A new approach to perturbative calculations

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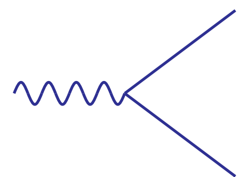
Introduction

- Algorithmic method for perturbative calculations
- Applicable to divergent integrals in dim. reg.: (loop, phase-space, . . . integrals).
- Method based on very simple mathematics
- I will study exclusive phase-space integrals:
  - Loop/incl. phase-space integrals have simpler boundaries.
  - Also studied by (Binoth and Heinrich)
- An example: $e^+e^- \to 2jets$ through $\mathcal{O}(\alpha_s^2)$
  - Simple but realistic test
  - First, fully differential, NNLO cross-section

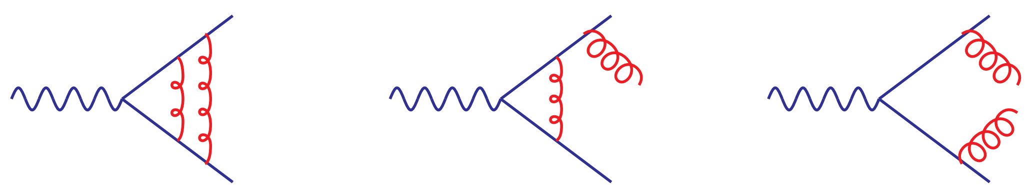
$e^+e^- \rightarrow jets$ **through NNLO**

- Leading order



- Next-to-leading order

- Next-to-next-to-leading order



NLO cross-section

- Loop-correction to $\gamma^* \to q\bar{q}$ is divergent

$$\sigma^V_{q\bar{q}} = \sigma^0_{q\bar{q}} \left\{ \frac{X_2}{\epsilon^2} + \frac{X_1}{\epsilon} + Finite_V \right\}$$

- Add together real-radiation corrections

$$\sigma_{q\bar{q}g} = \int dq d\bar{q} dg \delta^d(P - q - \bar{q} - g)\delta^+(q^2)\delta^+(\bar{q}^2)\delta^+(g^2)|\mathcal{M}_{q\bar{q}g}|^2$$

  - An integral over e.g. Mandelstam invariants

$$\begin{aligned} \sigma_{q\bar{q}g} &= \int ds_{q\bar{q}} ds_{\bar{q}g} ds_{qg} \delta\left(s_{q\bar{q}g} - s\right) \\ &\times \quad s^{\epsilon}_{\bar{q}g} s^{\epsilon}_{qg} \frac{f(s_{qg}, s_{\bar{q}g})}{s_{qg} s_{\bar{q}g}} \end{aligned}$$

  - which is also divergent: $(s_{qg}, s_{\bar{q}g}) \to 0$

Defining jets

- 3 jets if $s_{ij} > s_{min}$
- 2 jets if $s_{qg} < s_{min}$ or $s_{\bar{q}g} < s_{min}$

$$\sigma^R_{n\text{ jets}} = \int ds_{qg} ds_{\bar{q}g} ds_{q\bar{q}} s_{qg}^{-1+\epsilon} s_{\bar{q}g}^{-1+\epsilon} \delta\left(s_{\bar{q}qg} - s\right) \times f\left(s_{qg}, s_{\bar{q}g}\right) \mathcal{J}_n\left(s_{qg}, s_{\bar{q}g}, s_{q\bar{q}}\right)$$

- $\mathcal{J}_n$ should be treated as an arbitrary function
  - vary the definition of a jet
  - select jets with specific properties (e.g. energy)
  - impose cuts

A solved problem

- Subtract universal singular limits

$$\int ds_{ij} s_{ij}^{-1-\epsilon} \left[f(s_{ij}) \mathcal{J}(s_{ij}) - f(0) \mathcal{J}(0) \right] = Finite_R$$

- Integrate subtraction terms analytically

$$\int ds_{ij} s_{ij}^{-1-\epsilon} f(0) \mathcal{J}(0) \sim \frac{1}{\epsilon} f(0) \mathcal{J}(0)$$

Subtraction at NNLO

- Reasons for it:
  - Universal singular limits of matrix-elements
  - Calculate ϵ poles once and for all processes
- Reasons to try something different:
  - Subtraction terms: Difficult to find, and integrate
  - Solve the underlying mathematics problem.

NLO revisited

- Simplify phase-space boundaries:

$$s_{qg} + s_{\bar{q}g} + s_{q\bar{q}} = s$$

- Transform to variables with range $[0, 1]$

$$s_{qg} = \lambda_1 s, \quad s_{\bar{q}g} = (1 - \lambda_1)\lambda_2 s,$$
$$s_{q\bar{q}} = (1 - \lambda_1)(1 - \lambda_2) s$$

- Factorized singular structure

$$\sigma_R = \int_0^1 d\lambda_1 d\lambda_2 \frac{f(\lambda_1, \lambda_2)}{(\lambda_1 \lambda_2)^{1+\epsilon}} \mathcal{J}(\lambda_1, (1 - \lambda_1)\lambda_2, (1 - \lambda_1)(1 - \lambda_2))$$

Subtraction language: ' $+$ '

- Extracting the poles

$$\int_0^1 d\lambda f(\lambda)\lambda^{-1+\epsilon} = \frac{f(0)}{\epsilon} + \int_0^1 \frac{f(\lambda)-f(0)}{\lambda}\lambda^{\epsilon}$$

- systematically

$$\lambda^{-1+\epsilon} = \frac{\delta(\lambda)}{\epsilon} + \left[\frac{1}{\lambda}\right]_+ + \epsilon\left[\frac{\ln(\lambda)}{\lambda}\right]_+ + \frac{\epsilon^2}{2}\left[\frac{\ln^2(\lambda)}{\lambda}\right]_+ + \ldots$$

Real-radiation cross-section

$$\frac{\mathrm{d}^2\sigma_R}{\mathrm{d}\lambda_1\mathrm{d}\lambda_2} = \frac{64\pi\alpha_s}{3}\sigma_0\frac{\Gamma(1+\epsilon)}{(4\pi)^{d/2}}\left\{\frac{\delta(\lambda_1)\delta(\lambda_2)}{\epsilon^2} + \frac{1}{\epsilon}\left[-\frac{\delta(\lambda_1)}{[\lambda_2]_+} - \frac{\delta(\lambda_2)}{[\lambda_1]_+} + \left(1-\frac{\lambda_1}{2}\right)\delta(\lambda_2)\right.\right.$$
$$\left. + \left(1-\frac{\lambda_2}{2}\right)\delta(\lambda_1) - \delta(\lambda_1)\delta(\lambda_2)\right] + \left(\lambda_1 - 1 + \frac{1}{2}\frac{(2-2\lambda_1+\lambda_1^2)\ln(1-\lambda_1)}{\lambda_1}\right.$$
$$\left. - \left(1-\frac{\lambda_1}{2}\right)\ln(\lambda_1) + \left[\frac{1}{\lambda_1}\right]_+ + \left[\frac{\ln(\lambda_1)}{\lambda_1}\right]_+\right)\delta(\lambda_2) + \left(\lambda_2 - 1 + \frac{(2-2\lambda_2+\lambda_2^2)\ln(1-\lambda_2)}{\lambda_2}\right.$$
$$\left. - \left(1-\frac{\lambda_2}{2}\right)\ln(\lambda_2) + \left[\frac{1}{\lambda_2}\right]_+ + \left[\frac{\ln(\lambda_2)}{\lambda_2}\right]_+\right)\delta(\lambda_1) - \left(1-\frac{\lambda_1}{2}\right)\left[\frac{1}{\lambda_2}\right]_+$$
$$\left. - \left(1-\frac{\lambda_2}{2}\right)\left[\frac{1}{\lambda_1}\right]_+ + \left[\frac{1}{\lambda_1}\right]_+\left[\frac{1}{\lambda_2}\right]_+ - \frac{\pi^2}{6}\delta(\lambda_1)\delta(\lambda_2) + 1 - \lambda_1\left(1-\frac{\lambda_2}{2}\right)\right\}.$$

- I now have to do the integration:

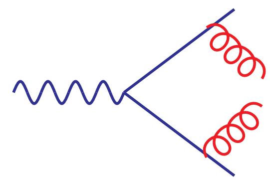
$$\int d\lambda_1 d\lambda_2 \frac{\mathrm{d}^2\sigma_R}{\mathrm{d}\lambda_1\mathrm{d}\lambda_2}\mathcal{J}_n(s_{qg}(\lambda_1,\lambda_2), s_{\bar{q}g}(\lambda_1,\lambda_2), s_{q\bar{q}}(\lambda_1,\lambda_2))$$

- J_n is kept unspecified; FORTRAN subroutine

Analyzing the pole structure

- $\frac{1}{\epsilon^2}\delta(\lambda_1)\delta(\lambda_2)$
  - Singularity from a soft gluon: $\mathcal{J}_n(0,0,1)$
- $\frac{\delta(\lambda_1)}{\epsilon}\left(\left[\frac{1}{\lambda_2}\right]_+ + \ldots\right) + (\lambda_1 \to \lambda_2)$
  - Singularity from a collinear gluon: $\mathcal{J}_n(0,\lambda,1-\lambda)$ or $\mathcal{J}_n(\lambda,0,1-\lambda)$.
- *Poles belong to a 2-jet configuration*
- *Cancel against the virtual corrections of the 2-jet configuration*

Real Radiation at NNLO



- Follow the NLO recipe:
  - Change variables: $s_{ij} \to \lambda_k$ with $0 < \lambda_k < 1$

    e.g. if $s_{min} < s_{ij} < s_{max}$ then $\lambda_k = \frac{s_{ij} - s_{min}}{s_{max} - s_{min}}$
  - Extract divergences

$$\lambda^{-1+\epsilon} = \frac{1}{\epsilon}\delta(\lambda) + \left[\frac{1}{\lambda}\right]_+ + \ldots$$

- Not quite...

$1 \to 4$ phase-space

- A nice factorization of the phase-space measure

$$\sigma_{RR} = \mathcal{N} \int d\lambda_1 \ldots d\lambda_5$$
$$[\lambda_1(1-\lambda_1)(1-\lambda_2)]^{1-2\epsilon} [\lambda_2\lambda_3\lambda_4(1-\lambda_3)(1-\lambda_4)]^{-\epsilon} [\lambda_5(1-\lambda_5)]^{-\frac{1}{2}-\epsilon}$$
$$|\mathcal{M}|^2 \mathcal{J}(s_{12}(\lambda), s_{13}(\lambda), \ldots)$$

- But terms in matrix-elements non-factorizable in this parametrization.

$$\begin{aligned} s_{234} &= \lambda_1 \\ s_{34} &= \lambda_1\lambda_2 \\ s_{23} &= \lambda_1(1-\lambda_2)\lambda_4 \\ s_{134} &= \lambda_2 + \lambda_3(1-\lambda_1)(1-\lambda_2) \\ s_{13} &= (1-\lambda_1)\left[\lambda_5(s_{13}^{+} - s_{13}^{-}) + s_{13}^{-}\right] \end{aligned}$$

Overlapping divergences

- A simple example:

$$\int_0^1 dxdy \frac{x^\epsilon y^\epsilon}{(x+y)^2}$$

- Divide into two integrations with definite variable ordering

$$I = \int_0^1 dx \int_0^x dy + \int_0^1 dy \int_0^y dx$$

- Singularities factorize in each sector when we map the integration region back to [0, 1]: e.g. in the $y < x$ sector I can set $y = \lambda x$.

$$I(y < x) = \int_0^1 dx d\lambda \frac{x^{-1+2\epsilon}\lambda^\epsilon}{(1+\lambda)^2}$$

- Then I can expand in plus distributions:

$$x^{-1+2\epsilon} = \frac{1}{2\epsilon}\delta(x) + \left[\frac{1}{x}\right]_+ + \dots$$

“Line” singularities

- Singularities can also occur inside the integration region:

$$s_{13} = (1-\lambda_1)(s_{13}^{-} + (s_{13}^{+} - s_{13}^{-})\lambda_5)$$

- e.g. At $\lambda_5 = 0$, a “line” singularity is produced:

$$\sim \frac{1}{s_{13}^{-}} \sim \frac{1}{|\lambda_3\lambda_4 - \lambda_2(1-\lambda_3)(1-\lambda_4)|^2}$$

- We bring the singularity at the endpoints by splitting the integration region and remapping back to [0, 1]:

$$\int_0^1 d\lambda_4 = \int_0^{\lambda_4^S} d\lambda_4 + \int_{\lambda_4^S}^1 d\lambda_4, \qquad \lambda_4^S = \frac{\lambda_2(1-\lambda_3)}{\lambda_3 + \lambda_2(1-\lambda_3)}$$

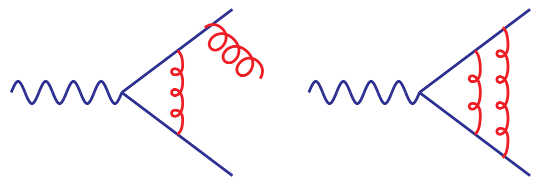
,

- Then apply sector decomposition.

Algorithm

- Describe the phase-space in terms of variables (λ) with range $[0, 1]$
- Move “line” singularities to the endpoints.
- Factorize overlapping singularities with sector decomposition.
- Expand in ϵ using ‘plus’ distributions.
- Evaluate numerically the coefficients of the ϵ expansion.

Loop corrections



- Feynman parameters have simple boundaries:

$$\int dx_1 \dots dx_n \delta(1 - x_1 - \dots - x_n) \dots$$

- bring boundaries to $[0, 1]$:

$$x_1 = \lambda_1, x_2 = \lambda_2(1 - \lambda_1), \dots x_n = (1 - \lambda_1) \dots (1 - \lambda_{n-1}).$$

- Apply sector decomposition.

Results

- For e.g. the JADE jet algorithm with $y_{cut} = 0.1$ we obtain:

$$R_{2jet} = 1 + \frac{\alpha_s}{\pi} C_1^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^2 C_2^{(2)}$$

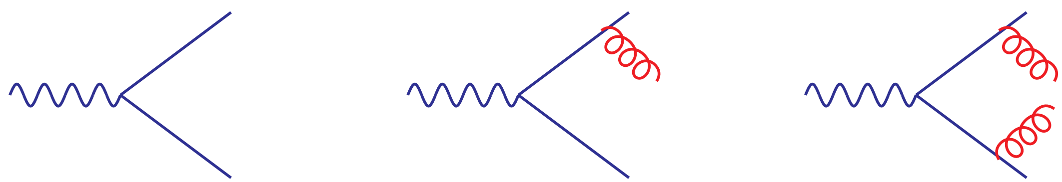
$$\begin{aligned} C_2^{(2)} &= \frac{10^{-6}}{\epsilon^4} + \frac{10^{-4}}{\epsilon^3} + \frac{10^{-3}}{\epsilon^2} + \frac{(-5 \pm 4) \times 10^{-2}}{\epsilon} \\ &+ \frac{(-0.3 \pm 4) \times 10^{-4}}{\epsilon} N_f + (-49.8 \pm 0.4) \\ &+ (1.798 \pm 0.002) N_f \ . \end{aligned}$$

- In about 4 hours on a $\$1000$ PC.

E_{max} distribution

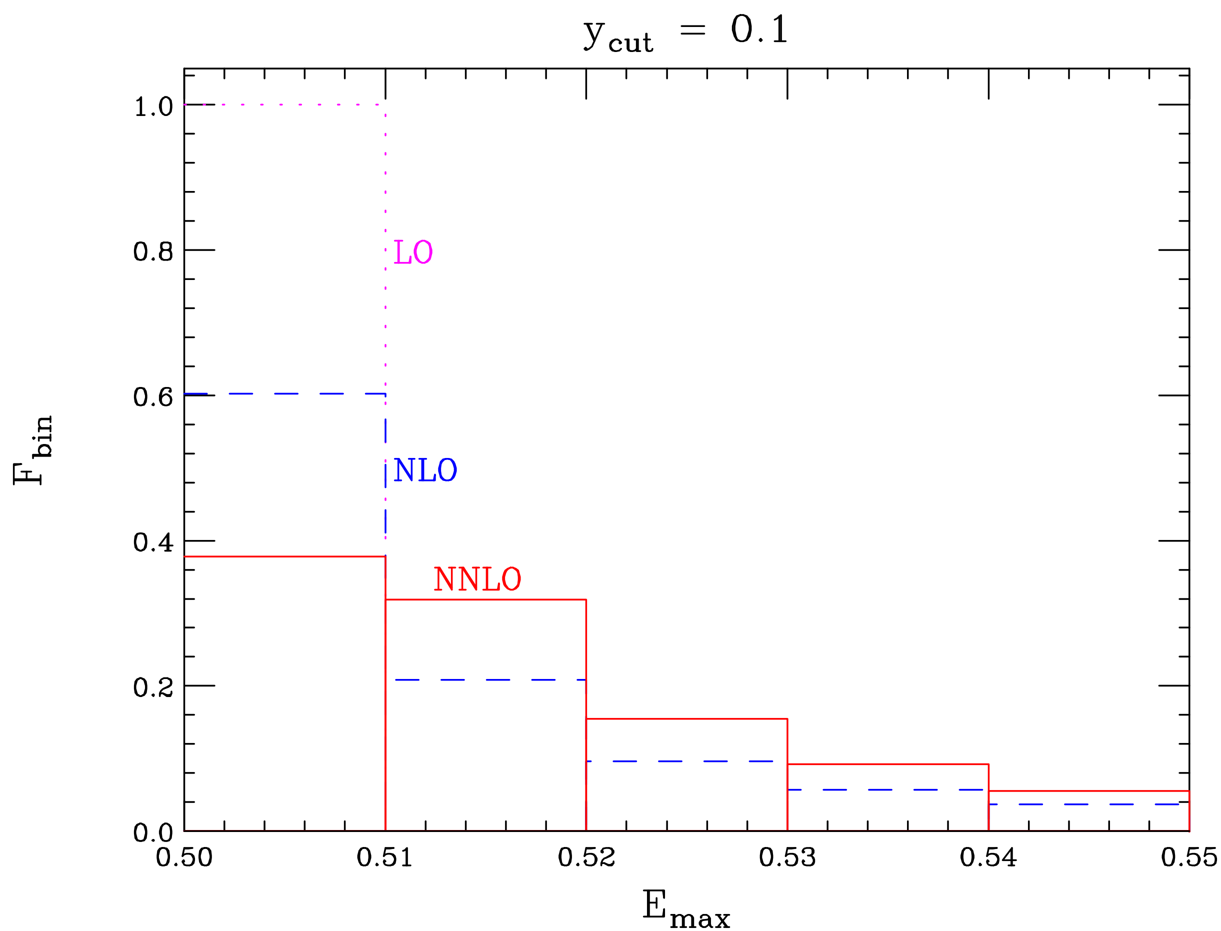
- Very easy to produce distributions, e.g.

$$E_{max} = \frac{1}{2s}\left(s + \left|m_{1jet}^2 - m_{2jet}^2\right|\right)$$



- At LO: $E_{max} = \frac{1}{2}$.
- At NLO and NNLO: $m_{jet} \neq 0$.

E_{max} distribution



Thresholds in Loops

- e.g. the one-loop box ($s > 0, t = -|t| < 0$).

$$I_4 = \mathcal{N} \int_0^1 \frac{\delta(1 - x_1 - x_2 - x_3 - x_4)}{(x_1 x_3 s - x_2 x_4 |t|)^{2+\epsilon}}$$

- Mapping to $[0, 1]$

$$I_4 = \mathcal{N} \int_0^1 d\lambda_1 d\lambda_2 d\lambda_3 \frac{(1-\lambda_1)^{-\epsilon}(1-\lambda_2)^{-1-\epsilon}}{(\lambda_1 \lambda_3 s - \lambda_2 (1-\lambda_1)(1-\lambda_3))^{2+\epsilon}}$$

- Line singularity at $\lambda_3^S = \frac{\lambda_2(1-\lambda_1)}{\lambda_1 s + \lambda_2(1-\lambda_1)|t|}$
- Bringing "line" to endpoints

$$I_4 = (-|t|)^{-\epsilon} I_a + s^{-\epsilon} I_b$$

  - I_a, I_b with sector decomposition
  - Imaginary parts from: $(-|t|)^{-\epsilon} = (|t|)^{-\epsilon} e^{-I\epsilon\pi}$.

Conclusions

- I have described a complete method for perturbative calculations
- It is based on:
  - Systematic factorization of overlapping singularities
  - Systematic subtraction of factorized singularities
  - Numerical techniques
- It can be used for any loop, phase-space, . . . integrals
- We computed the first fully differential cross-section at NNLO
- A very powerful tool; many applications to come.