Higher orders with Monte Carlo methods

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I: **Multileg NLO calculations**

II: **Local subtraction terms**

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V: **Results**

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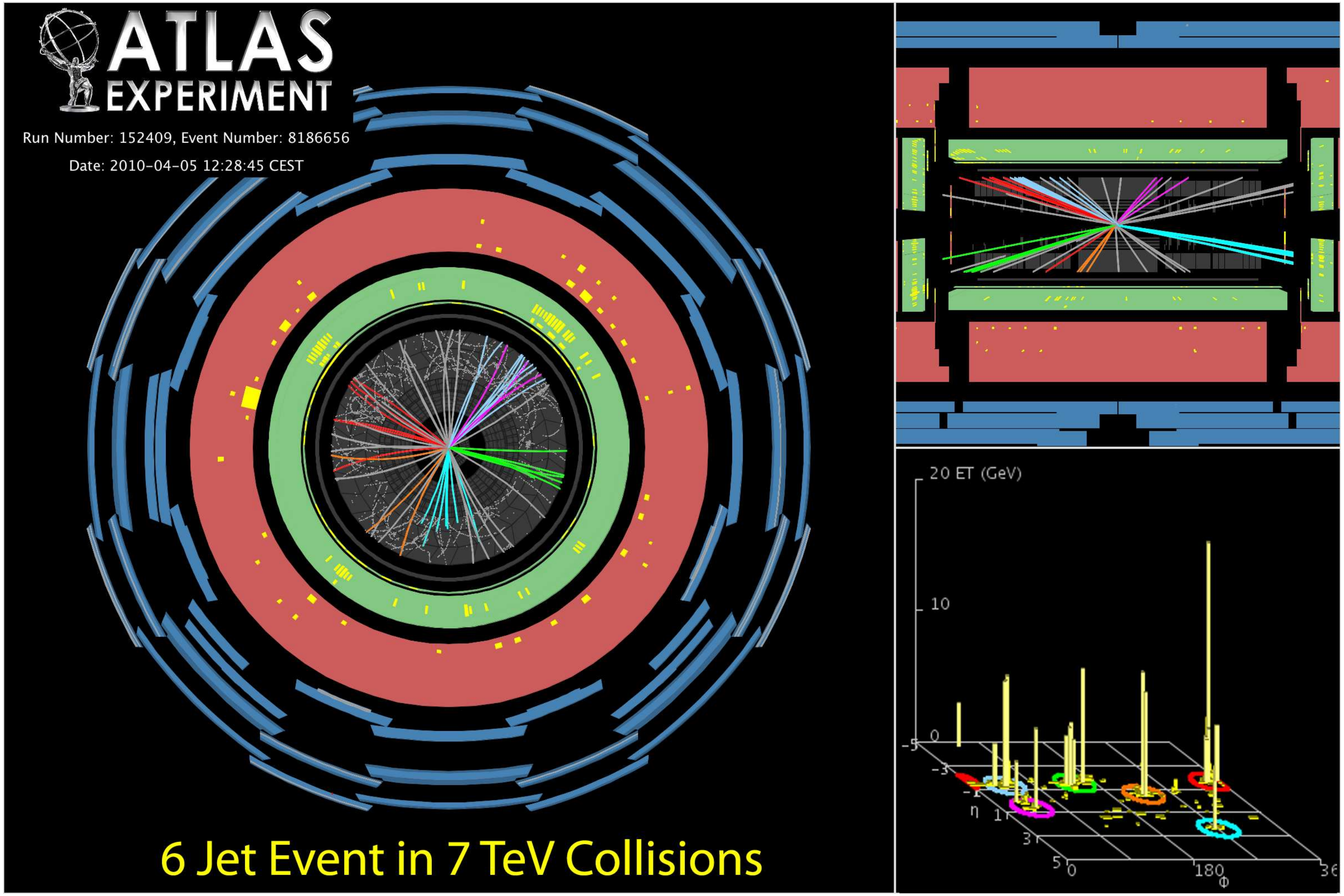
LHC physics



Jets: A bunch of particles moving in the same direction

Multileg NLO calculations

What one aims for:

- NLO calculations for multi-parton processes at the LHC.
  Multi-parton processes: 3, 4, 5, 6, ... partons in the final state.
- For a given process the program should be usable for any infrared-safe observable.
- Need to compute the virtual corrections and the real corrections.

Conventional set-up for NLO calculations

Need Born, real contribution and virtual contribution:

- Efficient methods known for Born and real contribution.
- Subtraction method or phase-space slicing is used to render the real contribution finite.
- Loop integrals in the virtual contribution are reduced (numerically) to a few master integrals.
- Integration over the loop momentum in the master integrals is done analytically.
- Integration over momenta of the final state particles is done by Monte Carlo.

Conventional set-up for the virtual part

Based on Feynman diagrams: avoid instabilities due to Gram determinants

Based on unitarity and cut techniques: avoid instabilities in solving linear systems of equations

Recent results:

- $pp \to W/Z + 4$ jets,
- $pp \to WW + 2$ jets,
- $pp \to t\bar{t} + 2$ jets,
- $e^+e^- \to 5$ jets,

Berger et al. (Blackhat collaboration), Ellis, Melnikov, Zanderighi, Melia, Rontsch, Bevilacqua, Czakon, Pittau, Papadopoulos, Worek, Bredenstein, Denner, Dittmaier, Pozzorini, Hirschi, Frederix, Frixione, Garzelli, Maltoni, ...

The subtraction method for the real emission

The subtraction methods subtracts out a simple term, which approximates the real emission in all singular limits:

$$\int_{n+1} d\sigma^R + \int_n d\sigma^V \quad = \quad \underbrace{\int_{n+1} \left(d\sigma^R - d\sigma^A\right)}_{\text{convergent}} + \underbrace{\int_n \mathbf{I} \otimes d\sigma^B}_{\text{singular}} + \int_n d\sigma^V$$

- Residue subtraction: Frixione, Kunszt and Signer, ’95; Del Duca, Somogyi, Trócsányi, ’05
- Dipole subtraction: Catani and Seymour ’96; Phaf and S.W. ’01; Catani, Dittmaier, Seymour and Trócsányi ’02; Nagy and Soper, ’07; Dittmaier and Kasprzik, ’08; Czakon, Papadopoulos and Worek, ’09; Chung, Kramer and Robens, ’10
- Antenna subtraction: Kosower, ’97; Gehrmann-De Ridder, Gehrmann, Glover, ’05

Computational costs

- Efficient methods like recursion relations known for Born and real contribution.
- Integration over momenta of the final state particles is done by Monte Carlo.
- Real emission (minus the subtraction terms) can be automated.

  S.W., '05, T. Gleisberg and F. Krauss, '07, M. Seymour and C. Tevlin, '08, K. Hasegawa, S. Moch and P. Uwer, '08, R. Frederix, T. Gehrmann and N. Greiner, '08, M. Czakon, C. Papadopoulos and M. Worek, '09.
- Insertion term $\mathbf{I} \otimes d\sigma^B$ is cheap.
- Virtual corrections usually reduced to a set of master integrals, which are known analytically.

  Evaluation time usually less or equal to real part.
- CPU-time for real emission sets time scale.

Our approach: Never change a winning team

Do the loop integrals numerically with Monte Carlo techniques !

- Can combine phase space integration ($3n-4$ dimensions) with loop integration (4 dimensions) in one Monte Carlo integration.
- Monte Carlo integration error scales with $1/\sqrt{N}$, independent of the dimension.

But: Loop integrals are divergent and need regularization. They are therefore calculated in $D = 4 - 2\varepsilon$ dimensions

$$\int d^{4-2\varepsilon}k \;\; f(k) \quad = \quad \frac{c_2}{\varepsilon^2} + \frac{c_1}{\varepsilon} + c_0 + O(\varepsilon)$$

Idea: Subtraction method.

$$\int d^{4-2\varepsilon}k \;\; f(k) \quad = \quad \underbrace{\int d^{4-2\varepsilon}k \;\; [f(k) - g(k)]}_{\text{convergent}} + \underbrace{\int d^{4-2\varepsilon}k \;\; g(k)}_{\text{simple}}$$

Subtraction method for loop integrals

Use subtraction also for the virtual part:

$$\int_{n+1} d\sigma^R + \int_n d\sigma^V = \underbrace{\int_{n+1} \left(d\sigma^R - d\sigma^A\right)}_{\text{convergent}} + \underbrace{\int_n \left(\mathbf{I} + \mathbf{L}\right) \otimes d\sigma^B}_{\text{finite}} + \underbrace{\int_n \left(d\sigma^V - d\sigma^{A'}\right)}_{\text{convergent}}$$

- In the last term $d\sigma^V - d\sigma^{A'}$ the Monte Carlo integration is over a phase space integral of n final state particles plus a 4-dimensional loop integral.

- All explicit poles cancel in the combination $\mathbf{I} + \mathbf{L}$.

- Divergences of one-loop amplitudes related to IR-divergences (soft and collinear) and to UV-divergences.

Numerical NLO QCD calculations

Proceed through the following steps:

1. Local subtraction terms for soft, collinear and ultraviolet singular part of the integrand of one-loop amplitudes

2. Contour deformation for the 4-dimensional loop integral.

3. Numerical Monte Carlo integration over phase space and loop momentum.

Not a new idea: Nagy and Soper proposed in ’03 this method, working graph by graph.

(see also: Soper; Krämer, Soper; Catani et al.; Kilian, Kleinschmidt)

What is new: The IR-subtraction terms can be formulated at the level of amplitudes, no need to work graph by graph.

The IR-subtraction terms are universal and amasingly simple.

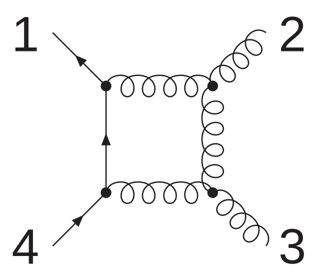
Primitive amplitudes

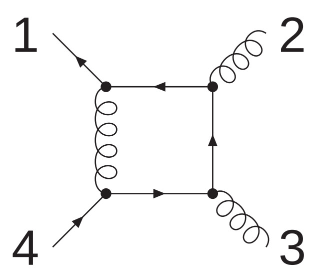
Colour-decomposition of one-loop amplitudes:

$$\mathcal{A}^{(1)} = \sum_j C_j A_j^{(1)}.$$

Primitive amplitudes distinguished by:

- fixed cyclic ordering
- definite routing of the fermion lines
- particle content circulating in the loop



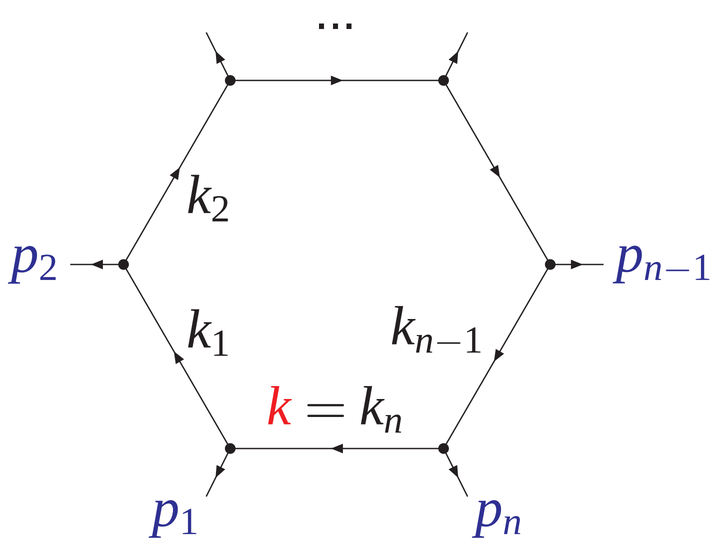


Z. Bern, L. Dixon, D. Kosower, ’95

Notation and kinematics

All momenta specified by p_1, ..., p_n and k:

$$k_i = k - (p_1 + \ldots + p_i)$$



For cyclic ordered amplitudes we have only n different propagators.

Write primitive one-loop amplitude as

$$A^{(1)}_{\text{bare}} = \int \frac{d^D k}{(2\pi)^D} G^{(1)}_{\text{bare}}, \qquad G^{(1)}_{\text{bare}} = P(k) \prod_{i=1}^{n} \frac{1}{k_i^2 - m_i^2 + i\delta}.$$

$P(k)$ is a polynomial in k.

Integrand can be calculated efficiently using recursion relations.

The infrared subtraction terms for the virtual corrections

Local unintegrated form:

$$G^{(1)}_{\text{soft+coll}} = -4\pi\alpha_s i \sum_{i\in I_g} \left(\frac{4p_i p_{i+1}}{k_{i-1}^2 k_i^2 k_{i+1}^2} - 2\frac{S_i g^{UV}_{i-1,i}}{k_{i-1}^2 k_i^2} - 2\frac{S_{i+1} g^{UV}_{i,i+1}}{k_i^2 k_{i+1}^2} \right) A_i^{(0)}.$$

with $S_q = 1$, $S_g = 1/2$. The function $g^{UV}_{i,j}$ provides damping in the UV-region:

$$\lim_{k\to\infty} g^{UV}_{i,j} = O\left(k^{-2}\right), \qquad \lim_{k_i || k_j} g^{UV}_{i,j} = 1.$$

Integrated form:

$$S_\varepsilon^{-1}\mu^{2\varepsilon}\int \frac{d^D k}{(2\pi)^D} G^{(1)}_{\text{soft+coll}} = \frac{\alpha_s}{4\pi}\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\sum_{i\in I_g}\left[\frac{2}{\varepsilon^2}\left(\frac{-2p_i\cdot p_{i+1}}{\mu^2}\right)^{-\varepsilon} + \frac{2}{\varepsilon}\left(S_i + S_{i+1}\right)\left(\frac{\mu^2_{\text{UV}}}{\mu^2}\right)^{-\varepsilon}\right] A_i^{(0)}$$

$$+O(\varepsilon),$$

M. Assadsolimani, S. Becker, S.W., ’09

UV-subtraction terms

In a fixed direction in loop momentum space the amplitude has up to quadratic UV-divergences.

Only the integration over the angles reduces this to a logarithmic divergence.

For a local subtraction term we have to match the quadratic, linear and logarithmic divergence.

The subtraction terms have the form of counter-terms for propagators and vertices.

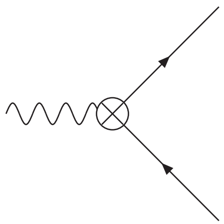
The complete UV-subtraction term can be calculated recursively.

S. Becker, Ch. Reuschle, S.W., JHEP 1012 (2010), 013, arxiv:1010.4187

UV-subtraction terms

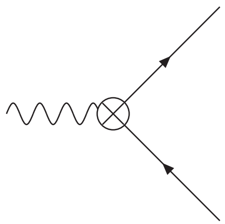
Example: The quark-gluon vertex.

Local unintegrated form:



$$= \; i g^3 S_\varepsilon^{-1} \mu^{4-D} \int \frac{d^D k}{(2\pi)^D i} \frac{2(1-\varepsilon)\,\bar{\not{k}}\gamma^\mu\bar{\not{k}} + 4\mu_{UV}^2\gamma^\mu}{\left(\bar{k}^2 - \mu_{UV}^2\right)^3}$$

Integrated form:



$$= \; i\frac{g^3}{(4\pi)^3}\gamma^\mu(-1)\left(\frac{1}{\varepsilon} - \ln\frac{\mu_{UV}^2}{\mu^2}\right) + O(\varepsilon)$$

We can ensure that the integrated expression is proportional to the Born.

Contour deformation

With the subtraction terms for UV- and IR-singularities one removes

- UV divergences
- Pinch singularities due to soft or collinear partons

Still remains:

- Singularities in the integrand, where a deformation into the complex plane of the contour is possible.
- Pinch singularities for exceptional configurations of the external momenta (thresholds, anomalous thresholds ...), integrable over phase space and loop space.

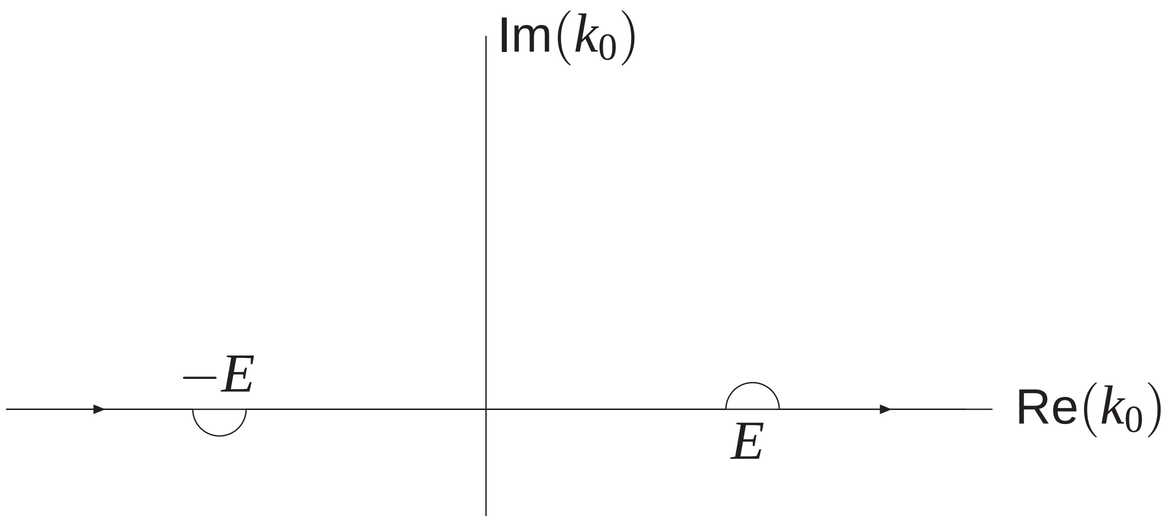
Contour deformation

$$I = \int d^4k \underbrace{[f(k) - g(k)]}_{h(k)}$$

$h(k)$ meromorphic function of four complex variables k_0, k_1, k_2, k_3.

Integration over a surface of (real) dimension 4 in $\mathbb{C}^4$.

I independent of the choice of the surface, as long as no poles are crossed.



What is the best choice for the surface, in order to minimize Monte Carlo integration errors ?

Contour deformation

We work with two methods for the contour deformation:

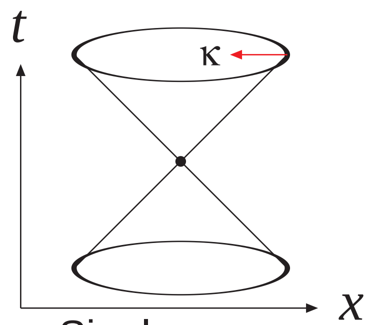
- Direct deformation, entirely in the space of the loop momentum.
  Integration is over the loop momentum k.
  At present only for massless particles.
  Gong, Nagy, Soper, '09; Becker, Reuschle, S.W., '12

- Additional Feynman parameters.
  Integration is over the loop momentum k and the Feynman parameters α.
  General, but slightly less efficient.
  Nagy, Soper, '06; Anastasiou, Beerli, Daleo, '07; Becker, Reuschle, S.W., '10

Direct contour deformation

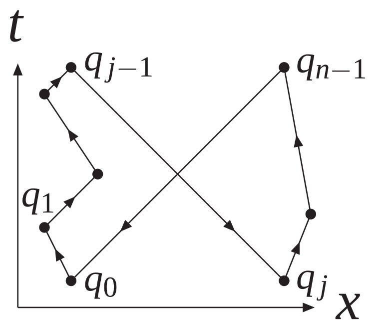
Deformation of the loop momentum:

$$k_{\mathbb{C}} \quad = \quad k_{\mathbb{R}} + i\kappa$$

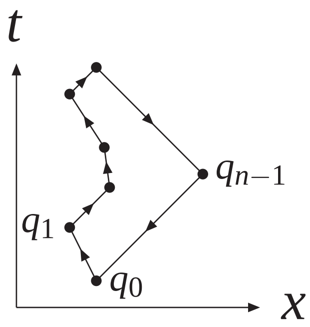


Single cone

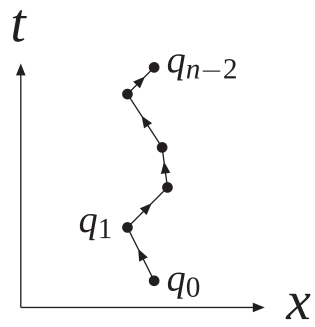
For n cones draw only the origins of the cones:



generic with 2 initial partons



initial partons adjacent



no initial partons

Efficiency

With the local subtraction terms and the contour deformation we obtain an integral, where the loop integration can – in principle – be performed with Monte Carlo methods.

However, the integrand is oscillating:

$$I = \int_0^1 dx \left[c + A \sin(2\pi x)\right], \qquad A \gg c$$

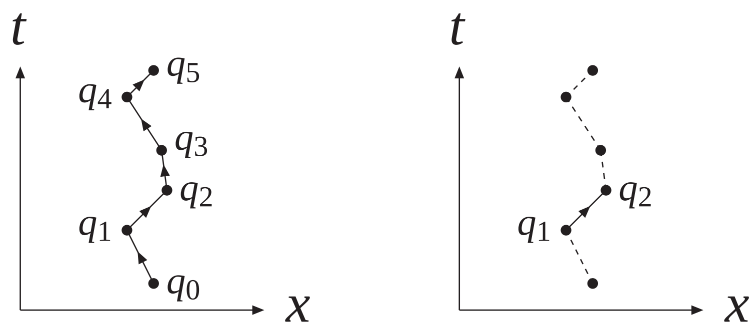
This leads to large Monte Carlo integration errors.

Solution: Antithetic variates: Evaluate the integrand at x and $(1-x)$.

Infrared channels

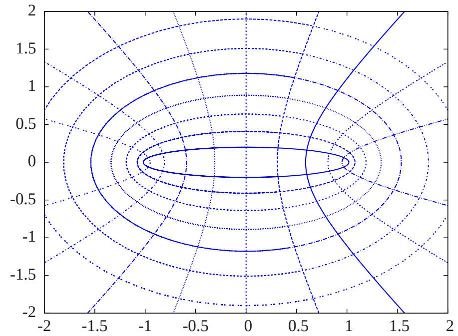
Non-holomorphic splitting:

$$I_{\text{int}} = \sum_i \int \frac{d^4k}{(2\pi)^4} w_i(k)\, f(k),$$



Weights:

$$w_i(k) = \frac{\left(\frac{1}{|k_i^2||k_{i+1}^2|}\right)^{\alpha}}{\sum_j \left(\frac{1}{|k_j^2||k_{j+1}^2|}\right)^{\alpha}},$$

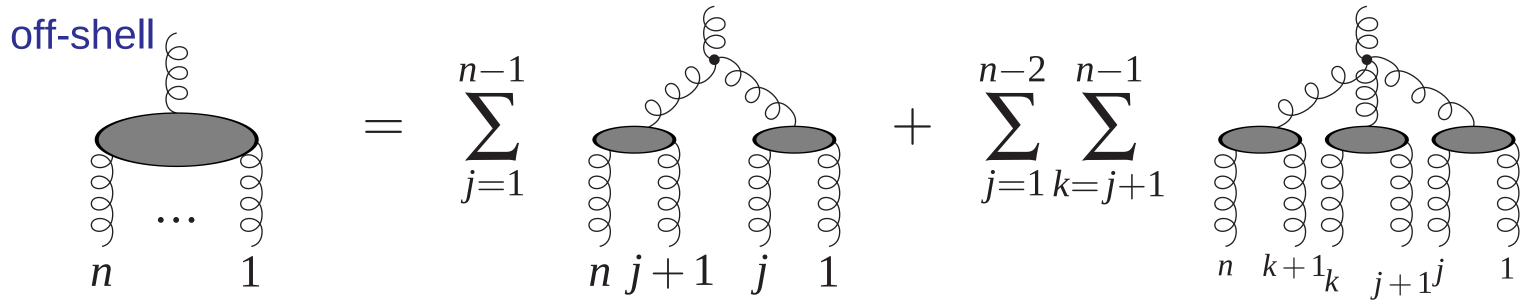


Coordinate system, where a line segment $[q_i, q_{i+1}]$ is singled out:
Generalised elliptical coordinates

Use technique of antithetic variates in these coordinates.

Recurrence relations

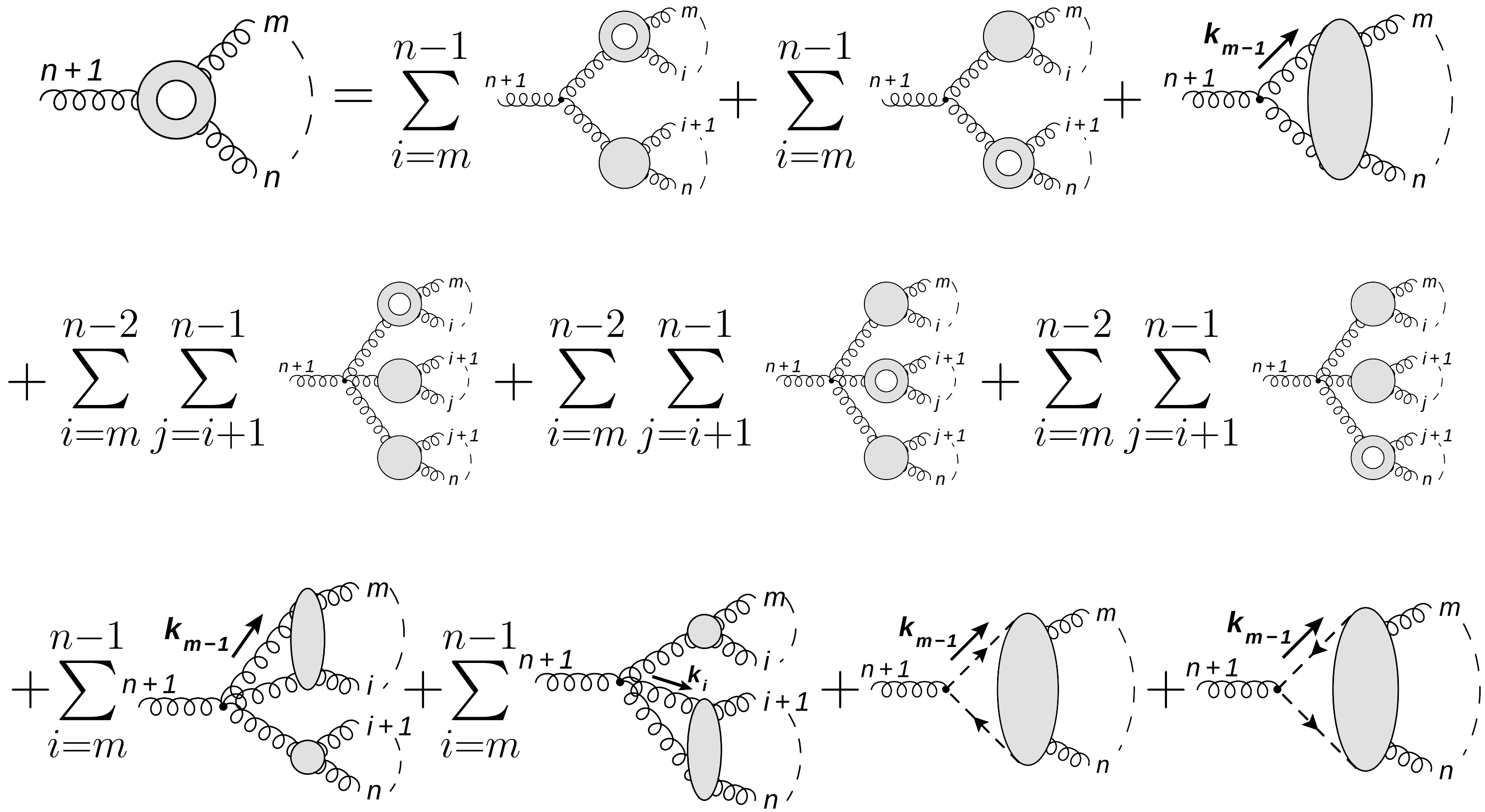
Off-shell currents provide an efficient way to calculate amplitudes:



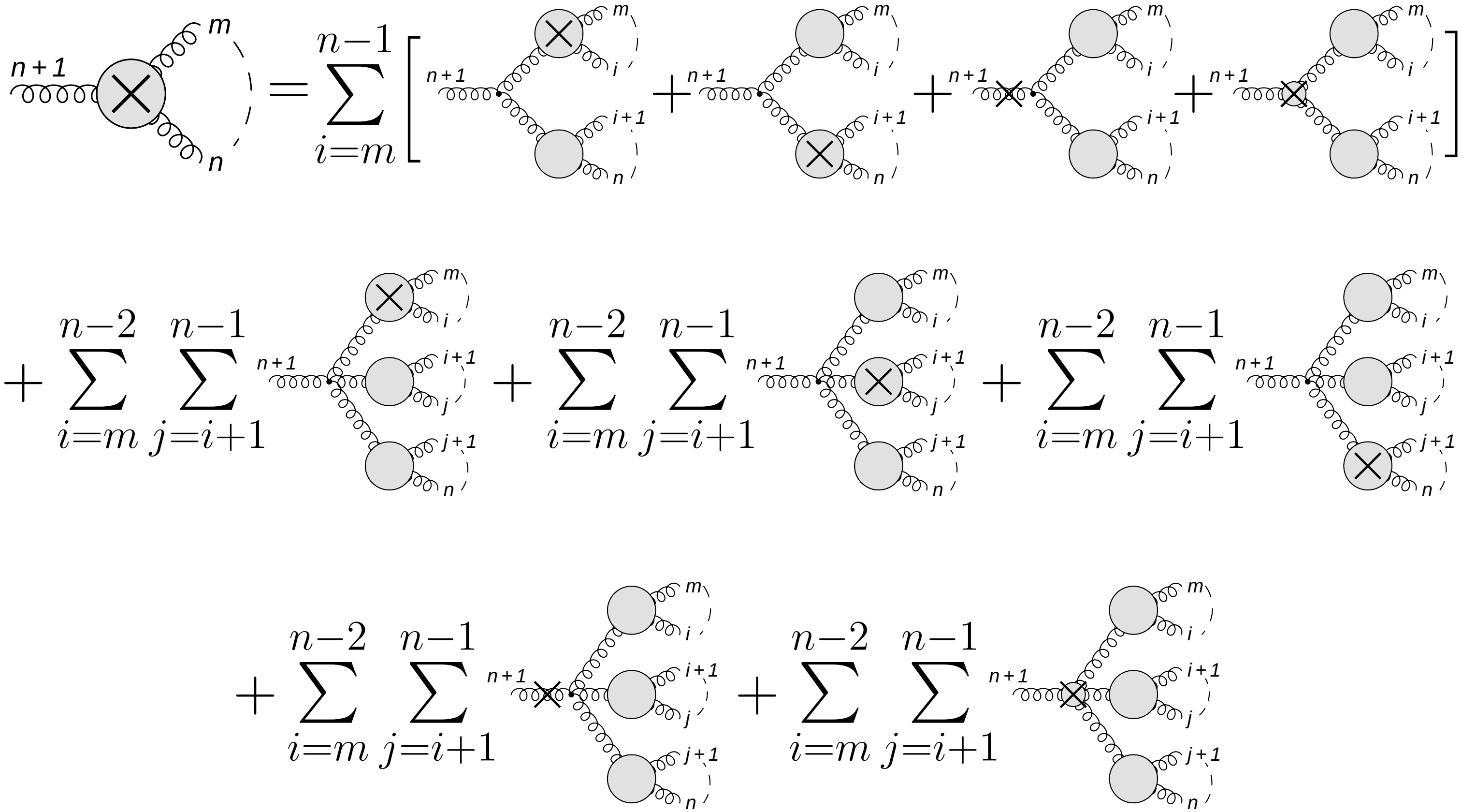
No Feynman diagrams are calculated in this approach !

F. A. Berends and W. T. Giele

The one-loop recurrence relations



The recurrence relation for the UV-counterterm



Results

We test our approach by calculating the NLO corrections for n-jet production in electron-positron annihilation in the leading colour approximation.

Results for $n = 2, 3, 4$ are well-known.

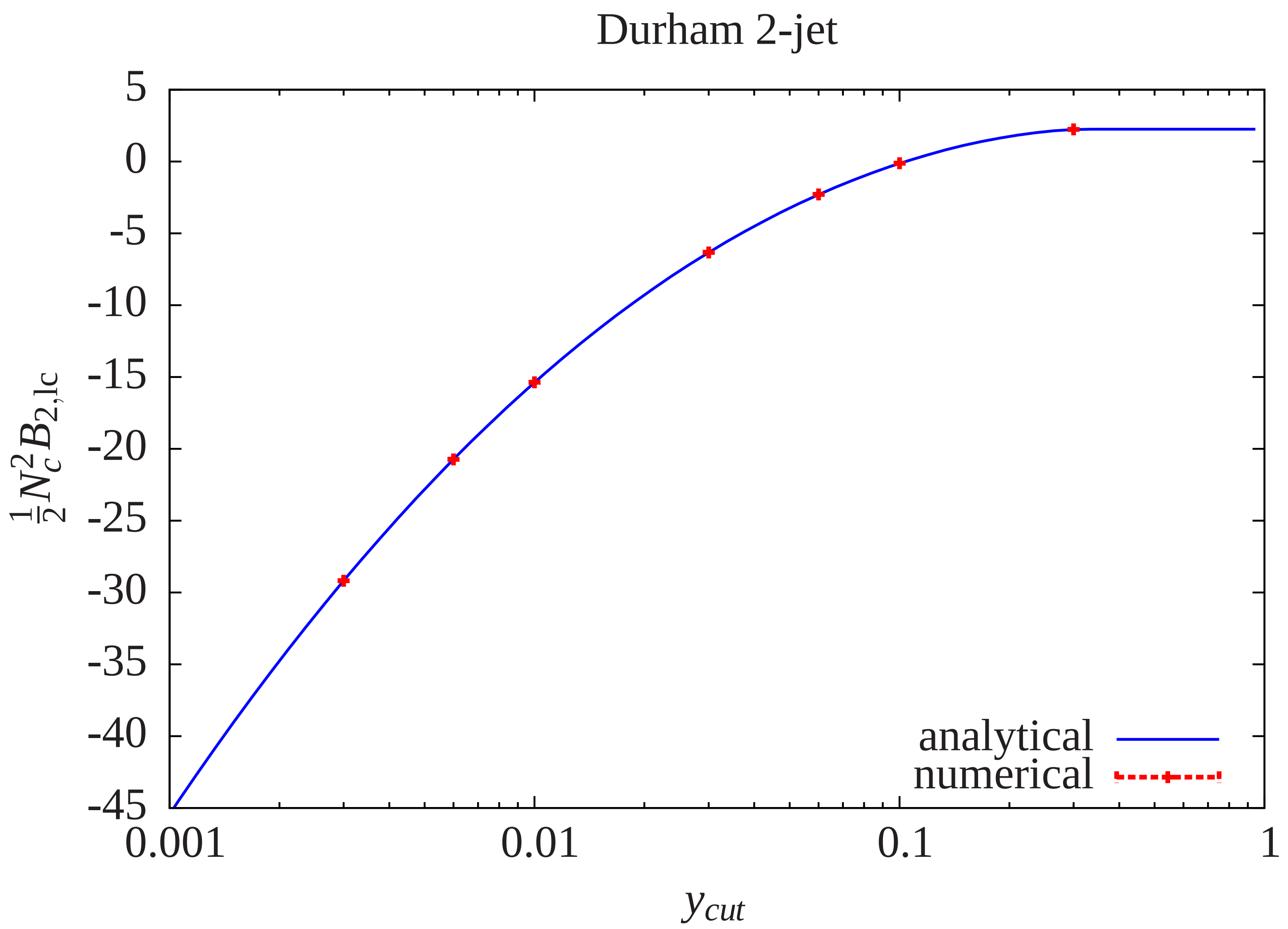
Results for $n = 5$ have been obtained recently.

R. Frederix, S. Frixione, K. Melnikov and G. Zanderighi, (arXiv:1008.5313).

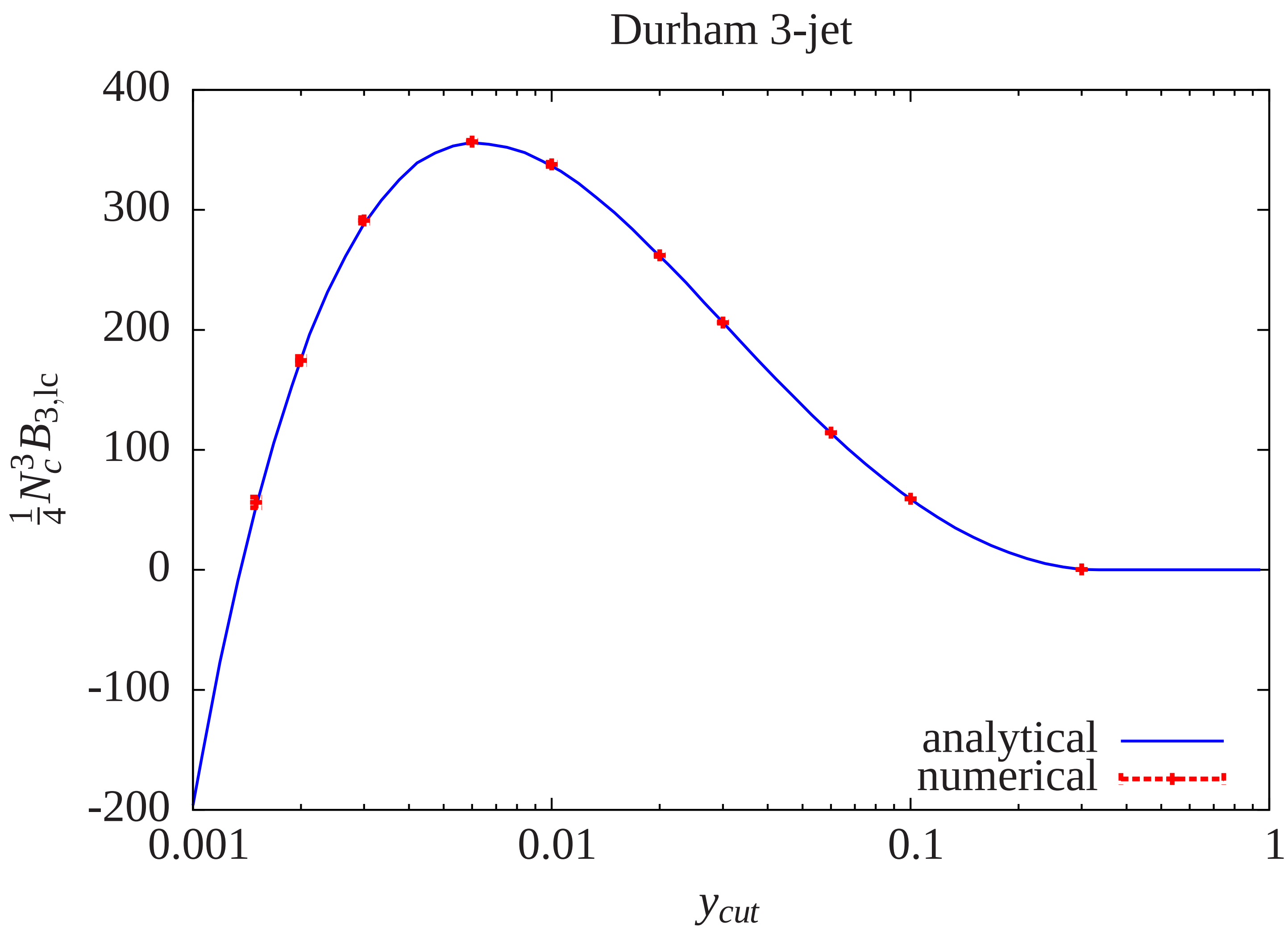
Jets are defined by the Durham jet algorithm.

In the program all parts work for arbitrary n.
CPU time scales polynomially with n.

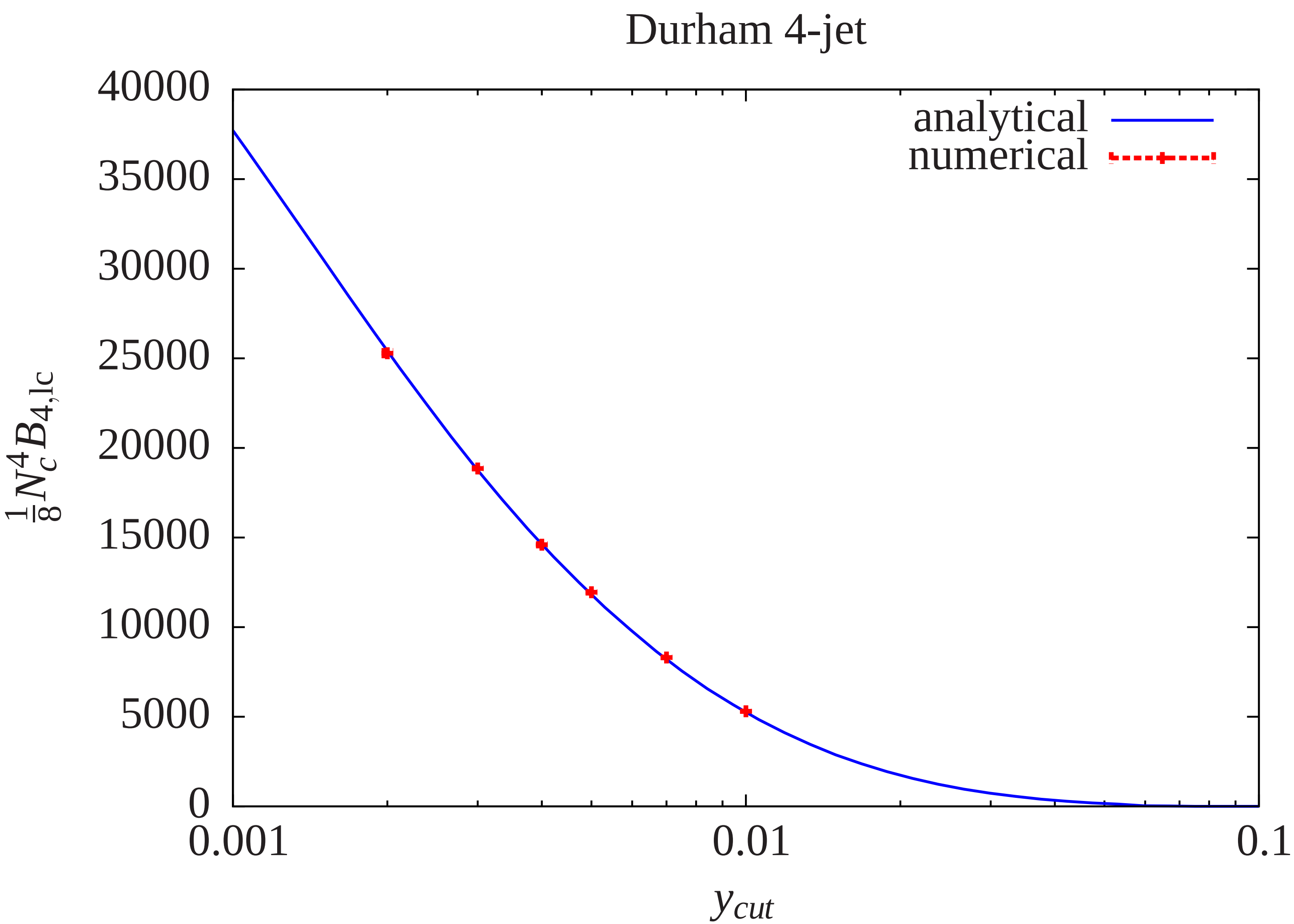
Durham 2-jet rate



Durham 3-jet rate



Durham 4-jet rate



Durham 5-, 6- and 7-jet rate

Perturbative expansion of the jet-rates:

$$\frac{\sigma_{n-\text{jet}}}{\sigma_0} = \left(\frac{\alpha_s}{2\pi}\right)^{n-2} A_n + \left(\frac{\alpha_s}{2\pi}\right)^{n-1} B_n + O(\alpha_s^n),$$

Leading-colour coefficient:

$$A_n = N_c \left(\frac{N_c}{2}\right)^{n-2} \left[A_{n,\text{lc}} + O\left(\frac{1}{N_c}\right)\right], \qquad B_n = N_c \left(\frac{N_c}{2}\right)^{n-1} \left[B_{n,\text{lc}} + O\left(\frac{1}{N_c}\right)\right].$$

Results for five, six and seven jets for the jet parameter $y_{cut} = 0.0006$:

$$\frac{N_c^4}{8} A_{5,\text{lc}} = (2.4764 \pm 0.0002) \cdot 10^4, \qquad \frac{N_c^5}{16} B_{5,\text{lc}} = (1.84 \pm 0.15) \cdot 10^6,$$

$$\frac{N_c^5}{16} A_{6,\text{lc}} = (2.874 \pm 0.002) \cdot 10^5, \qquad \frac{N_c^6}{32} B_{6,\text{lc}} = (3.88 \pm 0.18) \cdot 10^7,$$

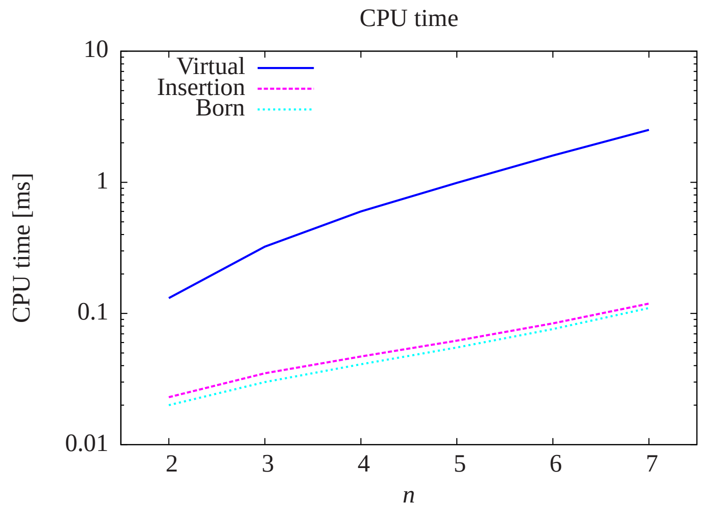
$$\frac{N_c^6}{32} A_{7,\text{lc}} = (2.49 \pm 0.08) \cdot 10^6, \qquad \frac{N_c^7}{64} B_{7,\text{lc}} = (5.4 \pm 0.3) \cdot 10^8.$$

First calculation of a physical observable involving a one-loop eight-point function!

CPU scaling behaviour

Scaling of the CPU time for one evaluation of the integrand (Born, virtual, insertion) with the number of external particles:

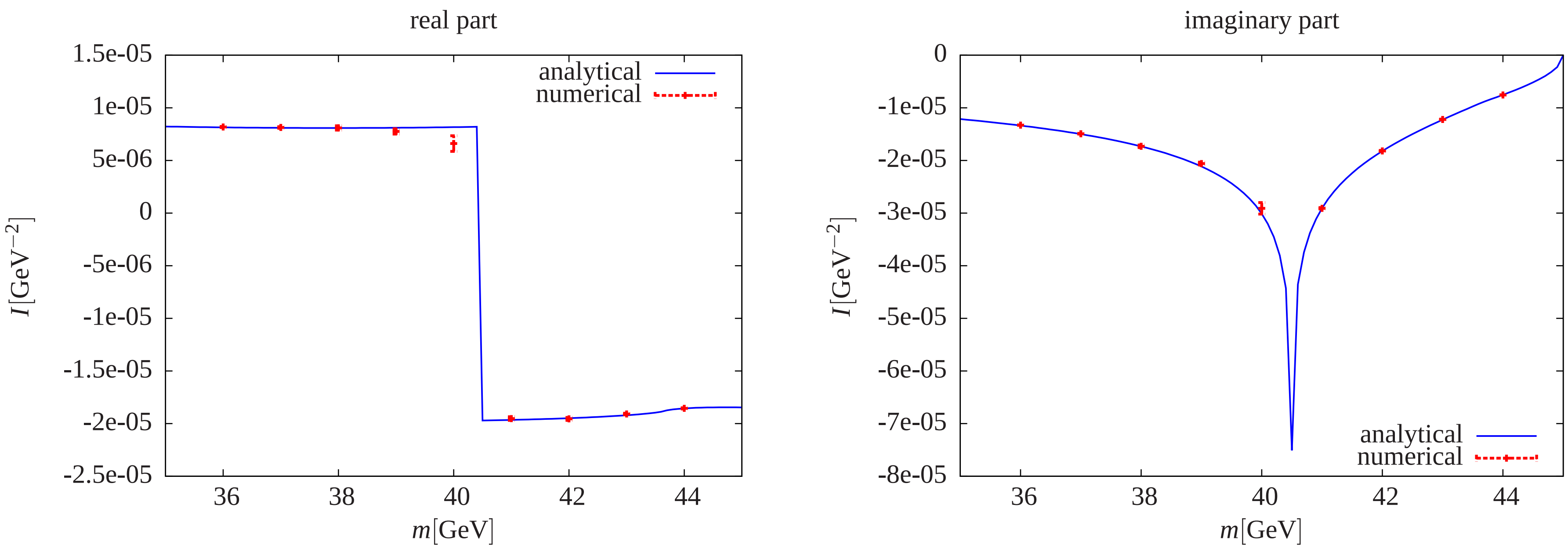
$$\text{CPU time} \quad \sim \quad n^4$$



- n^4-behaviour from recurrence relations
- helicity summation replaced by smooth integration over random polarisations
- Real part: Extension of the dipole formalism to random polarisations

  D. Götz, Ch. Schwan and S.W., ’12

Extension to massive particles



Comparison of the results obtained by Monte Carlo integration with the analytical results in the vicinity of a threshold.

S. Becker and S.W., '12

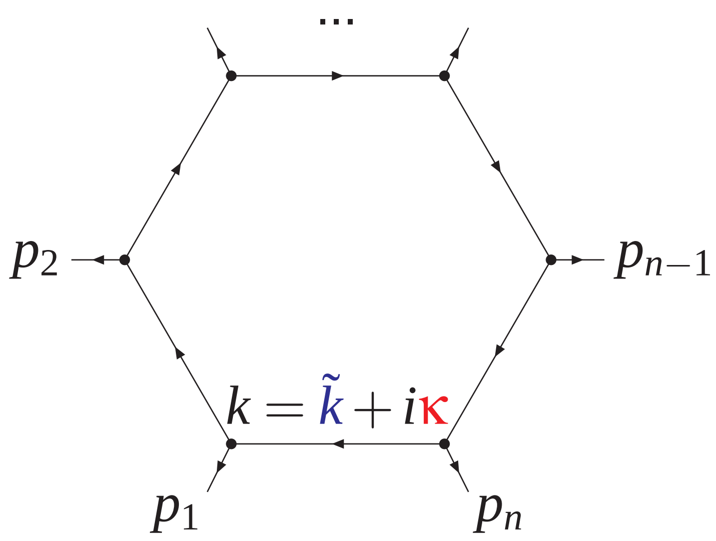
Extension to higher orders

- Subtraction terms

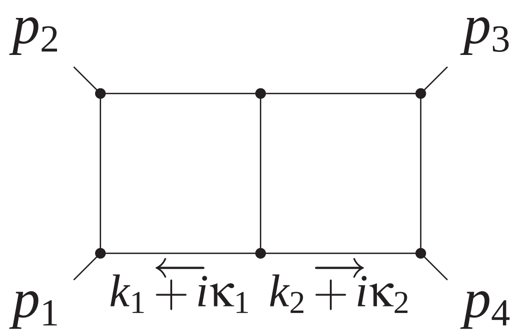
- Contour deformation

$$\int \frac{d^4k}{(2\pi)^4} f(k) \quad = \quad \int \frac{d^4\tilde{k}}{(2\pi)^4} \left| \frac{\partial k^\mu}{\partial \tilde{k}^\nu} \right| f(k(\tilde{k}))$$

κ vanishes whenever one loop momentum becomes soft.



Beyond one-loop

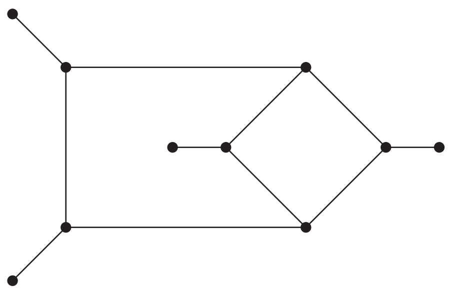


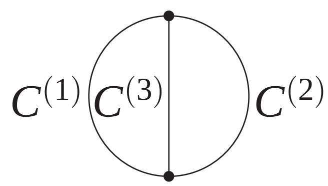
We have:

- 2 independent loop momenta
- 3 inequivalent cycles

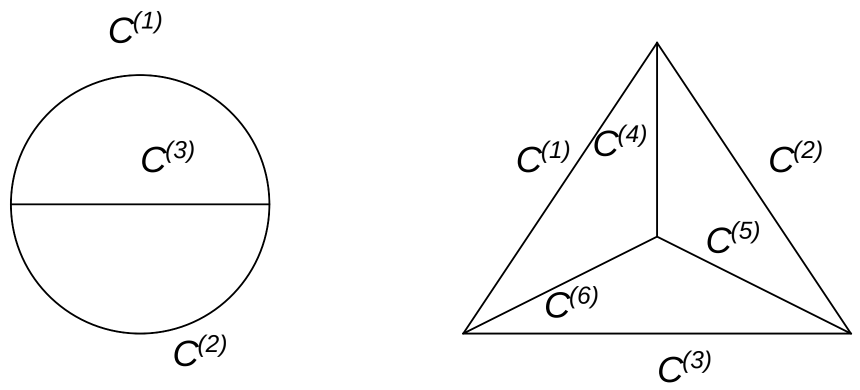
Chain diagrams

The momenta of the propagators in the same chain differ only by a linear combination of the external momenta.





Two and three loop chain diagrams

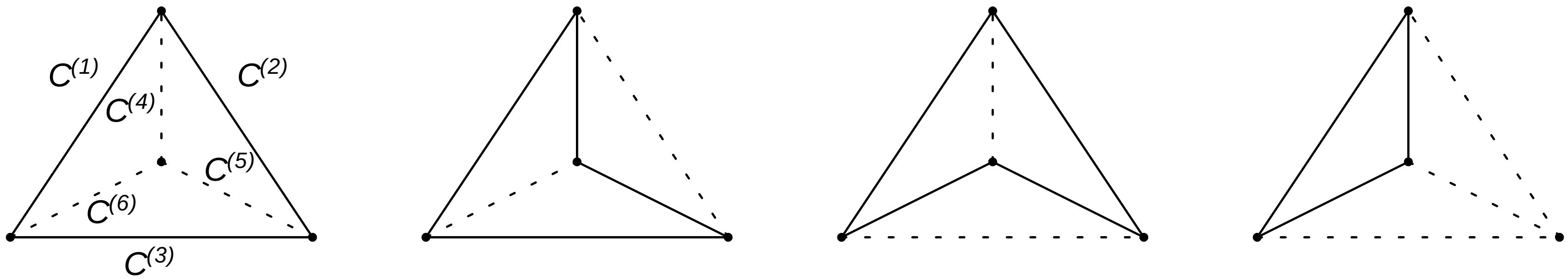


κ_i obtained as the sum of all deformation vectors for cycles containing propagator i.

Two-loop example:

$$\begin{aligned}\kappa_1 &= \kappa^{(12)} + \kappa^{(13)},\\ \kappa_2 &= \kappa^{(12)} + \kappa^{(23)},\end{aligned}$$

Three loops



$$
\begin{aligned}
\kappa_1 &= \kappa^{(123)} + \kappa^{(146)} + \kappa^{(1256)} + \kappa^{(1345)}, \\
\kappa_2 &= \kappa^{(123)} + \kappa^{(245)} + \kappa^{(1256)} + \kappa^{(2346)}, \\
\kappa_3 &= \kappa^{(123)} + \kappa^{(356)} + \kappa^{(1345)} + \kappa^{(2346)}.
\end{aligned}
$$

Verification and results

Comparison with analytical result (no internal masses, external legs off-shell):

- two- and three-loop propagator corrections
- two- and three-loop vertex functions (planar and non-planar)
- ladder diagrams (double box, triple box)

In addition:

- Two-loop six-point functions

Conclusions

- Numerical method for the computation of NLO corrections
  - Local subtraction terms for ultraviolet, soft and collinear divergences
  - Contour deformation
- First results for jet production in electron-positron annihilation
- More results on LHC processes to come
- Generalisation to higher orders