mathbb{R}^{2n}$

$(S^2, \omega) \circlearrowleft \mathrm{SDiff}(S^2)$

$\int_{S^2} \frac{\omega}{2\pi} = n \rightsquigarrow \mathcal{H} \circlearrowleft U(n)$

$\dim \mathcal{H} = n$

* Geometric quantization
* Deformation quantization
* "Brane quantization"

Geometric quantization:

(M, ω) Step 1: "prequantum line bundle" $L \to M$

Step 2: choose a polarization $\longleftarrow$!

Deformation quantization:

$A = C^\infty(M)$

$* : A \times A \to A[\hbar]$

$f * g = f \cdot g + i\hbar \partial_x f \partial_y g \, \omega^{-1} + \cdots$

Brane quantization

$$(M,\omega) \overset{\text{Step 1}}{\rightsquigarrow} (Y,\Omega)$$

complexification

$$\omega = \mathrm{Re}\,\Omega$$

Ex $M = S^2 : \; x^2+y^2+z^2=1$

$$(x,y,z) \in \mathbb{C}^3$$

$$\Omega = \frac{dx \wedge dy}{z}$$

Pick a "prequantum" line bundle $\mathcal{L} \to Y$
w/ curvature $\omega = \mathrm{Re}\,\Omega$

$$(M,\omega) \to (Y,\Omega,\mathcal{L}) \text{ or } (Y, I, \Omega, \mathcal{L})$$

Step 2 A-model of $(Y, \mathrm{Im}\,\Omega)$ with two branes

$\mathcal{B}'$ = Lagrangian A-brane

$\mathcal{B}_{cc}$ = Coisotropic A-brane

$$\mathcal{B}_{cc} \;\Big|\; \Sigma \;\Big|\; \mathcal{B}' \qquad \Rightarrow \quad \mathcal{H}$$

$\mathcal{B}'$ = Lagrangian A-brane supported on M

$\mathcal{B}_{cc}$ = coisotropic brane

$$(\omega_A^{-1} F)^2 = -\mathbb{1}$$

$$\Omega = \omega_J + i\,\omega_K$$

$$\omega = \omega_J|_M \qquad \Rightarrow \qquad \boxed{F = \omega_J}$$

$$\omega_K = \mathrm{Im}\,\Omega = \omega_A$$

$\mathcal{H}$ = quantization of (M, ω)

(cf. Aldi-Zaslow)

$$X(\sigma,\tau) = \underbrace{x + p\tau}_{\text{conjugate}} + \underbrace{\cdots}_{\text{non-zero modes}}$$

$$X(\sigma,\tau) = \overset{\Omega}{x} + \cdots$$

depends on $(Y, \mathcal{L}, \mathcal{L})$

$$\mathcal{H} \circlearrowleft \mathcal{A}$$

$$\mathcal{A} = \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc})$$

"Orbit method"

$M = \mathcal{O}_{\mathbb{R}}$ orbit of a real group $G_{\mathbb{R}}$

quantization of $M \rightsquigarrow \mathcal{H}$, representation of $G_{\mathbb{R}}$

(M, ω), $M = \mathcal{O}_{\mathbb{R}} \rightsquigarrow Y = \mathcal{O}_{\mathbb{C}}$ orbit of $G_{\mathbb{C}}$, complexification of $G_{\mathbb{R}}$

Ex $\mathcal{O}_{\mathbb{R}} = \{x^2+y^2+z^2=1\}$ of $G_{\mathbb{R}} = SU(2)$

$$\downarrow$$

$$Y = \{(x,y,z) \in \mathbb{C}^3 \mid x^2+y^2+z^2=1\}$$

$$= \mathcal{O}_{\mathbb{C}} \text{ of } G_{\mathbb{C}} = SL(2,\mathbb{C}).$$

$\Omega = \dfrac{dx \wedge dy}{z}$ is $G_{\mathbb{C}}$-invariant

$$A = \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) / \underset{\uparrow (x^2+y^2+z^2-1)}{\mathcal{I}}$$

~~$K_{\mathbb{R}}$ - invariant~~

$$\left\{ \text{representation of } G_{\mathbb{R}} \right\} \sim \left\{ A\text{-branes on } Y \right\}$$

$$\mathcal{H} = \mathrm{Hom}(\mathcal{B}', \mathcal{B}_{cc}) \longleftarrow \mathcal{B}'$$

$$\left\{ \begin{matrix} \text{admissible} \\ \text{representation of } G_{\mathbb{R}} \end{matrix} \right\} \sim \left\{ \begin{matrix} K_{\mathbb{R}}\text{-invariant} \\ A\text{-branes on } Y \end{matrix} \right\}$$

<u>B-B localization:</u>

A_K-model of Y

$\mathcal{B}'$ supported on M $\quad (A,B,A) \rightsquigarrow (B,A,A)$

$Y = T^*(G/T)$

<u>$\mathcal{B}_{cc}$</u> of type (A,B,A)

$\cancel{I = \omega_K^{-1} F = \omega_K^{-1} \omega_J}$

A_K-branes $\Leftrightarrow$ $\mathcal{D}$-mod(G/T)