A Calculational Formalism for One-Loop Integrals

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- Goals and Status.
- From Diagrams to Scalar Integrals.
- From Scalar Integrals to a Basis Set.
- Outlook.

W. Giele, Santa Barbara, 04/02/03

Goals and Status

- A numerical evaluation of one-loop matrix elements for Standard Model processes
  - The limiting factor on multiplicity set by computer resources
  - No numerical integrations, but basis set of known integrals (Pure numerical approach: D. Soper & Z. Nagy; G.J. van Oldenborgh & J.A.M. Vermaseren; A. Ferroglia, M. Passera, G. Passarino & S. Uccirati)
  - Numerical reduction of integrals to the basis set of known integrals

Goals and Status

- ✓ Finished basis algorithmic description for numerical implementation (W.G. & E.W.N. Glover: hep-ph/0402152)
  - Works for any internal and external mass configuration.
  - Calculates the finite part of the tensor loop integral

$$\int d\,l \frac{l_{\mu_1} l_{\mu_2} \cdots l_{\mu_m}}{(l+q_1)^2 (l+q_2)^2 \cdots (l+q_N)^2}; \quad m \leq N$$

  - The divergent part of the loop integral is expressed in terms of IR triangles for analytic evaluation.

Goals and Status

G. Zanderighi, W.G & E.W.N. Glover

✓ Implementation of algorithm for up to 6 legs

- Flushing out subtleties in algorithm
- Numerical accuracy of procedure
- First calibration point is : $\gamma_1\gamma_2 \rightarrow \gamma_3\gamma_4$

  A non-trivial helicity amplitude is:

$$M(+--+-) \sim -4\left\{\frac{t^2+u^2}{2s^2}\left[Ln^2\left(\frac{-u}{-t}\right)+\pi^2\right]+\frac{t-u}{s}Ln\left(\frac{-u}{-t}\right)+1\right\}$$

Analytic	Recursive
-0.847448212761-7.72950724975j	0.847448212761-7.72950724975j
-0.773231936093-7.38189821256j	0.773231936093-7.38189821256j
-0.00436614882507-0.553570755011j	0.00436614882487-0.55357075501j

Goals and Status

- Next calibration point for final algorithmic stability and correctness is: $d\sigma\left(\gamma_1\gamma_2 \rightarrow \gamma_3\gamma_4\gamma_5\gamma_6\right)$
- The final calibration point for algorithmic optimization will be: $d\sigma(g_1 g_2 \rightarrow g_3 g_4 g_5 g_6)$
- After this a more programmatic phase can start
- Many processes of interest can be implemented. E.g. $d\sigma\left(\mathrm{PP} \rightarrow W + 3\,\mathrm{jets}\right), d\sigma\left(\mathrm{PP} \rightarrow t\bar{t} + 2\,\mathrm{jets}\right), \ldots$
- Extension by implementing the formalism for more than 6 external lines

From Diagrams to Scalar Integrals

- The first step is to reduce an amplitude in scalar integrals
- Example: the 4 photon process is simply given by

$$M(\gamma_1\gamma_2\gamma_3\gamma_4) = m(1234) + m(1342) + m(1423)$$

$$m(1234) \sim \int d\,l \, \frac{Tr\left(\not\varepsilon_1\left(\not l + \not q_1\right)\not\varepsilon_2\left(\not l + \not q_2\right)\not\varepsilon_3\left(\not l + \not q_3\right)\not\varepsilon_4\left(\not l + \not q_4\right)\right)}{(l+q_1)^2(l+q_2)^2(l+q_3)^2(l+q_4)^2}$$

$$q_1 = k_1; q_2 = k_1 + k_2; q_3 = k_1 + k_2 + k_3; q_4 = 0$$

From Diagrams to Scalar Integrals

- To transform this into scalar integrals we use:
  - "Davydychev" decomposition (A.I. Davydychev)
  - Explicit helicity choices.
- The Davydychev decomposition translates a tensor integral into higher dimensional scalar integrals. E.g.:

$$\int d^D l \frac{l^{\mu_1} l^{\mu_2}}{(l+q_1)^2 (l+q_2)^2 (l+q_3)^2 (l+q_4)^2} =$$

$$-\frac{1}{2} g^{\mu_1 \mu_2} I(D+2;1,1,1,1)$$

$$+2 q_1^{\mu_1} q_1^{\mu_2} I_4(D+4;3,1,1,1) + (q_1^{\mu_1} q_2^{\mu_2} + q_1^{\mu_2} q_2^{\mu_1}) I_4(D+4;2,2,1,1) + \cdots$$

From Diagrams to Scalar Integrals

- The scalar integrals are given by

$$I_4(D;\nu_1\nu_2\nu_3\nu_4) \sim \int d^D l \frac{1}{\left[(l+q_1)^2\right]^{\nu_1}\left[(l+q_2)^2\right]^{\nu_2}\left[(l+q_3)^2\right]^{\nu_3}\left[(l+q_4)^2\right]^{\nu_4}}$$

$$\sim \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 \frac{\delta\left(\alpha_1+\alpha_2+\alpha_3+\alpha_4-1\right)\alpha_1^{\nu_1-1}\alpha_2^{\nu_2-1}\alpha_3^{\nu_3-1}\alpha_4^{\nu_4-1}}{\left(-\alpha\cdot S\cdot\alpha\right)^{D/2-\nu_1-\nu_2-\nu_3-\nu_4}}$$

From Diagrams to Scalar Integrals

- For instance one of the three diagrams becomes for the (-,-,+,+) helicity configuration:

$$m(1^-2^-3^+4^+) = 4\langle p3p4\rangle^2[p1p2]^2 \times \Big[6\tfrac{s_{13}}{s_{23}} I_4(12;1,4,1,2$$

$$-\tfrac{1}{s_{23}}\Big\{I_4(10;1,2,1,2)-I_4(10;1,2,2,1)-6\times I_4(10;1,3,1,1)-I_4(10;2,2,1,1)\Big\}$$

$$-6\times I_4(12;1,2,1,4)-I_4(12;1,2,2,3)-2\times I_4(12;1,2,3,2)-4\times I_4(12;2,2,1,3)$$

$$-3\times I_4(12;2,2,2,2)-2\times I_4(12;2,2,3,1)-2\times I_4(12;3,2,1,2)-8\times I_4(12;3,2,2,1)\Big]$$

From Scalar Integrals to Basis Set

- The remaining task is to evaluate the scalar integral $I_N(D;\nu_1\nu_2\cdots\nu_N) \sim I(D;\sigma); \sigma = \sum \nu_i \geq N$
- Here we convert a well established analytic method of recursively expressing the scalar integral into a numerical method

$$I(D;\sigma) \sim \sum b_i\, I(D_i;\sigma_i);\; D_i \leq D;\; \sigma_i < \sigma$$

- Repeating the recursion will lower N and D until we are left with irreducible scalar integrals: the basis set

From Scalar Integrals to Basis Set

- Recursion relations between scalar integrals have been known for a long time in 4 dimensions (Melrose 1965;W.L. van Neerven & J.A.M. Vermaseren):

$$I_5(D=4) \sim \sum b_i \; I_4(D=4)$$

- The extension to arbitrary dimensions was first formalized by Z. Bern, L.J. Dixon & D.A. Kosower:

$$I_5(D=4-2\varepsilon) \sim \sum b_i \; I_4(D=4-2\varepsilon) + \varepsilon B I_5(D=6-2\varepsilon)$$

- And further developed by many groups into the formulation we will use (J.M. Campbell, E.W.N. Glover and D.J. Miller; J. Fleischer, F. Jegerlehner & O.V. Tarasov; T. Binoth, J.P. Guillet & G. Heinrich; G. Duplancic & B. Nizic)

From Scalar Integrals to Basis Set

- The basic identity behind the recursion relations is the integration by part identity (K.G. Chtyrkin, A.L. Kataev & F.V. Tkachov)

$$\int d^D l \, \frac{\partial}{\partial l^\mu} \left(\frac{\left(\sum b_i\right) l^\mu + \left(\sum b_i q_i{}^\mu\right)}{d_1{}^{\nu_1} d_2{}^{\nu_2} \cdots d_N{}^{\nu_N}} \right) = 0; \; d_i = \left(l + q_i\right)^2; \; b_i \text{ arbitrary}$$

- This leads to the base equation

$$\sum_i \left(\nu_i - 1\right)\left(S\cdot b\right)_i I_N\left(D;\sigma\right) \sim \sum_i b_i I_{N_i}\left(D_i;\sigma_i\right)$$

where the kinematic matrix is given by:

$$S_{ij} = \left(q_i - q_j\right)^2$$

From Diagrams to Scalar Integrals

- For $N \leq 6$ we can invert the kinematic matrix to give the base recursion relation

$$\left(\nu_j - 1\right) I_N\left(D;\sigma\right) \sim \sum_i S_{ji}^{-1} I_{N_i}\left(D_i;\sigma_i\right)$$

$$\left(D_i, N_i, \sigma_i\right) < \left(D, N, \sigma\right)$$

- For N>6 the kinematic matrix cannot be inverted and the recursion relations change in character.

From Diagrams to Scalar Integrals

- Example:

$$2\times I_4(8;3,1,1,1) = -2\left(\sum_i S_{1i}^{-1}\right) I_4(8;2,1,1,1)$$
$$- S_{11}^{-1} I_4(6;1,1,1,1) - S_{12}^{-1} I_3(6;1,0,1,1)$$
$$- S_{13}^{-1} I_3(6;1,1,0,1) - S_{14}^{-1} I_3(6;1,1,1,0)$$

- The integral $I_4(8;2,1,1,1)$ will again be reduced
- The other integrals are already base integrals
- The S_{ij}^{-1} are just determined by the numerical matrix inversion of the kinematic matrix $S_{ij} = (q_i - q_j)^2$

From Scalar Integrals to Basis Set

- After applying the recursion relation repeatedly we arrive at a base set of integrals:
  - The 6-dimensional 5-point function $I_5(6;1,1,1,1,1)$
    This finite integral does not contribute at NLO to any physical cross sections (**Z. Bern, L.J. Dixon & D.A. Kosower; T. Binoth, J.P. Guillet & G. Heinrich**)
  - The 6 dimensional 4-point function, e.g. massless case

$$I_4(6;1,1,1,1) \sim \frac{1}{s_{12}+s_{23}}\left(\log^2\left(\frac{-s_{12}}{-s_{23}}\right)+\pi^2\right)$$

  - Triangles (some of which can be IR/UV divergent)

$$I_3\left(D;\nu_1\nu_2\nu_3\right)=I_3^{\mathrm{DIV}}\left(D;\nu_1\nu_2\nu_3\right)+I_3^{\mathrm{FIN}}\left(D;\nu_1\nu_2\nu_3\right)$$

From Scalar Integrals to Basis Set

- Any process (or Feynman diagram) can now numerically be reduced to the basis set

$$m(1^-2^-3^+4^+) = 4\langle p3p4\rangle^2 [p1p2]^2 \times \left[6\,\tfrac{s_{13}}{s_{23}}\, I_4(12;1,4,1,2) - \cdots\right]$$

$$= K_4 I_4(6;1,1,1,1) + \sum K_3(D;\nu_1\nu_2\nu_3) I_3^{\text{fin}}(D;\nu_1\nu_2\nu_3)$$

- The divergent part cancels in this case.
- The kinematic coefficients are calculated numerical in the recursion algorithm
- The analytic expressions of the base functions are evaluated numerical in the recursion algorithm

From Scalar Integrals to Basis Set

- In general we have divergent integrals.
- The UV divergent part of tensor integrals (rank four 4-point and lower points) is trivially separated in the Davydychev decomposition:

$$\int d^D l \frac{l_{\mu_1} l_{\mu_2} l_{\mu_3} l_{\mu_4}}{(l+q_1)^2 (l+q_2)^2 (l+q_3)^2 (l+q_4)^2}$$

$$\sim \frac{1}{4}\left(g_{\mu_1\mu_2} g_{\mu_3\mu_4} + g_{\mu_1\mu_3} g_{\mu_4\mu_2} + g_{\mu_1\mu_4} g_{\mu_2\mu_3}\right) I_4^{UV}(8-2\varepsilon;1,1,1,1) + \cdots$$

$$I_4^{UV}(8-2\varepsilon;1,1,1,1) = I_4(6;1,1,1,1) - \frac{1}{1-2\varepsilon}\Big(b_1 I_3^{UV}(6-2\varepsilon;0,1,1,1) + b_2 I_3^{UV}(6-2\varepsilon;1,0,1,1)$$

$$+ b_3 I_3^{UV}(6-2\varepsilon;1,1,0,1) + b_4 I_3^{UV}(6-2\varepsilon;1,1,1,0)\Big)$$

From Scalar Integrals to Basis Set

- The remaining divergences are the IR divergences.
- Ofter, we know what they are:
  E.g. for the 6-gluon color ordered amplitude we get (W.G. & E.W.N. Glover):

$$m^{(1)}(123456) \sim \frac{\alpha_S N}{2\pi} \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \left(\sum_{ij} \frac{\left(-s_{ij}\right)^{-\varepsilon}}{\varepsilon^2} \right) m^{(0)}(123456) + F^{(1)}(123456) + \mathrm{O}\left(\frac{1}{N}\right)$$

(given the appropriate separation of $I_3^{IR} \sim I_3^{DIV} + I_3^{FIN}$)

From Scalar Integrals to Basis Set

- But we might want to calculate the divergent part analytical (e.g. the color suppressed part in the 6-gluon amplitude)
- This can be done without using the recursion relations (S. Dittmaier, W.G & E.W.N. Glover). The coefficients of the IR divergent triangles can be calculated analytical:

$$\int d\,l \frac{l_{\mu_1} l_{\mu_2} \cdots l_{\mu_m}}{(l+q_1)^2 (l+q_2)^2 \cdots (l+q_N)^2} \rightarrow T^{(D;\nu_1\nu_2\nu_3)}_{\mu_1\mu_2\cdots\mu_m}\left(q_1 q_2 \cdots q_N\right) I_3^{DIV}\left(D;\nu_1\nu_2\nu_3\right)$$

- Where the tensor T is known for any (m,N).

From Scalar Integrals to Basis Set

- So, generically we get

$$M^{NLO}(123456) \to V(123456)M^{LO} + F(123456)$$

$$F(123456) = \sum K_5 I_5\left(6;1,1,1,1,1\right) + \sum K_4 I_4(6;1,1,1,1) + \sum K_3 I_3^{fin}(D;\nu_1\nu_2\nu_3)$$

$$V(123456) = \sum K_3^{DIV} I_3^{DIV}(D;\nu_1\nu_2\nu_3) \sim \alpha_S N c_\Gamma \left\{ \sum \frac{\left(-s_{ij}\right)^{-\varepsilon}}{\varepsilon^2} + \frac{3}{2} n_{q\bar{q}} \frac{1}{\varepsilon} + \mathrm{O}\left(\frac{1}{N^2}\right) \right\}$$

- $K_5 = 0$ and $\left(K_3, K_4\right)$ are calculated numerical
- The analytic expressions for I_4 and I_3^{FIN} are evaluated numerical
- I_3^{DIV} used in algebraic evaluation

Outlook

- We are in the middle of implementing an algorithm for calculating up to $2 \to 4$ and $1 \to 5$ Standard Model processes:
  - Finished first validation of recursive algorithm: $\gamma\gamma \to \gamma\gamma$
  - Working on second validation: $\gamma\gamma \to \gamma\gamma\gamma\gamma$
  - Finally for algorthmic optimalization we need to calculate the diagrammatically most complicated process : $gg \to gggg$

Outlook

- After the validation/optimalization of the algorithm we can start working on specific MC programs for Run 2/LHC/LC
- Calculating the one loop diagrams within this scheme is straightforward :
  1. Apply Davydychev decomposition to the Feynman diagrams: this gives Lorenz structures multiplying scalar integrals
  2. Use the recursion algorithm to numerically evaluate the scalar integrals to give an evaluation of finite virtual
  3. Calculate the divergent part of virtual and combine with soft real emission using slicing/subtraction/sector decomposition to construct the MC

Outlook

- Potential future extensions:
  - Extend the algorithm to go beyond 6 external particles. (PC farms project)
  - Construct a formalism for 2-loop.
  - (Grid computing)