Some Aspects of Multi-Loop Calculation

J.H. Kühn

Universität Karlsruhe, Institut für Theoretische Teilchenphysik

I. Two-Loop Form Factor and Sudakov Logarithms
[Feucht, Penin, Smirnov]

II. Four-Loop Tadpoles - A Status Report
[Chetyrkin, Faisst, Mastrolia, Sturm]

III. Massless Propagators:
The Long March Towards $R(s)$ at $O(\alpha_s^4)$
[Baikov,Chetyrkin]

I. Two-Loop Form Factor and Sudakov Logarithms

Large logarithmic corrections to exclusive electroweak processes

at high energies: $Q^2 >> M^2$

- NNLL terms for four-fermion processes were evaluated
- Large subleading corrections were observed

$\Rightarrow$ Next step: N^3LL for 4-fermion processes

Important ingredients:

A) Complete two-loop result for Form Factors in massive Abelian theory

$$\mathcal{F}(\alpha, M, Q) = 1 + \left(\frac{\alpha}{4\pi}\right) f^{(1)} + \left(\frac{\alpha}{4\pi}\right)^2 f^{(2)} + \quad \cdots$$

Drop power suppressed terms; define $\mathcal{L} = \ln(Q^2/M^2)$;

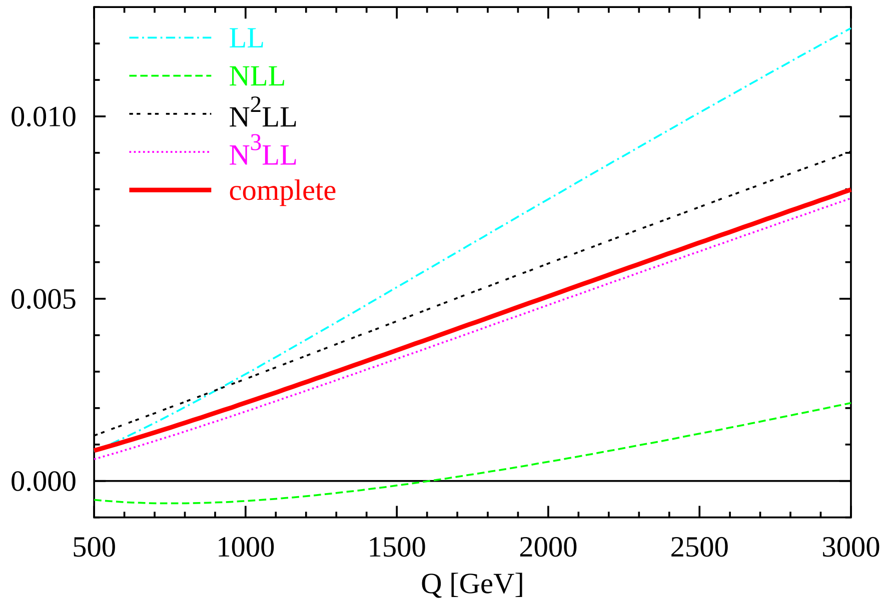
$$f^{(1)} = -\mathcal{L}^2 + 3\mathcal{L} - \frac{7}{2} - \frac{2}{3}\pi^2$$

Explicit two-loop calculation (Feucht, Penin, Smirnov) *via* expansion by region (Smirnov,...)

$$\begin{aligned} f^{(2)} \quad = \quad & \frac{1}{2}\mathcal{L}^4 - 3\mathcal{L}^3 + \left(8 + \frac{2}{3}\pi^2\right)\mathcal{L}^2 - \left(9 + 4\pi^2 - 24\zeta(3)\right)\mathcal{L} \\ & + \frac{25}{2} + \frac{52}{3}\pi^2 + 80\zeta(3) - \frac{52}{15}\pi^4 - \frac{32}{3}\pi^2 \ln^2 2 + \frac{32}{3}\ln^4 2 + 256\,\mathrm{Li}_4\left(\frac{1}{2}\right) \end{aligned}$$

Large subleading terms $(\mathcal{L}^3, \mathcal{L}^2)$ in agreement with earlier results.

Remaining terms suppressed.



⇒ Complete resummed result in N^4LL:

$$\mathcal{F}(\alpha, M, Q) = \exp\left\{\frac{\alpha}{4\pi}\left[-\mathcal{L}^2 + \left(3 - \frac{\alpha}{4\pi}\left(-\frac{3}{2} + 2\pi^2 - 24\zeta(3)\right) + \mathcal{O}(\alpha^2)\right)\mathcal{L}\right]\right\}\mathcal{F}(\alpha, M, M).$$

B) Complete two-loop result for theory with mass gap $(Q^2 >> M^2 >> \lambda^2)$ and two couplings, α, α':

$$\mathcal{F}(\alpha,\alpha',\lambda,M,Q) = \tilde{\mathcal{F}}(\alpha,\alpha',M,Q)\cdot\mathcal{F}_{\alpha'}(\lambda,Q) + \mathcal{O}(\lambda^2/M^2)$$

where:

$\mathcal{F}_{\alpha'}(\lambda,Q)$ contains all infrared divergencies ;

$$\tilde{\mathcal{F}}(\alpha,\alpha',M,Q) = 1 + \left(\frac{\alpha}{4\pi}\right) f^{(1)} + \left(\frac{\alpha}{4\pi}\right)^2 f^{(2)} + \frac{\alpha\,\alpha'}{(4\pi)^2} f^{(1,1)} \;;$$

$f^{(1)}, f^{(2)}$ as above ;

$$f^{(1,1)} = \left(3 - 4\pi^2 + 48\zeta(3)\right)\mathcal{L} - 2 + \frac{20}{3}\pi^2 - 84\zeta(3) - \frac{7}{45}\pi^4 \,.$$

$\Rightarrow$ main ingredients for 4-fermion processes up to N^3LL.

II. Four-Loop Tadpoles

A) Motivation

One-scale tadpoles (+ expansions) lead to analytic results for many important observables

e.g. @ three-loop:

- *top* contribution to $\delta\rho$ in order $G_F m_t^2 \alpha_s^2$, $(G_F m_t^2)^2 \alpha_s$, $(G_F m_t^2)^3$

  ($\Rightarrow m_t$ from precision e.w. data)

- $R(s)$ for massive quarks in $O(\alpha_s^2)$

- *charm* and *bottom* mass from moments

Moments: $\mathcal{M}_n^{\rm th} \equiv \frac{12\pi^2}{n!}\left(\frac{\rm d}{{\rm d}q^2}\right)^n \Pi_c(q^2)\Big|_{q^2=0} = \frac{9}{4}Q_c^2\left(\frac{1}{4m_c^2}\right)^n \bar{C}_n$

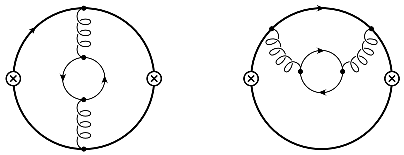
$\bar{C}_n$ depend on the charm quark mass through

$$
\begin{aligned}
l_{m_c} &\equiv \ln(m_c^2(\mu)/\mu^2) \\
\bar{C}_n &= \bar{C}_n^{(0)} + \frac{\alpha_s(\mu)}{\pi}\left(\bar{C}_n^{(10)} + \bar{C}_n^{(11)} l_{m_c}\right) \\
&\quad + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left(\bar{C}_n^{(20)} + \bar{C}_n^{(21)} l_{m_c} + \bar{C}_n^{(22)} l_{m_c}^2\right)
\end{aligned}
$$

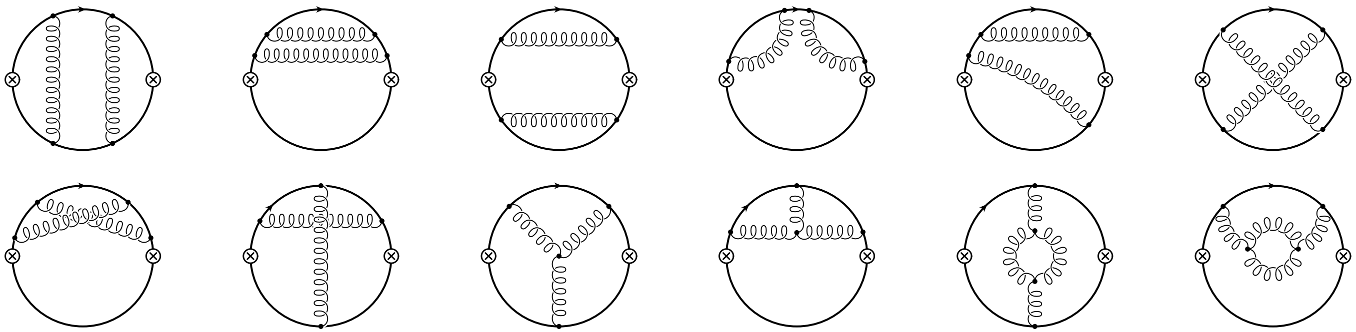
Dispersion Relation: $\Pi_c(q^2) = \frac{q^2}{12\pi^2}\int {\rm d}s\, \frac{R_c(s)}{s(s-q^2)}$ + subtraction

$\Rightarrow \quad \mathcal{M}_n^{\rm exp} = \int \frac{{\rm d}s}{s^{n+1}} R_c(s) \qquad \Rightarrow \quad \mathcal{M}_n^{\rm exp} = \mathcal{M}_n^{\rm th} \qquad \Rightarrow \quad m_c$

All three-loop (NNLO) – one-scale tadpole amplitudes can be calculated with “arbitrary” power of propagators (Broadhurst; Chetyrkin, JK, Steinhauser); FORM-program MATAD (Steinhauser)



Three-loop diagrams contributing to $\Pi_l^{(2)}$ (inner quark massless) and $\Pi_F^{(2)}$ (both quarks with mass m).



Purely gluonic contribution to $O(\alpha_s^2)$

Results (Steinhauser, JK)

$$m_c(m_c) = 1.304\ (27)\ GeV$$

$$m_b(m_b) = 4.19\ (5)\ GeV$$

analysis relies on lowers ($n = 1, 2$) moments;
$n = 3, 4$ have a smaller experimental error, but larger theoretical uncertainty;

$\Rightarrow$ 4-loop Tadpoles

B) Toward an Analytical Solution (Chetyrkin, Mastrolia, Sturm)

Ingredients:

- generation of diagrams (QGRAPH) simple
- classification of diagrams $\Rightarrow$ standard representation

  (detect symmetries, massless tadpoles, factorized topologies)

! reduction to Master Integrals in progress

Integration-by-Parts identities:

explicit solution, like MATAD, impossible

$\Rightarrow$ huge number of linear equations, $\mathcal{O}(100\,000)$,

SOLVE (Remiddi) or variants

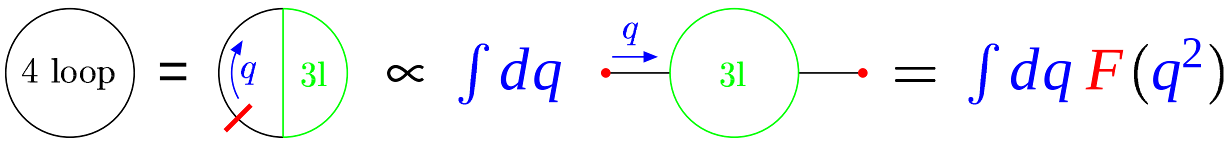
n_l^2-terms just obtained:

$$\frac{35794}{729} - \frac{992}{27}\zeta(3) + \frac{q^2}{4m^2}\left(\frac{2699072}{32805} - \frac{7168}{135}\zeta(3)\right)$$

C) Numerical Approach (Chetyrkin, Faisst)

Idea: perform the integration over 3 loop-momenta analytically and the 4^{th} numerically

Diagramatically:



resulting problem: find the function $F(q^2)$ up to three-loop level

Reconstruction of $F(q^2)$

Three-loop two-point function with one internal mass scale!

algebraic programs:

- asymptotic expansion: EXP (Seidensticker)
- algebraic evaluation of tadpoles: MATAD (Steinhauser)
- massless propagators: MINCER (Larin, Tkachov, Vermaseren)
- high and low energy expansion of $F(q^2)$

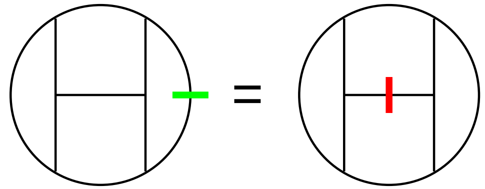
$\Rightarrow$ reconstruction of $F(q^2)$ through Padé

Test Example:

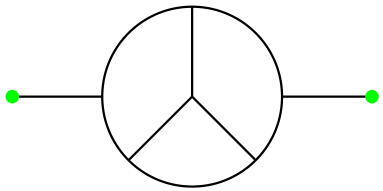
(scalar, massive propagators)

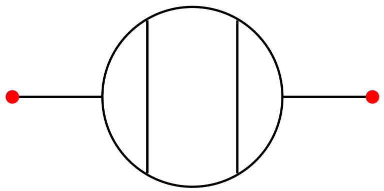
Result available with high precision

Laporta hep-ph/0210336



The two options lead to to the following 3-loop propagator diagrams:

1. 

2. 

Difference between Laporta's and our result

Laporta's: 1.34894 80217 09708 . . .

Result for cut 1:

$\times 10^{-5}$		high energy input 3	4	5	6	7	8
low energy input	3	-191.42436	-6.01348	0.44044	0.09165	0.00151	-0.01738
	4	-5.15001	1.94591	-0.10397	-0.00344	0.00052	0.00029
	5	-16.06059	0.74717	0.01367	-0.00101	0.00388	-0.00025

Result for cut 2:

$\times 10^{-5}$		high energy input 3	4	5	6	7	8
low energy input	3	-579.78184	8.92004	-0.08593	0.01259	0.00456	0.00010
	4	-186.75432	1.21461	-0.04923	-0.00242	-0.00008	-0.00004
	5	-41.00154	0.53958	-0.10894	0.00332	0.00009	-0.00002

III. Massless Propagators

$R(e^+e^-)$ and R_τ at order α_s^4

R_τ: analysis uses estimates of the α_s^4-term, based on PMS etc.

$R(e^+e^-)$: "gold plated" determination of α_s (inclusive, NNLO) ;

but:

- LEP: theory-uncertainty comparable with experimental error
- GIGA-Z: theory error dominant
- B-factory: millions of events;

10 GeV $\Rightarrow$ larger sensitivity to α_s
larger sensitivity to higher order

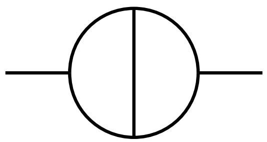
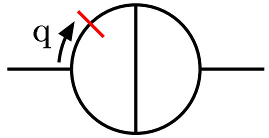
Strategy

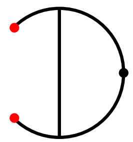
α_s^4 requires absorptive part of 5-loop correlator

$\widehat{=}$ divergent part $(1/\varepsilon)$ of 5-loop correlator

A finite part of 4-loop $\Rightarrow$ div. part of 5-loop

systematic, automatized algorithm (Chetyrkin)

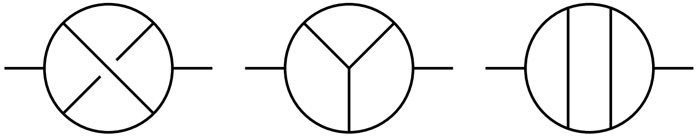
div  $\widehat{=} \int dq^2$ 

requires 

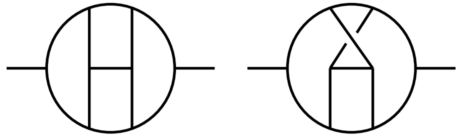
B finite part of 4-loop massless propagators difficult!

compare 3- and 4-loop calculation

3 topologies without insertions

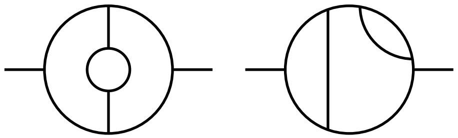


11 topologies without insertion



. . .

14 topologies with+without insertions



. . .

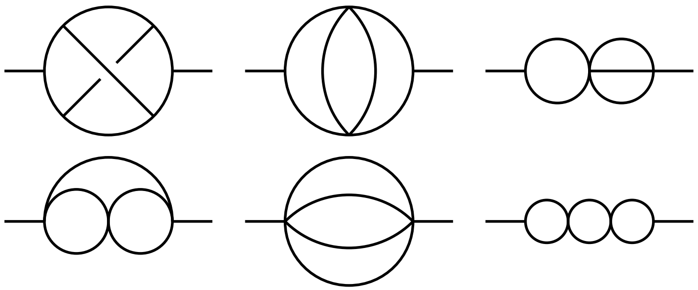
∼150 topologies with+without insertions

. . .

reduction to master integrals:

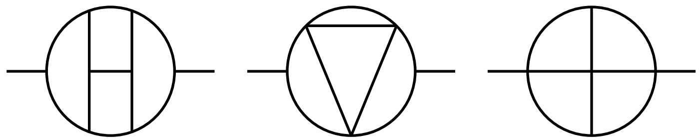
MINCER

6 master integrals



reduction to master integrals ???

28 master integrals



. . .

MINCER: 3-loop (Larin, Tkatchov, Vermaseren)

recursion relations based on integration by parts identities!

reduction algorithm and program constructed “manually” for 14 topologies.

4-loop:

more complicated identities

$\sim$ 150 topologies . . .

straightforward generalization of MINCER diffi cult

$\Rightarrow$ fully automatized construction of program; new concept?

C Baikov: recursion relations can be solved “mechanically” in the limit of large dimension d:

consider amplitude f:

f(topology, power of prop, d)
$= \sum_{\alpha=\text{masters}} C^{(\alpha)}$(topology, power of prop, d) $\star f^{(\alpha)}(d)$

$f^{(\alpha)}$: 28 masters, analytically or numerically solvable

$C^{(\alpha)}$: rational function $\frac{P^n(d)}{Q^m(d)}$, to be calculated

expand $C^{(\alpha)}$:

$C^{(\alpha)} = \sum_k c_k^{(\alpha)}$(topology, power of prop)$(1/d)^k + \ldots$

suffi ciently many terms $c_k^{(\alpha)} \Rightarrow C^{(\alpha)}$

$m,\ n$ depend on power of propagators!

evaluation of $c_k^{(\alpha)}$:

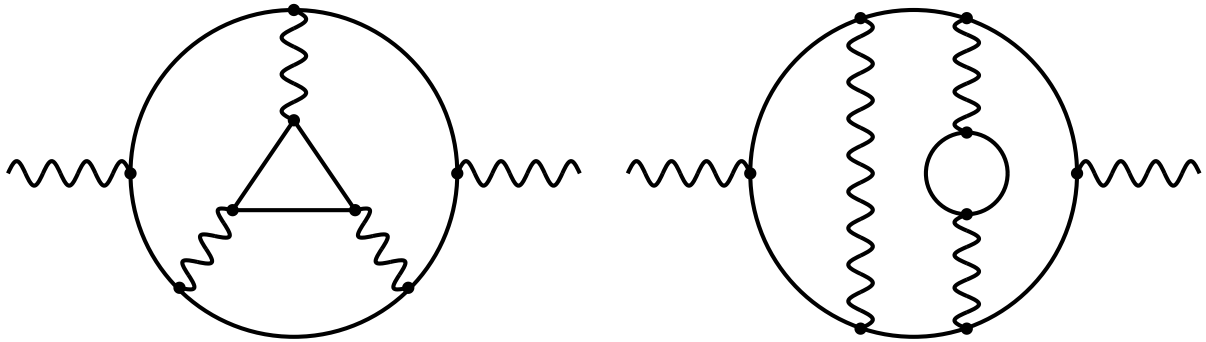
handling of polynomials of 9 variables of degree k

$\frac{(9+k)!}{9!\,k!}$ terms $\quad k = 40 \;\Rightarrow\; 2 \cdot 10^9$ terms

$k = 24 \;\Rightarrow\; 4 \cdot 10^7$ terms (4 GB disk $\rightarrow$ 40 GB)

weeks of runtime

addition information on structure of $P^n(d)$, $Q^m(d)$ may lead to drastic reduction of hardware requirements status: four-loop n_f-terms done

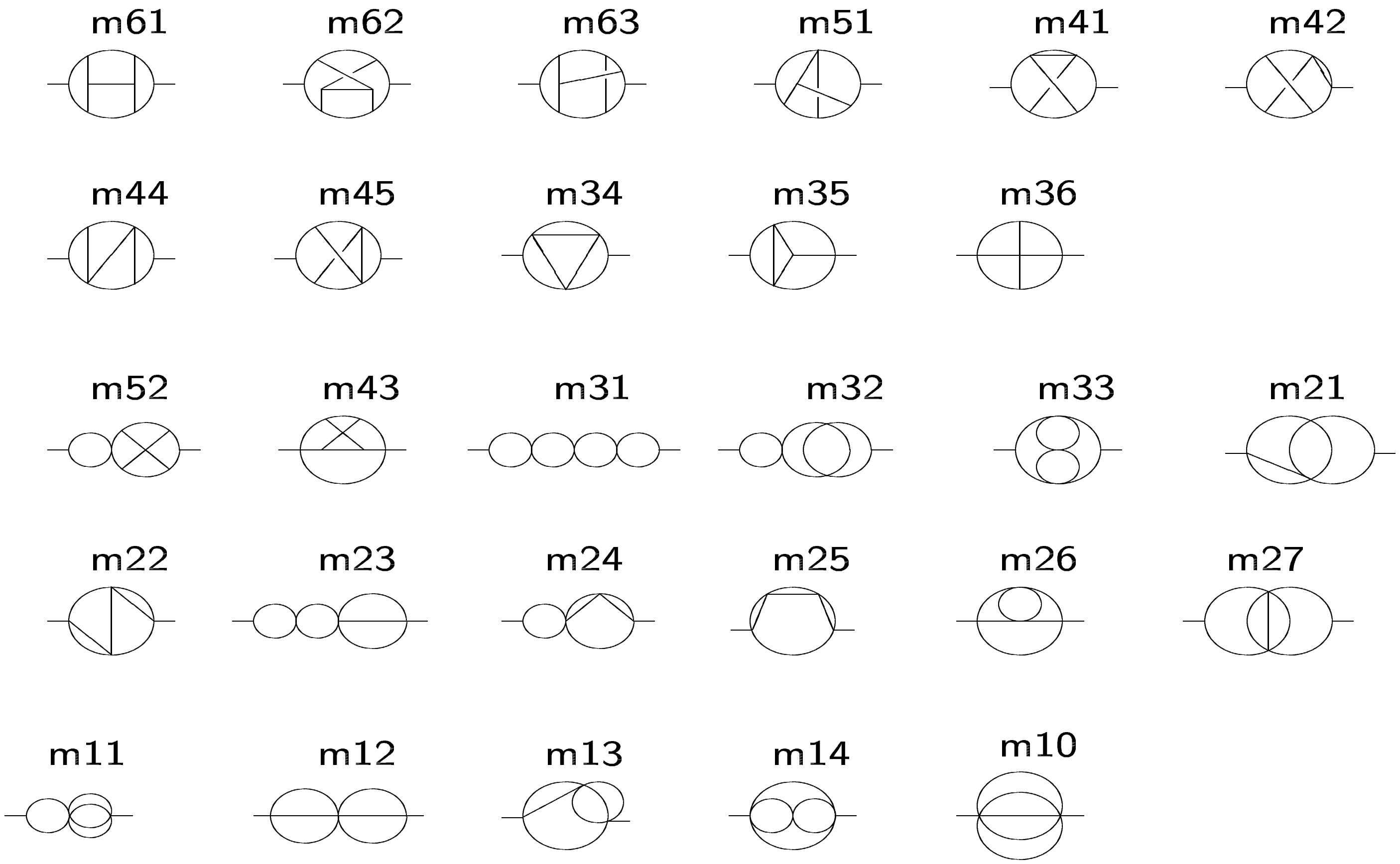


$\Rightarrow$ leading and subleading n_f terms for $R_{e^+e^-}$, R_τ, m^2/s-terms:

$\alpha_s^4 n_f^3$ (renormalon chain)

$\alpha_s^4 n_f^2$ new

NEW All relevant Master Integrals solved ("glue and cut" (Chetyrkin, Tkachov))



NEW Complete 4-loop mass correction

Define

$$\Pi_{\mu\nu} = (-g_{\mu\nu}\, q^2 + q_\mu q_\nu)\Pi(q^2)$$
$$\Pi(q^2) = \Pi_0(q^2) + \frac{m^2}{q^2}\Pi_2(q^2)$$

finite part of Π_2 in $O(\alpha_s^n)$ + RG

$\Rightarrow$ log part in $O(\alpha_s^{n+1})$

$\Rightarrow$ Im part in $O(\alpha_s^{n+1})$

Result for constant part in Π_2

$$\Pi_2 = -8 - \frac{64}{3} a_s + a_s^2 \left\{ \frac{95}{9} n_f + -\frac{18923}{54} - \frac{784}{27} \zeta_3 + \frac{4180}{27} \zeta_5 \right\}$$
$$+ a_s^3 \left\{ \left[-\frac{5161}{1458} - \frac{8}{27} \zeta_3 \right] n_f^2 + \left[\frac{62893}{162} + \frac{424}{27} \zeta_3^2 - \frac{4150}{243} \zeta_3 + \frac{20}{3} \zeta_4 - \frac{28880}{243} \zeta_5 \right] n_f + k_{2,0}^{[V]3} \right\}$$

$$k^{[V]3} = -\frac{10499303}{1944} + \frac{66820}{81} \zeta_3 - \frac{7225}{27} \zeta_3^2 + \frac{281390}{81} \zeta_5 - \frac{1027019}{648} \zeta_7$$

Numerically,

$$\Pi_2 = -8 - 21.333 a_s + a_s^2 (10.56 n_f - 224.80)$$
$$+ a_s^3 (-3.896 n_f^2 + 274.37 n_f - 2791.81)$$

Result for r_2^V

Define

$$R(s) = 3\left\{r_0^V + \frac{m^2}{s} r_2^V\right\} + \cdots = 3\left\{\sum_{i\geq 0} a_s^i \left(r_0^{V,i} + \frac{m^2}{s} r_2^{V,i}\right)\right\} + \ldots$$

$$r_2^V = 12\, a_s + a_s^2 \,(-4.3333\, n_f + 126.5) + a_s^3 \left(1.2182\, n_f^2 - 104.167\, n_f + 1032.14\right)$$
$$+ a_s^4 \left(-0.20345\, n_f^3 + 49.0839\, n_f^2 + \; r_{2,1}^{V,4}\, n_f \; + r^{V,4}{}_{2,0}\right)$$

For, say, $n_f = 4$ we get

$$r_2^V/12 \text{ (exact)} = a_s + 9.09722 a_s^2 + 52.913 a_s^3 + 128.499 a_s^4$$

$$r_2^V/12 \text{ (PMS)} = a_s + 9.09722 a_s^2 + 52.913 a_s^3 + 177 a_s^4$$

$$r_2^V/12 \text{ (FAC)} = a_s + 9.09722 a_s^2 + 52.913 a_s^3 + 197 a_s^4$$

Summary/Outlook

- Sudakov Logarithms

  maybe important in the TeV-range

  subleading terms are large

  N^4LL is available for Form Factors

  N^3LL is in sight for 4-fermion processes

- 4-loop Massive Tadpoles

  will lead to an excellent determination of quark masses and open the door to many other applications

- 4-loop Massless Propagators

  first (partial) results available

  will lead to the most precise and theoretically safe result for α_s